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THE EMBEDDING OF HOMEOMORPHISMS OF THE PLANE IN CONTINUOUS FLOWS

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A study of fundamental regions of the plane under an orientation preserving, fixed point free, self-homeomorphism of the plane is made under the conditions that there are finitely many fundamental regions R_i under f, if $x \in R_i - \operatorname{Int} R_i$, then $x \in C \subset R_i - \operatorname{Int} R_i$ where C is a proper flowline, and if x_1 and x_2 are in $\operatorname{Int} R_i$, then $x_1 \sim x_2 \bmod \operatorname{Int} R_i$. The topological structure of the fundamental regions is determined. Using these results, it is shown that in certain cases the embedding problem can be reduced to a problem of extending a continuous flow defined on an open set to the closure of the set.

In the last section, sufficient conditions for self-homeomorphisms of the plane and the closed unit disc with one fixed point to be embedded in continuous flows are given.

Let X be a topological space and let G denote a topological group. The ordered triple (X, G, π) is a *dynamical system* if (1) $\pi: X \times G \rightarrow X$ is continuous, (2) $\pi(\pi(x, g_1), g_2) = \pi(x, g_1 + g_2)$, and (3) $\pi(x, e) = x$ for every $x \in X$. If G is the additive group of real numbers, then (X, G, π) is called a continuous flow. If G is the additive group of integers, then (X, G, π) is a discrete flow.

This paper is concerned with the following problem. If G is a given topological group and (X, G^*, π^*) is a given dynamical system, where G^* is a subgroup of G with the relative topology, then find a dynamical system (Y, G, π) such that (1) $\pi(y, g^*)$ for $g^* \in G^*$ is invariant on a subset Z of Y, (2) Z is homeomorphic to X, and (3) π on $Z \times G^*$ is topologically equivalent to π^* on $X \times G^*$, i.e. there is a homeomorphism $h: X \rightarrow Z$ such that $h^{-1}(\pi(h(x), g^*)) = \pi^*(x, g^*)$. If G is the additive group of real numbers and G^* is the additive group of integers, then the problem is that of embedding a self-homeomorphism in a continous flow.

If the space X is allowed to be enlarged in order to accommodate π, the problem is referred to as the *unrestricted problem*. If $X = Y = Z$ the problem is referred to as the *restricted problem* [5].

If G is the additive group of real numbers and G^* is the integers, then the unrestricted problem is easily solved [5], [7], [8] for any topological space X and any self-homeomorphism.

The restricted problem is only partially solved. Fine and Schweigert [2] and Fort [6] proved that a self-homeomorphism T of an interval can be embedded in a continuous flow if and only if it is order preserving. Utz [11] gave a constructive proof of this theorem.

Foland and Utz [5] solved the restricted problem where T is an orientation preserving self-homeomorhism of a circle. Foland [3] has shown that if T is an almost periodic, orientation preserving self-homeomorphism of a closed 2-cell, then T can be embedded in a continuous flow on the 2-cell. Moreover, if T is almost periodic but not periodic, then the embedding is unique.

Andrea [1] has given sufficient conditions for a fixed point free self-homeomorphism of the Euclidean plane to be embedded in a continuous flow.

2. Fundamental regions of the plane. In this section numerous properties of fundamental regions of the plane will be given, and using these properties it will be possible to give a description of the plane when it consists of n fundamental regions.

In this section f will be a self-homeomorphism of the plane which is orientation preserving and which has no fixed points.

DEFINITION 1. [1]. If p and q are points in the plane, then $p \sim q$ if p and q are endpoints of some curve segment C (the homeomorph of the closed unit interval) for which $f^n(C) \rightarrow \infty$ as $n \rightarrow \pm \infty$.

THEOREM 1. [1] *The relation $\sim$ is an equivalence relation.*

DEFINITION 2. [1] The fundamental regions of the plane under f are the equivalence classes of the equivalence relation $\sim$.

DEFINITION 3. If A is a subset of the plane, then $x \sim y \bmod A$ if x and y are endpoints of some curve segment $C \subset A$ such that for every compact set K in the plane there is a natural number N such that $f^n(C) \cap K = \phi$ when $|n| > N$.

DEFINITION 4. [1]. A proper flowline for f is a subset F of the plane such that

(a) $f(F) = F$,

(b) F is a one-to-one continuous image of the real line, and

(c) $F \cup \{\infty\}$ is a Jordan curve on the sphere $R^2 \cup \{\infty\}$.

The following three conditions will be assumed in this section and in §3.

(1) There exists exactly n fundamental regions R_i under f.

(2) If $x \in R_i - \operatorname{Int} R_i$, then $x \in C \subset R_i - \operatorname{Int} R_i$ where C is a proper flowline.

(3) If $x_1, x_2 \in \operatorname{Int} R_i$, then $x_1 \sim x_2 \bmod \operatorname{Int} R_i$.

We will let R or R with a subscript denote fundamental regions of the plane under f.

It follows from the following remark that each fundamental region is f-invariant.

REMARK 1. [1]. Suppose that $\{E_i\}$ is a finite collection of disjoint arcwise connected sets. If each $f(E_i) = E_j$, then $f(E_i) = E_i$ for every i.

REMARK 2.
(a) R is arcwise connected,
(b) R is unbounded, and
(c) if C is a simple closed curve contained in R, then $\operatorname{Int}(C) \subset R$.

Proof. (a) and (b) follow directly from the definitions. Suppose that (c) does not hold. Then there is $x \in \operatorname{Int} C$ such that $x \notin R$. Suppose $x \in S$ where S is a fundamental region under f. Since $f^n(x) \to \infty$, as $n \to \infty$, $f^n(x) \notin C \cup \operatorname{Int} C$ for some n. But $f^n(x) \in S$ and S is arc wise connected. Therefore, there is an arc L joining x and $f^n(x)$ where $L \subset S$. But that is not possible since $L \cap C \neq \phi$.

REMARK 3. If there exists $x \in R_i$—$\operatorname{Int} R_i$, then $x \in C$, where C is defined by (2), and C divides the plane into two open half planes H_1 and H_2 such that

(a) $x_1, x_2 \in \operatorname{Int} R_i$ implies x_1 and x_2 are in the same half plane,

(b) if $\operatorname{Int} R_i \neq \phi$ and if $x \in C$, then there is a neighborhood U of x such that $U \cap H_1 \subset \operatorname{Int} R_i$ where $\operatorname{Int} R_i \subset H_1$, and (c) if $\operatorname{Int} R_i = \phi$, then $R_i = C$.

