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CONTINUOUS DEPENDENCE ON PARAMETERS AND BOUNDARY DATA FOR NONLINEAR TWO-POINT BOUNDARY VALUE PROBLEMS

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Sufficient conditions are given for the continuous dependence of solutions to the two-point boundary value problem

(1) $$x'' = f(t, x, x'; \mu)$$

(2) $$x(a) = \alpha \qquad x(b) = \beta$$

on the boundary data and the parameter μ.

Previous results given by Gaines and Klaasen for continuous dependence on the boundary data have assumed continuity on f and uniqueness to two-point BVP'S. Klaasen has also shown assuming uniqueness to two-point BVP'S and the existence of a C^2-solution to (1)$-$(2) that there exist solutions $x(t; \alpha', \beta')$ to (1) with the boundary conditions

$$x(a) = \alpha' \qquad x(b) = \beta'$$

for all (α', β') sufficiently close to (α, β). Furthermore, $x(t; \alpha', \beta')$ is a uniformly continuous function of (α', β') at (α, β) on $[a, b]$. This same result is shown to be valid under weaker uniqueness conditions. Sufficient conditions are also given for existence and continuous dependence on the parameter, μ, of solutions to (1)$-$(2).

We consider the BVP

(1) $$x'' = f(t, x, x'; \mu)$$

(2) $$x(a) = \alpha \qquad x(b) = \beta$$

and assume

I. $f(t, x_1, x_2; \mu)$ is continuous on $[a, b] \times R^3$

II. There exists a solution $x_0(t)$ to the BVP

(3) $$x'' = f(t, x, x'; \mu_0)$$

(4) $$x(a) = \alpha_0 \qquad x(b) = \beta_0$$

such that if $x(t)$ is any other solution to (3) and $x(t_i) = x_0(t_i)$, $i = 1, 2$, for $a \leqq t_1 < t_2 \leqq b$, then $x(t) \equiv x_0(t)$ on $[t_1, t_2]$.

Following Jackson, [3], we make the following definition.

DEFINITION 2.1. For any constant $C > 0$, let

$$F^*(t, x, x'; \mu) = \begin{cases} f(t, x, C; \mu) & \text{if} \quad x' \geqq C \\ f(t, x, x'; \mu) & \text{if} \ |x'| \leqq C \\ f(t, x, -C; \mu) & \text{if} \quad x' \leqq -C \end{cases}$$

and define for $u(t) \leqq v(t)$ on $[a, b]$

$$(5) \qquad F(t, x, x'; \mu) = \begin{cases} F^*(t, v(t), x'; \mu) + [x - v(t)]^{1/2} & \text{if } x \geqq v(t) \\ F^*(t, x, x'; \mu) \quad \text{if } u(t) \leqq x \leqq v(t) \\ F^*(t, u(t), x'; \mu) - [u(t) - x]^{1/2} & \text{if } x \leqq u(t) \,. \end{cases}$$

Then $F(t, x, x'; \mu)$ is called the modification of $f(t, x, x'; \mu)$ with respect to $u(t)$, $v(t)$ and C.

LEMMA 2.2. *Under Conditions I and II, given $\varepsilon > 0$ there exists constants $\sigma > 0$, $C_0 > 0$, $C_0' > 0$ such that for any $t_0 \in [a, b]$ and any α, β, μ with $|x_0(t_0) - \alpha| \leqq \varepsilon$, $|x_0'(t_0) - \beta| \leqq \varepsilon$ and $|\mu - \mu_0| < \varepsilon$ every solution $x(t; \mu)$ to IVP*

$$(1) \qquad x'' = f(t, x, x'; \mu)$$

$$(6) \qquad x(t_0) = \alpha \qquad x'(t_0) = \beta$$

or IVP

$$(7) \qquad x'' = F^*(t, x, x'; \mu)$$

$$(6) \qquad x(t_0) = \alpha \qquad x'(t_0) = \beta$$

exists on $[t_0 - \delta, t_0 + \delta]$ and satisfies $|x(t; \mu)| \leqq C_0'$, $|x'(t; \mu)| \leqq C_0$ on $[t_0 - \delta, t_0 + \delta]$.

Proof. The proof is an easy application of the Peano existence theorem ([2], Theorem 2.1,) where

$$C_0 = \max_{[a,b]} |x_0'(t)| + \varepsilon + 1, \ C_0' = \max_{[a,b]} |x_0(t)| + \varepsilon + 1 \,.$$

THEOREM 2.3. *Assume Conditions I and II. Then there exists a $\delta > 0$ such that for all μ with $|\mu - \mu_0| < \delta$, a solution $x(t; \mu)$ to BVP (1) – (4) exists. Furthermore, $x(t; \mu) \to x_0(t)$ in C^1-norm on $[a, b]$ as $\mu \to \mu_0$.*

Proof. Let $\{\mu_n\}$ be any sequence converging to μ. It suffices to show that there is a subsequence $\{\mu_n^*\}$ such that for all μ_n^*, a solution $x(t; \mu_n^*)$ to *BVP* (1) – (4) exists.