Proof. (a) Let x_1 and $x_2 \in \operatorname{Int} R_i$. Let $x_1 \in H_1$ and $x_2 \in H_2$. By (3), there is an arc A joining x_1 and x_2 such that $A \subset \operatorname{Int} R_i$. Therefore, $A \cap C \neq \phi$, which is impossible since $C \subset R_i - \operatorname{Int} R_i$.

(b) Let $x \in C$. Let $x_1 \in \operatorname{Int} R_i$. Then there is a curve $L \subset R_i$ such that x_1 and y are endpoints of L, where $y \in C$, and $L \cap C = \{y\}$. Since $f(C) = C$ and f is fixed point free, there are integers m and n such that $f^n(y)$ and $f^m(y)$ are separated on C by x.

If $f^n(L) \cap f^m(L) \neq \phi$, let L_1 be the smallest closed subsegment of $f^n(L)$ that contains $f^n(y)$ such that $L_1 \cap f^m(L) \neq \phi$. Let L_2 be the smallest closed subsegment of $f^m(L)$ that contains $f^m(y)$ such that $L_2 \cap L_1 \neq \phi$. If C_1 is the closed subsegment of C from $f^m(y)$ to $f^n(y)$, then $C_1 \cup L_1 \cup L_2$ forms a simple closed curve. Let $\varepsilon = \min \{d(x, L_1), d(x, L_2)\}$. Then if $S = \{z\colon d(z, x) < \varepsilon\}$, $S \cap H_1$ is contained in the simple closed curve $C_1 \cup L_1 \cup L_2$, which, by Remark 2, is contained in R_i.

There is an arc $K \subset \operatorname{Int} R_i$ such that $f^n(x_1)$ and $f^m(x_1)$ are its endpoints. If $f^n(L) \cap f^m(L) = \phi$, let L_1 be the smallest closed subsegment of $f^n(L)$ containing $f^n(y)$ such that $L_1 \cap K = \{z_1\}$. Let L_2 be the smallest closed subsegment of $f^m(L)$ containing $f^m(y)$ such that $L_2 \cap K = \{z_2\}$. Let K_1 be the subsegment of K such that z_1 and z_2 are its endpoints. Now $C_1 \cup K_1 \cup L_1 \cup L_2$ is a simple closed curve. Let $S = \{u: d(x, u) < \varepsilon\}$ where $\varepsilon = \min\{d(x, K_1), d(x, L_1), d(x, L_2)\}$. Then $S \cap H_1$ is contained in the simple closed curve $C_1 \cup L_1 \cup L_2 \cup K_1$ which is contained in R_i.

(c) If $x \in R_i$, then by (2) there is a proper flowline C such that $x \in C$. If $y \in R_i - C$ then there is a proper flowline C' such that $y \in C'$.

If $C \cap C' \neq \phi$, let L_1^+ be the closed half line from y containing $f(y)$ and L_1^- the closed half line from y containing $f^{-1}(y)$. Let C_1 and C_2 be the smallest closed subsegments of L_1^- and L_1^+ respectively such that $C_i \cap C = \{x_i\}$ for $i = 1, 2$. Such segments do exist because $C' \cap C \neq \phi$, $y \notin C$, and the properties of f. Thus the segment from x_1 to x_2 on C, C_1 and C_2 form a simple closed curve and every point in its interior must be in R_i, which is not possible.

If $C \cap C' = \phi$, we can join C to C' by a segment $A \subset R_i$ such that $A \cap C = \{x_1\}$ and $A \cap C' = \{x_2\}$. Let C_i be the closed subsegments of C and C' respectively from x_i to $f(x_i)$ for $i = 1, 2$. Let A_1 be the smallest closed subsegment of $f(A)$ such that $A_1 \cap (A \cup C_2) \neq \phi$. Now C_1, A_1, and $(A \cup C_2)$ from x_1 to $A_1 \cap (A \cup C_2)$ is a simple closed curve and the result follows.

REMARK 4. If C_1 and C_2 contained in R_i are curves defined as in (2) such that $C_1 \neq C_2$, then $C_1 \cap C_2 = \phi$. Also, if C_1 is a curve defined as in (2), then R_i is contained in the closed half plane defined by C_1 containing Int R_i.

Proof. Suppose $C_1 \neq C_2$, but $C_1 \cap C_2 \neq \phi$. Suppose $y \in C_1 - C_2$. Let A be a closed subsegment of C_1 containing y such that x_1 and x_2 are endpoints of A, x_1 and $x_2 \in C_1 \cap C_{2'}$ and $(A - \{x_1, x_2\}) \cap C_2 = \phi$ for $i = 1, 2$. Now A together with the subsegment of C_2 from x_1 to x_2 forms a simple closed curve K. Let $z \in$ the interior of K. Now $f^n(z) \to \infty$ as $n \to \infty$, thus $f^n(z) \in$ the interior of $f^n(K)$ but $f^n(z) \notin$ interior K for some integer n. Now z and $f^n(z) \in \operatorname{Int} R_i$ by Remark 2, but z and $f^n(z)$ can not be connected by an arc in Int R_i which is contrary to (3).

Suppose there is $x \in R_i$ but in the opposite open half plane, determined by C_1, from Int R_i. Then by the above paragraph and (2), there is a proper flowline C_2 such that $x \in C_2$ and $C_1 \cap C_2 = \phi$. Now the proof of Remark 3c can be applied to show that there must be a point in the same half plane as x which is in Int R_i, which is not possible.

THEOREM 2. [1]. *The self-homeomorphism f is equivalent to a* *translation if and only if it has exactly one fundamental region.*

REMARK 5. Let C be a proper flowline of f. Then there is a homeomorphism g such that $g|_c = f|_c$ and such that g is topologically equivalent to a translation.

Proof. Since C is a proper flowline of f there is a selfhomeomorphism h of the plane where $h(C) = L$ and where L is the y-axis. Define $T\colon P \to P$, where P is the plane, by

$$T(0, y) = hfh^{-1}(0, y), \text{ where } (0, y) \in L$$
$$T(x, y) = (x, (hfh^{-1}(0, y))_y) \text{ where } (x, y) \notin L$$

and where $(hfh^{-1}(0, y))_y$ denotes the y coordinate of $hfh^{-1}(0, y)$. Then T is an orientation preserving self-homeomorphism of the plane with no fixed points and with exactly one fundamental region. Therefore, $g = h^{-1}Th$ has this property, $g|_c = f|_c$, and g is equivalent to a translation by Theorem 2.

REMARK 6. If $R - \operatorname{Int} R$ contains two distinct proper flow lines C_1 and C_2, then R is the closed strip bounded by C_1 and C_2.

Proof. By Remark 3, if C_1 and C_2 are distinct proper flow lines in R-$\operatorname{Int} R$, then $\operatorname{Int} R \neq \phi$. Let H_j^1 and H_j^2 be the two closed half planes defined by C_j. By Remark 4, it can be assumed that $R \subset H_1^2$. Therefore, $C_2 \subset H_1^2 - C_1$. Let H_2^1 be the half plane containing H_1^1. Then, $R \subset H_2^1$ and therefore $R \subset H_2^1 \cap H_1^2$. Let $x_3 \in H_2^1 \cap H_1^2 - (C_1 \cup C_2)$. Let B be a closed segment joining C_1 and C_2 such that $x_3 \in B$, $B \cap C_1 = \{y\}$, and $B \cap C_2 = \{z\}$. Since $y \sim z$, there is an arc $A \subset R$ joining y and z such that $f^n(A) \to \infty$ as $n \to \pm\infty$. Let A_1 be a closed subsegment of A such that $A_1 \cap C_1 = \{y_1\}$ and $A_1 \cap C_2 = \{z_1\}$. There is an integer $N > 0$ such that if $n > N$, $f^n(y_1)$ and $f^{-n}(y_1)$ are separated on C_1 by y, and $f^n(z_1)$ and $f^{-n}(z_1)$ are separated on C_2 by z. Since $f^n(A_1) \to \infty$ as $n \to \pm\infty$, there is an $M > N$ such that $f^M(A_1)$, $f^{-M}(A_1)$, and the closed subsegments of C_1 and C_2 from $f^M(y_1)$ to $f^{-M}(y_1)$ and $f^M(z_1)$ to $f^{-M}(z_1)$ respectively form a simple closed curve which contains x_3 in its interior. But that implies $x_3 \in \operatorname{Int} R$.

REMARK 7. If $x \in (R_i - \operatorname{Int} R_i) \cap \bar{R}_k$, then there is an arc A joining a point of R_k to x, where $A - \{x\} \subset R_k$. Further, if $x \in C$, where C is defined as in (2), then for every $w \in C$, there is a neighborhood U of w such that $U \cap (H' - C) \subset R_k$, where H' is the closed half plane defined by C which contains R_k.

Proof. Let $x \in C_i$, where C_i is defined by (2). Let the closed half planes defined by C_i be H_i^1 and H_i^2. Suppose $R_k \subset H_i^2$. By Remark 6, there are at most $2n$ distinct curves C_j defined as in (2). Let $\alpha = \min_{C_j \subset (H_i^2 - C_i)} \{d(x, C_j)\}$. Consider the open disc S with center at x and radius $\alpha/2$. Since $x \in \bar{R}_k$, there exists $y \in R_k \cap S$. Suppose $z \in S \cap (H_i^2 - C_i)$, where $z \notin R_k$. Since $(\bigcup_j R_j) \cap S = S$, there exists $z_0 \in S \cap (H_i^2 - C_i)$ where $z_0 \in R_{k_0} - \operatorname{Int} R_{k_0}$. But then there would exist a curve C_{k_0} defined by (2), where $C_{k_0} \cap S \cap (H_i^2 - C_i) \neq \phi$, which is not possible. The proof of the last part is similar to the proof of Remark 3b.

REMARK 8. If C is an arcwise connected component of $\bar{R} - \operatorname{Int} R$, then C is a proper flow line contained in exactly one fundamental region.

Proof. Let $x \in C$. Then $x \in (R_j - \operatorname{Int} R_j) \cap \bar{R}_k$ for some fundamental regions R_j and R_k. If $\operatorname{Int} R_j \neq \phi$, by Remarks 3 and 7, there is a neighborhood U of x such that $U \subset R_j \cup R_k$. Thus, R is R_j or R_k. Let $x \in C_1$, where C_1 is given by (2). Remarks 3 and 7 imply $C_1 \subset C$. If $w \in C - C_1$, there must be an arc in $\bar{R} - \operatorname{Int} R$ from w to C_1, which is not possible by Remarks 3 and 7. If $\operatorname{Int} R_j = \phi$ and if $R_j = R$ the result follows from Remark 3. Otherwise if $x \in C$, then $x \in R_j \cap \bar{R}$. Now applying Remark 7, it follows that $C_1 = C$.

THEOREM 3. [1] *The plane under f can not have exactly two fundamental regions.*

The notation $\#(R)$ will be used to represent the number of arcwise connected components of $\bar{R} - \operatorname{Int} R$.

REMARK 9. If $\#(R) = 1$, then R is a proper flow line or a closed half plane.

Proof. If $\operatorname{Int} R = \phi$, then R is a proper line by Remark 2. If $\operatorname{Int} R \neq \phi$, then by Remarks 3, 7 and 8, R is an open half plane or a closed half plane.

Suppose R is not closed. Then $\bar{R} - \operatorname{Int} R \subset R_2$. Thus, there exists $x \in (R_2 - \operatorname{Int} R_2) \cap \bar{R}$. Let $x \in C$, where C is defined as in (2). Let H_1^2 and H_1^1 be the closed half planes defined by C. By Remark 2, it can be assumed that $R \subset H_1^1$, and then, by Remarks 3 and $4 R_2 \subset H_1^2$. Using Remark 5, define g such that $g|_c = f|_c$. Define $F: P \rightarrow P$, where P is the plane, by

$$F(x) = \begin{cases} f(x) & \text{if } x \in H_1^1 \\ g(x) & \text{if } x \in H_1^2 \,. \end{cases}$$

Then F is an orientation preserving self-homeomorphism of the plane with no fixed points, and the plane under F has exactly one fundamental region, since, by Theorem 3, it can not have two fundamental regions. Let A be any arc joining $y \in R$ to x, where $A \subset H_1^1$. Then $f^n(A) = F^n(A) \to \infty$ as $n \to \pm\infty$. Therefore, $x \in R$, and it follows that R is closed.

REMARK 10. If $\sharp(R) = 2$, then R is either an open strip or a closed strip.

Proof. Let C_1 and C_2 be the two arcwise connected components of $\bar{R} - \operatorname{Int} R$. It follows that the interior of the strip bounded by C_1 and C_2 is contained in R, and that R is contained in the closed strip bounded by C_1 and C_2.

Suppose $x_1 \in R - \operatorname{Int} R$. Then $x_1 \in C_1$ or $x_1 \in C_2$. Suppose $x_1 \in C_1$. By Remark 8, $C_1 \subset R$.