Denote by $\bar{x}_m(t; \mu_n)$ a solution to (1) satisfying

$$(8) \qquad x(a) = \alpha_0 \qquad x'(a) = x_0'(a) + 1/m \,.$$

Pick $\varepsilon = 1$ and let δ, C_0, C_0' be the constants assured by Lemma 2.2. For n sufficiently large, say $n \geqq N$, $|\mu_n - \mu_0| \leqq 1$, and hence the sequences $\{\bar{x}_m(t; \mu_n)\}_{n=N}^{\infty}$ and $\{\bar{x}_m'(t; \mu_n)\}_{n=N}^{\infty}$ are uniformly bounded and equicontinuous on $[a, a+\delta]$. By Ascoli's theorem there exists a subsequence $\{\bar{x}_m^*(t; \mu_n)\}$, which converges in C^1-norm on $[a, a+\delta]$ to a solution $\bar{x}_m(t; \mu_0)$ of IVP (3) − (8). Letting $m \to \infty$, there exists a subsequence, $\{\bar{x}_m^*(t; \mu_0)\}$, which converges in C^1-norm to a solution $z_0(t)$ of (3) satisfying the initial conditions

$$x(a) = \alpha_0 \qquad x'(a) = x_0'(a)\ . \tag{9}$$

Case 1. $z_0(t) \equiv x_0(t)$ on $[a, a+\delta]$. Given any $\varepsilon > 0$, $\varepsilon < 1$, there exists an M and N such that for all $m \geqq M$ and all $n \geqq N$,

$$\|\bar{x}_m(t; \mu_n) - x_0(t)\|_{C^1} < \varepsilon \text{ on } [a, a+\delta]$$

and

$$\bar{x}_m(a+\delta; \mu_n) > x_0(a+\delta)\ .$$

By a similar procedure there exists a sequence $\{\underline{x}_m(t; \mu_n)\}$ of solutions to equation (1) satisfying

$$x(a) = \alpha_0 \qquad x'(a) = x_0'(a) - 1/m\ , \tag{10}$$

such that for all m and n sufficiently large, (say $m \geqq M$, $n \geqq N$ without loss of generality),

$$\|\underline{x}_m(t; \mu_n) - x_0(t)\|_{C^1} < \varepsilon \quad \text{on } [a, a+\delta]$$

and

$$\underline{x}_m(a+\delta; \mu_n) < x_0(a+\delta)\ .$$

By our uniqueness assumption in II, there exists an $N_m' \geqq N$ such that for all $n \geqq N_m'$, $\bar{x}_m(t; \mu_n) \geqq \underline{x}_m(t; \mu_n)$.

Now let $F_m(t, x, x'; \mu_n)$ be the modification of $f(t, x, x'; \mu_n)$ with respect to $\bar{x}_m(t; \mu_n)$, $\underline{x}_m(t; \mu_m)$ and C_0 for any $n \geqq N_m'$. By Theorem 2.5, [3], there exists solutions $\bar{y}_1(t; \mu_n; j)$ and $\underline{y}_1(t; \mu_n; j)$ to BVP

$$x'' = F_m(t, x, x'; \mu_n)$$

$$x(a) = \alpha_0 \qquad x(a+\delta) = x_0(a+\delta) \pm 1/j\ ,$$

respectively, for all $j \geqq J$ where

$$\underline{x}_m(a+\delta; \mu_n) \leqq x_0(a+\delta) - 1/j < x_0(a+\delta) + 1/j \leqq \bar{x}_m(a+\delta; \mu_n)\ .$$

Furthermore,

$$\underline{x}_m(t; \mu_n) \leqq \bar{y}_1(t; \mu_n; j) \leqq \bar{x}_m(t; \mu_n) \quad \text{on } [a, a+\delta]$$

and hence

$$|\bar{y}_1'(a; \mu_n, j) - x_0'(a)| \leqq 1/m ,$$

implying by Lemma 2.2

$$|\bar{y}_1'(t; \mu_n; j)| \leqq C_0 \quad \text{on } [a, a+\delta] .$$

Thus $\bar{y}_1(t; \mu_n; j)$ is in fact a solution to

$$x'' = f(t, x, x'; \mu_n) . \tag{11}$$

Similarly $\underline{y}_1(t; \mu_n; j)$ is a solution to (19).