Using Remark 5, define g_i where g_i agrees with f on C_i, $i = 1, 2$. Define F: $P \to P$, where P is the plane, by

$$F(x) = \begin{cases} g_1(x) & \text{if } x \in \text{closed half plane defined by } C_1 \text{ not con-} \\ & \text{taining } \operatorname{Int} R \\ f(x) & \text{if } x \in \text{closed strip bounded by } C_1 \text{ and } C_2 \\ g_2(x) & \text{if } x \in \text{closed half plane defined by } C_2 \text{ not con-} \\ & \text{taining } \operatorname{Int} R. \end{cases}$$

Then F is an orientation preserving self-homeomorphism of the plane, and the plane has at most two fundamental regions under F. Therefore, by Theorem 3, it has exactly one. It now follows that $C_2 \subset R$.

REMARK 11. If $\sharp(R) \geqq 3$, then either R is open or else R contains just one of the components of $\bar{R} - \operatorname{Int} R$.

Proof. If R contains at least two of the components of $\bar{R} - \operatorname{Int} R$, then by Remark 6, $\sharp(R) = 2$.

Consider the homeomorphism defined in Figure 1. Each point x lies on a curve. Let $f(x)$ be in the direction of the arrow along the curve one unit. The homeomorphism f satisfies (1), (2), and (3), and $\sharp(R_4) = 3$ but R_4 is not open.

REMARK 12. If R_1 and R_2 are not separated and R_2 and R_3 are not separated, then R_1 and R_3 are separated by R_2.

Proof. Since R_1 and R_2 are not separated, either there exists $x_2 \in (R_2 - \operatorname{Int} R_2) \cap \bar{R}_1$, or there exists $x_2 \in (R_1 - \operatorname{Int} R_1) \cap \bar{R}_2$.

Case I. Suppose $x_2 \in (R_2 - \operatorname{Int} R_2) \cap \bar{R}_1$. Let $x_2 \in C_2$, where C_2 is defined by (2). Let H_2^1 and H_2^2 be the closed half planes defined by

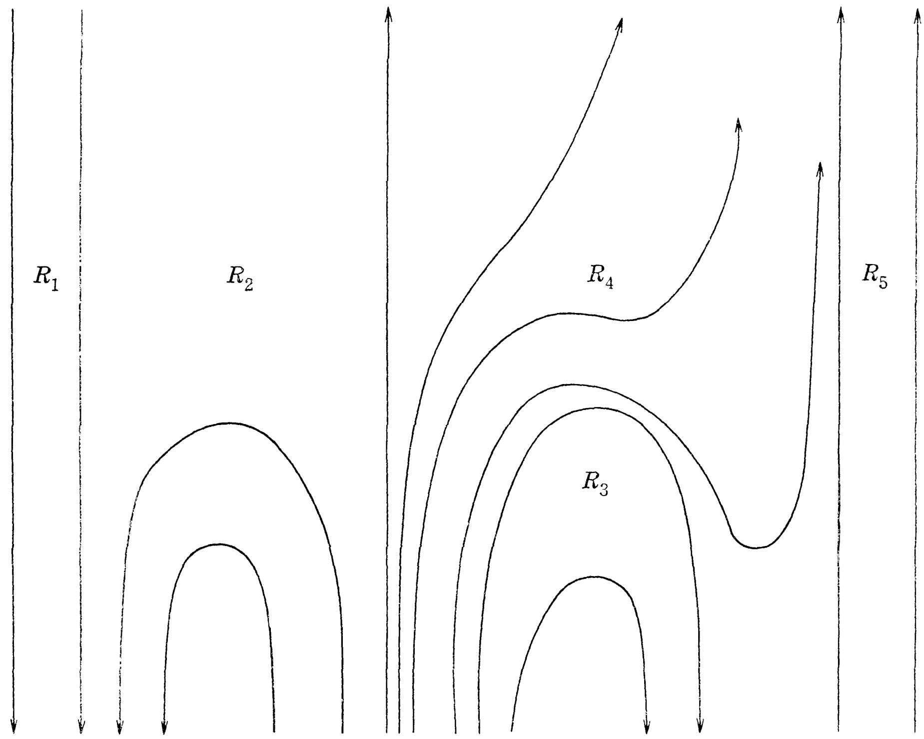

FIGURE 1

C_2. Suppose for definiteness that $R_1 \subset H_2^1 - C_2$ and then by Remark 3, $R_2 \subset H_2^2$. By Remark 7, for every $w \in C_2$, there is a neighborhood U of w such that $U \cap (H_2^1 - C_2) \subset R_1$. Since R_3 and R_2 are not separated, $R_3 \subset H_2^2 - C_2$. Therefore, R_1 and R_3 are separated by R_2.

Case II. The proof is similar to Case I.

THEOREM 4. *Suppose R_i and R_{i+1} are not separated for $i = 1, 2, \cdots, n-1$. Then*
(a) *R_i and R_k are separated for $k \neq i-1, i+1$,*
(b) *R_1 and R_n are closed half planes,*
(c) *R_{2k-1} is closed, R_{2k} is open for $k = 1, 2, \cdots$, and*
(d) *n is odd.*

The proof of this theorem follows easily from the above remarks.

The above remarks can be applied to describe the plane if there are n fundamental regions.

For example, suppose the plane $P = R_1 \cup R_2 \cup R_3$. Since P is connected, suppose R_1 and R_2 are not separated. Then either R_1 and R_3 or R_2 and R_3 are separated. Suppose R_2 and R_3 are not separated.

By Remark 12, R_1 and R_3 are separated by R_2. Thus, by Remark 9, R_1 and R_3 are closed and $R_i - \operatorname{Int} R_i = C_i$, where C_i is a proper flowline, for $i = 1, 3$. Therefore, R_2 is open.

In the case $n = 4$, 5, or 6, there are respectively 1, 3, or 4 possible arrangements of the fundamental regions.

3. Self-homeomorphism of the plane with no fixed points.

In this section it will be shown that in certain cases, the embedding problem can be reduced to a problem of extending a continuous flow defined on an open set to the closure of the set. Sufficient conditions for this extension will be given. Some properties of the homeomorphisms on the fundamental regions will also be given.

In order to show that an open connected subset B of the plane is homeomorphic to the plane, it is enough to show that every simple closed curve in B separates B and at least one of the components has compact closure in B [12]. Thus, Theorem 5 follows.

THEOREM 5. *If R_i and R_{i+1} are not separated for $i = 1, 2, \cdots, k-1$, then* $\operatorname{Int}(\cup_{j=1}^{k} R_j)$ *is homeomorphic to the plane.*

Proof. By the earlier remarks, we can assume that R_1 and R_k have nonempty interior, for otherwise R_1 or $R_k \not\subseteq \operatorname{Int}(\cup_{j=1}^{k} R_j)$. By (3), $\operatorname{Int} R_j$ is connected. Since R_i and R_{i+1} are not separated, suppose for definiteness that there exists $x \in (R_i - \operatorname{Int} R_i) \cap \bar{R}_{i+1}$. By Remarks 3 and 7, there is a neighborhood U of such that $U \subset R_{i-1} \cup R_i \cup R_{i+1}$ if $i \neq 1$ or $R_i \cup R_{i+1}$ if $i = 1$. Therefore, $x \in \operatorname{Int}(\cup_{j=1}^{k} R_j)$. Thus, $\operatorname{Int}(\cup_{j=1}^{k} R_j)$ is connected.

Let B be a simple closed curve such that $B \subset \operatorname{Int}(\cup_{j=1}^{k} R_j)$. Let $x \in \operatorname{Int} B$. Since $f(x) \to \infty$ as $n \to \infty$, it follows that there is an integer n such $f^n(x) \notin B \cup \operatorname{Int} B$. But, since $x \sim f^n(x)$, there is an arc A, where $A \subset R_i$ if $x \in R_i$ and where A joins x and $f^n(x)$. Therefore, $\{x\}$ and $A \cap B$ are in the same fundamental region. Thus, $B \cup \operatorname{Int} B \subset \operatorname{Int}(\cup_{j=1}^{k} R_j)$. It follows that $B \cup \operatorname{Int} B$ is compact in $\operatorname{Int}(\cup_{j=1}^{k} R_j)$. Therefore, $\operatorname{Int}(\cup_{j=1}^{k} R_j)$ is homeomorphic to the plane.

THEOREM 6. *Suppose R_i and R_{i+1} are not separated for $i = 1, \cdots, k-1$. Then, there is a homeomorphism g:* $\operatorname{Int}(\cup_{i=1}^{k} R_i) \to P_2$, *where P_2 is a plane, and P_2 under gfg^{-1} has at most k fundamental regions.*

Proof. The homeomorphism g exists by Theorem 5.

Let $x_1, x_2 \in R_j \cap \operatorname{Int}(\cup_{i=1}^{k} R_i)$. If $R_j \subset \operatorname{Int}(\cup_{i=1}^{k} R_i)$, then by definition of R_i, there is a curve segment $L \subset \operatorname{Int}(\cup_{i=1}^{k} R_i)$ joining x_1 and x_2 for which $f^n(L) \to \infty$ as $\pm n \to \infty$. Since $\operatorname{Int} R_j \subset \operatorname{Int}(\cup_{i=1}^{k} R_i)$, if $x_1, x_2 \in$

$\operatorname{Int} R_j$, then by (3), there is a curve segment $L \subset \operatorname{Int}(\cup_{i=1}^k R_i)$ for which $f^n(L) \to \infty$ as $n \to \pm\infty$. If $x_1 \in R_j - \operatorname{Int} R_j$, then by (2), $x_1 \in C_1 \subset R_j - \operatorname{Int} R_j$. By Remarks 3 and 7, $C_1 \subset \operatorname{Int}(\cup_{i=1}^k R_i)$. Let $y \in R_j$ such that $y \notin C_1$. Since $x_1 \sim y$, there is an arc L_1 joining x_1 and y such that $f^n(L_1) \to \infty$ as $n \to \pm\infty$. By Remark 3, there is a subarc L of L_1 such that L joins x_1 to an interior point of R_j and such that $L \subset \operatorname{Int}(\cup_{i=1}^k R_i)$. Therefore, if $x_1, x_2 \in R_j \cap \operatorname{Int}(\cup_{i=1}^k R_i)$, then there is a curve $L \subset \operatorname{Int}(\cup_{i=1}^k R_i)$ which $f^n(L) \to \infty$ as $n \to \pm\infty$. Let C be any compact set in P_2. Then, $g^{-1}(C)$ is a compact set in P_1. Thus, there is a natural number N such that if $|n| > N$, then $f^n(L) \cap g^{-1}(C) = \phi$. Now, $g(L)$ is a curve segment joining $g(x_1)$ and $g(x_2)$ in plane P_2. Since $f^n(L) \cap g^{-1}(C) = \phi$, $gf^ng^{-1}g(L) \cap C = \phi$. Therefore, $g(x_1)$ and $g(x_2)$ are in the same fundamental region.

COROLLARY. *If $k = 2$, then P_2 has exactly one fundamental region under gfg^{-1}.*

The following two embedding theorems can be considered as a continuation of the work started by Andrea [1].

THEOREM 7. *If the plane under f has exactly one fundamental region and if C is a proper flowline, then f can be embedded in a continuous flow Π, such that $\Pi(C, R) = C$.*

Proof. The homeomorphism f is topologically equivalent to a translation. Hence, there is a curve K such that K separates the plane, $f^n(K) \cap f^m(K) = \phi$ for $n \neq m$, if z is a point of the plane $f^n(z)$ is in the strip bounded by K and $f(K)$ for some unique integer n, and $K \cap C = \{x_0\}$. The plane, P_1, is homeomorphic to another plane P_2 under a homeomorphism h such that $h(K)$ and $(hf)K$ are parallel straight lines and $h(C)$ from $h(x_0)$ to $(hf)x_0$ is a straight line. Now if $x \in K$ and if $0 \leqq t < 1$ define

$$\Pi_1(x, t) = y$$

where $h(y) = (1 - t)\,h(x) + t(hf)\,x$. If $x \in K$ and t is any real number, define

$$\Pi_2(x, t) = f^n(\Pi_1(x, s)) ,$$

where $n + s = t$, n is an integer, and $0 \leqq s < 1$. If x is in the interior of the strip bounded by K and $f(K)$ or if $x \in K$, and if t_1 is any real number, define

$$\Pi_3(x, t_1) = \Pi_2(y, t_2 + t_1)$$

where $h(x) = (1 - t_2)(hf)y + t_2 h(y)$ and $y \in K$. If $x \in P_1$ and t is any real number, define

$$\Pi(x, t) = \Pi_3(f^{-m}(x), t + m) ,$$

where $f^{-m}(x)$ is in the strip bounded by K and $f(K)$. It follows that Π is a continuous flow and that $\Pi(x, n) = f^n(x)$ for every $x \in P_1$ and every integer n.