Now fix j. The sequences $\{\bar{y}_1(t; \mu_n; j)\}_{n=N_m}^{\infty}$, and $\{\bar{y}_1'(t; \mu_n; j)\}_{n=N_m}^{\infty}$, are uniformly bounded and equicontinuous, and hence a subsequence converges to a solution $\bar{y}_1(t; \mu_0; j)$ of (3) satisfying the boundary conditions

$$x(a) = \alpha_0 \qquad x(a+\delta) = x_0(a+\delta) + 1/j . \tag{12}$$

Similarly a subsequence of $\{\bar{y}_1(t; \mu_0; j)\}_{j=J}^{\infty}\}$ converges in C^1-norm on $[a, a+\delta]$ to $x_0(t)$ by II. By Lemma 2.2, then, there exists a $J' \geqq J$ such that $\bar{y}_1(t; \mu_0; j)$ may be extended to $[a, a+2\delta]$ for all $j \geqq J'$. Then also for n sufficiently large the solutions $\bar{y}_1(t; \mu_n; j)$ may be extended to $[a, a+2\delta]$.

Similarly for n and j sufficiently large the solutions $\underline{y}_1(t; \mu_n; j)$ may be extended to $[a, a+2\delta]$. We now use the solutions $\bar{y}_1(t; \mu_n; j)$ and $\underline{y}_1(t; \mu_n; j)$ in place of $\bar{x}_m(t; \mu_n)$ and $\underline{x}_m(t; \mu_n)$ and argue as above on the interval $[a, a+2\delta]$.

We continue in this way until we may assume that there are sequences of solutions $\bar{y}(t; \mu_n; j)$ and $\underline{y}(t; \mu_n; j)$ to equation (11) satisfying

$$x(a) = \alpha_0 \qquad x(b) = \beta_0 \pm 1/j , \qquad \text{respectively} , \tag{13}$$

with $\bar{y}(t; \mu_n; j) \geqq \underline{y}(t; \mu_n; j)$ on $[a, b]$. Using the modification of $f(t, x, x'; \mu_n)$ with respect to $\bar{y}(t; \mu_n; j)$, $\underline{y}(t; \mu_n; j)$, and C_0, and arguing as above we may assume that for some subsequence $\{\mu_n^*\}$ of $\{\mu_n\}$ there exists solutions $x(t; \mu_n^*)$ to BVP (11) − (4).

Case 2. $z_0(t) \not\equiv x_0(t)$ on $[a, a+\delta]$. Then by II,

$$z_0(a+\delta) > x_0(a+\delta) .$$

Let

$$0 < \varepsilon < \min(1, z_0(a+\delta) - x_0(a+\delta)) .$$

Similarly, if $\underline{x}_m(t; \mu_0)$ does not converge to $x_0(t)$ on $[a, a+\delta]$, then for n and m sufficiently large

$$\underline{x}_m(a + \delta; \mu_n) < x_0(a + \delta) - \varepsilon/3 .$$

Now we may obtain the solutions $\bar{y}_1(t; \mu_n; j)$ and $\underline{y}_1(t; \mu_n; j)$ and proceed as in Case 1.

Case 3. In Case 1 we considered the possibility that the sequences $\{\bar{x}_m(t; \mu_0)\}$ and $\{\underline{x}_m(t; \mu_0)\}$ had subsequences which converged to $x_0(t)$ on $[a, a + \delta]$. In Case 2, these sequences converged to functions not identically equal to $x_0(t)$ on $[a, a + \delta]$. In Case 3, then, we must consider the possibility that one of these sequences converges to $x_0(t)$ and the other converges to some function not identically equal to $x_0(t)$ on $[a, a + \delta]$. The proof for Case 3 is thus just a combination of the proofs of Case 1 and Case 2.

To complete the proof of the theorem we must show that $x(t; \mu_n^*) \rightarrow x_0(t)$ in C^1-norm on $[a, b]$ as $n \rightarrow \infty$. By construction we have

$$| x(t; \mu_n^*) | \leqq \max_{[a, b]} | x_0(t) | + 1$$

and

$$| x'(t; \mu_n^*) | \leqq C_0 \quad \text{on } [a, b] .$$

Thus the sequences $\{x(t; \mu_n^*)\}$ and $\{x'(t; \mu_n^*)\}$ are uniformly bounded and equicontinuous on $[a, b]$ and hence by Ascoli's Theorem there exists a further subsequence which converges in C^1-norm on $[a, b]$ to a solution of BVP (3) – (4) which by II must be $x_0(t)$. Similarly any subsequence of $\{x(t; \mu_n^*)\}$ has a further subsequence which converges to $x_0(t)$ implying that the original sequence itself must converge to $x_0(t)$.

Note. We have proven only a weak form of continuous dependence on the parameter; i.e., we have shown only that the solutions $x(t; \mu_n)$ must converge to $x_0(t)$. It is still unknown whether all solutions to BVP (11) – (4) must converge to $x_0(t)$ as $\mu \rightarrow \mu_0$.