THEOREM 8. *Suppose R_1 and R_i are not separated for $i = 2, \cdots, n$. Then*

(a) *R_1 is open*

(b) *$f|_{R_1}$ can be embedded in a continuous flow, and*

(c) *if $f|_{\bar{R}_1}$ can be embedded in a continuous flow Π_1, then f can be embedded in a continuous flow Π, where $\Pi(x, r) = \Pi_1(x, r)$ if $(x, r) \in \bar{R}_1$ X Reals.*

(d) *If $f|_{R_1}$ can be embedded in a continuous flow Π_1, such that Π_1 restricted to $R_1 \times [0, 1]$ is uniformly continuous, then f can be embedded in a continuous flow Π.*

Proof. (a) By Remark 12, R_j and R_k are separated for $j \neq k$ and $j, k \neq 1$. Thus by Remark 9, R_j is closed for $j \neq 1$. Thus, $R_1 =$ Plane $- (\cup_{j=2}^{n} R_j)$ is open.

(b) By Theorem 5, there is a homeomorphism $g: R_1 \to P_2$ where P_2 is a plane. By Theorem 6, P_2 has exactly one fundamental region under gfg^{-1}. Thus, gfg^{-1} is topologically a translation and thus can be embedded in a continuous flow. Therefore $f|_{R_1}$ can be embedded in a continuous flow.

(c) If $x \in (R_2 - \operatorname{Int} R_2) \cap \bar{R}_1$ then $x \in C_2 \subset R_2 - \operatorname{Int} R_2$, where C_2 is a proper flowline. By Remarks 3 and 7, $C_2 \subset \operatorname{Int}(R_1 \cup R_2)$. It follows that $\operatorname{Int}(R_1 \cup R_2) = R_1 \cup R_2$, and from the proof of Remark 9 that $R_2 - \operatorname{Int} R_2 = C_2$. From Theorem 5, $R_1 \cup R_2$ and plane P_2 are homeomorphic, under a homeomorphism g. By the corollary to Theorem 6, P_2 has exactly one fundamental region under gfg^{-1}. Therefore, gfg^{-1} can be embedded in a continuous flow σ such that $\sigma(g(C), R) = g(C)$, by Theorem 7. It also can be shown that we can choose σ such that $\Pi_1(x, r) = g^{-1}(\sigma(g(z), r)$ for $g(z) = x \in C$. Now define $\Pi'_2: (R_1 \cup R_2) \times R \to (R_1 \cup R_2)$ by

$$\Pi'_2(z, r) = g^{-1}(\sigma(g(z), r)) .$$

Then Π'_2 is a continuous flow and $\Pi'_2(z, n) = f^n(z)$, where $z \in R_1 \cup R_2$. Define $\Pi_2: (R_1 \cup R_2) \times R \to (R_1 \cup R_2)$ by

$$\Pi_2(z, r) = \begin{cases} \Pi'_2(z, r) & \text{if } z \in R_2 \\ \Pi_1(z, r) & \text{if } z \in \bar{R}_1 . \end{cases}$$

Then Π_2 is a continuous flow on $R_1 \cup R_2$ since Π'_2 and Π_1 agree on C_2. In the same way, define Π_i: $(R_1 \cup R_i) \times R \to (R_1 \cup R_i)$ for $i = 2, 3, \cdots, n$. Define Π: $P_1 \times R \to P_1$ by

$$\Pi(z, r) = \Pi_i(z, r) \text{ if } z \in R_i \text{ for } i = 1, 2, \cdots, n .$$

The fact that Π is a continuous flow follows from the fact that R_i and R_j are closed and separated for $i \neq j$; $i, j \neq 1$.

(d) It is enough to show that $f|_{\bar{R}_1}$ can be embedded in a continuous flow. Define Π: $\bar{R}_1 \times \text{Reals} \to \bar{R}_1$ by

$$\Pi(x, r) = \lim_{n\to\infty} f^k(\Pi_1(x_n, s)) ,$$

where $x_n \to x$ as $n \to \infty$, $x_n \in R_1$, and $r = k + s$, where k is an integer and $0 \leqq s < 1$.

Let (x, r) and $\varepsilon > 0$ be given. Then $r = k + s$, where k is an integer and $0 \leqq s < 1$. Let $x_n \to x$ as $n \to \infty$, where $x_n \in R_1$. By the uniform continuity of Π, on $R_1 \times [0, 1]$, there is a $\delta > 0$ such that if $d(x_n, x_m) < \delta$, then $d(\Pi_1(x_n, s), \Pi_1(x_m, s)) < \varepsilon$. Since $x_n \to x$, there is a natural number N such that if $n, m > N$, then $d(x_n, x_m) < \delta$. Therefore, $\lim_{n\to\infty} f^k\Pi_1(x_n, s)) = f^k(\lim_{n\to\infty} \Pi_1(x_n, s))$ exists.

Suppose $x_n \to x$ and $\bar{x}_m \to x$, where $x_n, \bar{x}_m \in R_1$ for every n, m. Let $\lim_{n\to\infty} \Pi_1(x_n, s) = A$, and $\lim_{n\to\infty} \Pi_1(\bar{x}_m, s) = B$. By the uniform continuity of Π_1 on $R_1 \times [0, 1]$, there is a $\delta > 0$ such that if $d(y, \bar{y}) < \delta$ then $d(\Pi_1(y, s), \Pi_1(\bar{y}, s)) < \varepsilon/3$. Since $x_n \to x$ and $\bar{x}_m \to x$ as $n, m \to \infty$, there is a natural number N such that if $n, m > N$, then $d(x_n, x) < \delta/2$ and $d(\bar{x}_m, x) < \delta/2$. Therefore $d(x_n, \bar{x}_m) \leqq d(x_n, x) + d(\bar{x}_m, x) < \delta$ if $n, m > N$. Thus, $d(A, B) \leqq d(A, \Pi_1(x_n, s)) + d(\Pi_1(x_n, s), \Pi_1(\bar{x}_m, s) + d(\Pi(\bar{x}_m, s), B) < \varepsilon$ for large m and n. Since $r = k + s$, where k is an integer and $0 \leqq s < 1$, is a unique representation, Π is well defined.

Since $\Pi_1(x_n, 0) = x_n$ if $x_n \in R_1$, $\Pi(x, 0) = x$ for all $x \in \bar{R}_1$.

Let $x_n \to x$ as $n \to \infty$, $r_1 = k_1 + s_1$, $r_2 = k_2 + s_2$, and $r_1 + r_2 = k_3 + s_3$, where k_i are integers and $0 \leqq s_i < 1$ for $i = 1, 2, 3$, and where $x_n \in R_1$. Then, $\Pi(x, r_1 + r_2) = \lim_{n\to\infty} f^{k_3}(\Pi_1(x_n, s_3) = \lim_{n\to\infty} f^{k_2}(\Pi_1(x_n, s_3 + r_1 - s_3 + s_2) = \lim_{n\to\infty} f^{k_2}(\Pi_1(x_n, r_1 + r_2)) = \lim_{n\to\infty} f^{k_2}(\Pi_1(x_n, k_1 + s_1 + s_2)) = \lim_{n\to\infty} f^{k_2}(\Pi_1(\Pi_1(x_n, k_1 + s_1), s_2) = \lim_{n\to\infty} f^{k_2}(\Pi_1(f^{k_1}(\Pi_1(x_n, s_1)), s_2) = \Pi(\Pi(x, r_1), r_2)$.

Let $(x, g) \in \bar{R}_1 \times [0, 1]$. Let $\varepsilon > 0$ be given. Choose a neighborhood $N \times U$ of (x, g) in $\bar{R}_1 \times [0, 1]$ so that if $(x_1, g_1), (x_2, g_2) \in (N \times U) \cap (R_1 \times [0, 1])$ then $d(\Pi_1(x_1, g_1), \Pi_1(x_2, g_2)) < \varepsilon$. By the definition of Π, there exists $x_1 \in R_1 \cap N$ such that $d(\Pi_1(x_1, g), \Pi(x, g)) < \varepsilon$. Suppose $(\bar{x}, \bar{g}) \in N \times U$. Then there exists $x_2 \in R_1 \cap N$ so that $d(\Pi_1(x_2, \bar{g}), \Pi(\bar{x}, \bar{g})) < \varepsilon$. Thus, $d(\Pi(x, g), \Pi(\bar{x}, \bar{g})) \leqq d(\Pi(x, g), \Pi_1(x_1, g)) + d(\Pi_1(x_1, g), \Pi_1(x_1, \bar{g})) + d(\Pi_1(x_1, \bar{g}), \Pi_1(x_2, \bar{g}) + d(\Pi_1(x_2, \bar{g}), \Pi(\bar{x}, \bar{g})) < \varepsilon + \varepsilon + \varepsilon = 3\varepsilon$. Thus Π is continuous on $\bar{R}_1 \times [0, 1]$. Therefore, by continuity of f^n for every integer n, Π is continuous on $\bar{R}_1 \times$ Reals.

It follows from Theorem 8 and the remarks §2, that if the plane has either three or four fundamental regions under f, then the embedding problem is reduced to the problem of embedding $f|_{R_1}$ in a flow that can be extended to $\bar{R}_1$.

REMARK 13. If $B \subset R_k$ is a proper flowline and B separates R_j and R_i, $i \neq k \neq j$, then each closed half plane defined by B contains at least two fundamental regions other than R_k.

Proof. Use Remark 5 to define a homeomorphism g such that $g(x) = f(x)$ if $x \in B$ and such that g is topologically equivalent to a translation. Let H^1 be the half plane defined by B that contains R_j, and let H^2 be the half plane defined by B that contains R_i. Define $F: P_1 \to P_1$, where P_1 is the plane, by

$$F(x) = \begin{cases} g(x) & \text{if } x \in H^1 \\ f(x) & \text{if } x \in H^2 . \end{cases}$$

Suppose that R_i is the only fundamental region other than R_k that has a point in H^2. Let $z \in H^1 - B$, and let $x \in B$. Join z and x by a curve $J \subset H^1$. Then $F^n(J) = g^n(J) \to \infty$ as $n \to \pm\infty$. Therefore, the plane has at most two fundamental regions under F, and thus it has exactly one. Now join $y \in R_i$ to $x \in B$ by a curve $K \subset H^2$. Then $f^n(K) = F^n(K) \to \infty$ as $n \to \pm\infty$. Thus, x and y are in the same fundamental region under f.

COROLLARY. *If $n = 3$ or 4, then B does not separate any two fundamental regions if neither contains B.*

REMARK 14. Suppose there are exactly three fundamental regions under f, and R_i and R_{i+1} are not separated for $n = 1, 2$. If $y \in R_2$, $z \in R_1$, and U is any open set such that $R_3 \subset U$, then for every arc C connecting y and z, there is an integer N such that $f^N(C) \cap U \neq \phi$.

Proof. By Theorem 4, R_2 separates R_1 and R_3, and R_2 is open. Thus, Int $(R_1 \cup R_2) = R_1 \cup R_2$. By Theorem 5, $R_1 \cup R_2$ is homeomorphic to the plane. Suppose $g: R_1 \cup R_2 \to P_2$, where P_2 is the plane and g is a homeomorphism. By Theorem 6, P_2 has exactly one fundamental region under gfg^{-1}. Suppose that for some $y \in R_2$, $z \in R_1$, and for some open set $U \supset R_3$, there is an arc C connecting y and z such that $f^n(C) \cap U = \phi$ for every integer n. Let K be any compact set in P_1. Then $(P_1 - U) \cap K$ is compact. Since $f^n(C) \cap U = \phi$ for every integer n, it follows that $f^n(C) \cap (P_1 - U) \cap K = \phi$ implies $f^n(C) \cap K = \phi$. Now, $gf^n g^{-1}(g(C)) \to \infty$ as $n \to \pm\infty$. Since $(P_1 - U) \cap K = K_1 \subset R_1 \cup R_2$,

$g(K_1)$ is compact in P_2. Thus, there is a natural number N such that if $|n| > N$, then $gf^n(C) \cap g(K_1) = \phi$. But that implies $f^n(C) \cap K_1 = \phi$. Therefore, y and z are in the same fundamental region, which is a contradiction.

REMARK 15. Suppose $C_j \subset R_1$, where C_j is a proper flowline for $j = 1, 2$; H_j^k, $k = 1, 2$, are the two closed half planes defined by C_j; and $C_1 \subset \operatorname{Int} H_2^1$ and $C_2 \subset \operatorname{Int} H_1^2$. Then $H_2^1 \cap H_1^2 \subset R_1$.

Proof. The proof is similar to the proof of Remark 6.

REMARK 16. Suppose $C_i \subset R_2$, where C_i is a proper flowline for $i = 1, 2$. If C_1 separates R_1 and R_3, then C_2 separates R_1 and R_3.

Proof. Suppose C_1 separates R_1 and R_3, but C_2 does not. Let H_i^k, $k = 1, 2$, be the closed half plane defined by C_i. Suppose $R_1 \subset H_1^1$ and $R_3 \subset H_1^2$. Since C_2 does not separate R_1 and R_3, and since each R_i is arcwise connected, suppose R_1 and R_3 are contained in H_2^1. If $C_1 \cap C_2 = \phi$, suppose $C_2 \subset H_1^2$. Then $R_3 \subset H_2^1 \cap H_1^2$. By Remark 15, $R_3 \subset H_2^1 \cap H_1^2 \subset R_2$. If $C_2 \subset H_1^1$, then $R_1 \subset H_1^1 \cap H_2^1$. By Remark 15, $R_1 \subset H_1^1 \cap H_2^1 \subset R_2$. If $C_1 \cap C_2 \neq \phi$, let $y \in C_1 \cap C_2$. Then $f^n(y) \in C_1 \cap C_2$ for every integer n. In this case there is a point $x \in R_1 \cup R_3$, where x is in the region bounded by C_1 and C_2. Since that is not possible, C_2 separates R_1 and R_3.

The following Remark easily follows from the above results.

REMARK 17. Suppose R_i and R_{i+1} are not separated for $i = 1, 2$. Then, there is a homeomorphism g: $\operatorname{Int}(\bigcup_{i=1}^3 R_i) \rightarrow P_2$, where P_2 is a plane. If P_2 has exactly three fundamental regions under gfg^{-1} if $y \in R_2 \cap \operatorname{Int}(\bigcup_{i=1}^3 R_i)$, if $z \in R_1 \cap \operatorname{Int}(\bigcup_{i=1}^3 R_i)$, and if U is any open set such that $R_3 \subset U$, then for every arc $B \subset \operatorname{Int}(\bigcup_{i=1}^3 R_i)$ connection y and z there is an integer N such that $f^N(B) \cap U \neq \phi$.

The above results gives us some information about the possible behavior of a self-homeomorphism of the plane with n fundamental regions. Consider, for example, the case where the plane $P_1 = \bigcup_{i=1}^4 R_i$, where R_1 and R_i are not separated for $i = 2, 3, 4$. By Corollary to Remark 13, R_1 does not contain a proper flowline B such that B separates R_j and R_k for $j \neq k$, $j, k \neq 1$. By Theorem 5, $\operatorname{Int}(R_1 \cup R_2 \cup R_i) = R_1 \cup R_2 \cup R_i$ is homeomorphic to the plane for $i = 3, 4$. By Remark 17, if $y \in R_2 - \operatorname{Int} R_2$, if U is any neighborhood of y, and if V_i is an open set such that $V_i \supset R_i$, then there is an integer N such that $f^N(U) \cap V_i \neq \phi$.

4. Self-homeomorphisms of the plane and the closed disc with one fixed point. In this section sufficient conditions to embed self-homeomorphisms of the plane and the closed disc will be given.

THEOREM 9. *Let f be an orientation preserving self-homeomorphism of the plane with one fixed point, x_0. Suppose $f^n(x) \to \infty$ as $n \to -\infty$ and $f^n(x) \to x_0$ as $n \to \infty$ for $x \neq x_0$. Then f can be embedded in a continuous flow.*

Proof. The proof is an immediate consequence of the theorem by Homma and Kinoshita [10] that says that such a homeomorphism is topologically equivalent to $x' = (1/2)x$, $y' = (1/2)y$.

We conclude this paper with the following embedding theorem.

THEOREM 10. *Let f be an orientation preserving self-homeomorphism of the closed unit disc D. Suppose*

(1) *if $x \in D$, then $f^{\pm n}(x) \to x_0$ as $n \to \infty$, where $x_0 \in D - \operatorname{Int} D$* and *$f(x_0) = x_0$, and*

(2) *if $x_1, x_2 \in D - \{x_0\}$, there is an arc $A \subset D$ such that A joins x_1 and x_2 and $f^{\pm n}(A) \to x_0$ as $n \to \infty$.*

Then f can be embedded in a continuous flow.

Proof. Let L be a straight line tangent to the circle $C = D - \operatorname{Int} D$ at x_0. Let L_1 be the ray along L from x_0. The subspace $P_1 - L_1$ is homeomorphic to a plane P_2. Suppose g is a homeomorphism of $P_1 - L_1$ onto P_2 sending $C - \{x_0\}$ onto the y-axis of plane P_2. If $x \in \operatorname{Int} D$, let H_1 be the closed half plane defined by the y-axis of plane P_2 containing $g(x)$. It follows that $g^{-1}(H_1) = D - \{x_0\}$. Now define F: $P_2 \to P_2$ by

$$\begin{aligned} F(x, y) &= (x, (gfg^{-1}(0, y)_y) && \text{if } (x, y) \in P_2 - H_1 \\ &= gfg^{-1}(x, y) && \text{if } (x, y) \in H_1\,. \end{aligned}$$

F is orientation preserving self-homeomorphism of plane P_2 with no fixed points. If $z \in P_2 - H_1$, then clearly z is in the same fundamental region as the y-axis of P_2. Let $x_1, x_2 \in H_1$ and let K be any compact set in P_2. Since $g^{-1}(K)$ is a compact set in P_1 and $x_0 \notin g^{-1}(K)$, it follows that there is an arc A joining $g^{-1}(x_1)$ and $g^{-1}(x_2)$ such that $f^{\pm n}(A) \cap g^{-1}(K) = \phi$ if $n > N$ for some N. But that implies $F^{\pm n}(g(A)) \cap K = \phi$ if $n > N$. Thus P_2 has exactly one fundamental region under F, and consequently can be embedded in a flow Π which leaves the y-axis of P_2 invariant by Theorem 7. It follows that $\Pi(H_1, r) = H$ for every real number r. Thus, f is restricted to $D - \{x_0\}$ can be embedded in a flow σ_1. Now define σ: $D \times R \to D$ by

$$\sigma(d, r) = \sigma_1(d, r) \qquad \text{if } d \in D - \{x_0\},\ r \in R\ ,$$
$$\sigma(x_0, r) = x_0 \qquad \text{if } r \in R\ .$$

σ is a continuous flow if it is continuous at x_0. Let U be an open neighborhood of x_0 in D. Then $D - U$ is compact in P_1. Thus $g(D - U)$ is compact in P_2. There exists a compact set $K \subset P_2$ such that if $x \in P_2 - K$ and if $|t - t_0| < 1$ then $\Pi(x, t) \in P_2 - g(D - U)$. From that it follows that if $x \in [g^{-1}(P_2) - g^{-1}(K)] \cap D$ and if $|r - r_0| < 1$, then $\sigma(x, r) \in U$. Thus, σ is continuous at (x, r_0).

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