Theorem 2.3 is of interest when considering nonlinear eigenvalue problems. If an eigenvalue problem satisfies Conditions I and II, then the set of eigenvalues is dense in itself.

We now seek sufficient conditions for existence and continuous dependence of solutions to BVP's in which we vary not only the parameter but also the boundary data.

LEMMA 3.4. *Assume I and II. Then there exists sequences of solutions* $\{u_n(t)\}$, $\{v_n(t)\}$, $\{w_n(t)\}$ *and* $\{z_n(t)\}$ *that converge to* $x_0(t)$ *in the* C^1*-norm on* $[a, b]$ *and such that*

$$u_n(a) = x_0(a) \qquad u_n(b) > u_{n+1}(b) > x_0(b) \qquad \textit{for all } n ,$$

$$v_n(b) = x_0(b) \qquad v_n(a) > v_{n+1}(a) > x_0(a) \qquad \textit{for all } n ,$$
$$w_n(a) = x_0(a) \qquad w_n(b) < w_{n+1}(b) < x_0(b) \qquad \textit{for all } n ,$$

and

$$z_n(b) = x_0(b) \qquad z_n(a) < z_{n+1}(a) < x_0(a) \qquad \textit{for all } n ,$$

Proof. The proof is contained in the proof of Theorem 3.7, [5].

THEOREM 2.5. *Under Conditions I and II, there exist sequences* $\{\bar{x}_n(t)\}$ *and* $\{\underline{x}_n(t)\}$ *of solutions to* (3) *which converge to* $x_0(t)$ *in* C^1*-norm on* $[a, b]$ *from above and below, respectively, with*

$$\bar{x}_n(a) > \bar{x}_{n+1}(a) > x_0(a) > \underline{x}_{n+1}(a) > \underline{x}_n(a)$$

and

$$\bar{x}_n(b) > \bar{x}_{n+1}(b) > x_0(b) > \underline{x}_{n+1}(b) > \underline{x}_n(b)$$

for all n.

Proof. We will show the existence of $\{\bar{x}_n(t)\}$. Let S_1 be the set of all $t_0 \in [a, b]$ such that there exists a second solution $x_1(t)$ to IVP

$$x'' = f(t, x, x'; \mu_0) \tag{3}$$

$$x(t_0) = x_0(t_0) \qquad x'(t_0) = x_0'(t_0) \tag{14}$$

with $x_1(t) > x_0(t)$ on $(\lambda_1^-, t_0) \cap [a, b]$ for some $\lambda_1^- < t_0$. Let $t_1 = \inf S_1$. If $S_1 = \phi$, let $t_1 = b$. Similarly, let S_2 be the set of all $t_0 \in [a, b]$ such that there exists a second solution $x_2(t)$ to IVP (3) – (14) with $x_2(t) > x_0(t)$ on $(t_0, \lambda_2^+) \cap [a, b]$ for some $\lambda_2^+ > t_0$, and let $t_2 = \sup S_2$. If $S_2 = \phi$, let $t_2 = a$.

Case 1. $t_1 < t_2$. Then there are solutions $x_i(t)$, $i = 1, 2$ to IVP (3) – (14) for $t_0 \in [t_1, t_2]$ satisfying

$$x_1(t) > x_0(t) \qquad \text{for some } t < t_0$$

and

$$x_2(t) > x_0(t) \qquad \text{for some } t > t_0 .$$

By Lemma 2.2 all solutions to IVP (3) – (14) exist on $[t_0 - \delta, t_0 + \delta]$ for some $\delta > 0$. Assume $t_0 + \delta > t_2$ and hence $x_2(t_0 + \delta) > x_0(t_0 + \delta)$. By Knesser's Theorem, ([2], Theorem 4.1, page 15), there exist solutions $\{u_n(t)\}$ to IVP (3) – (14) on $[t_0, t_0 + \delta]$ such that $u_n(t_0 + \delta) \to x_0(t_0 + \delta)$ as $n \to \infty$. By Kamke's convergence theorem, ([2], Theorem 3.2, page 14), and our uniqueness assumption, there exists a subsequence $\{u_n^*(t)\}$ which converges to $x_0(t)$ in C^1-norm on $[t_0, b]$ with

$$u_n(b) > u_{n+1}(b) > x_0(b) \qquad \text{for all } n .$$

If $x_2(t) \equiv x_0(t)$ on $[t_0, t_0 + \delta]$ we continue to the right until there is an interval $[t_3, t_3 + \delta]$ such that $x_2(t) \equiv x_0(t)$ on $[t_0, t_3]$ and

$$x_2(t_3 + \delta) > x_0(t_3 + \delta) .$$

Then argue as above.

Similarly there are solutions $v_n(t)$ to IVP (3) – (14) which converge to $x_0(t)$ in C^1-norm on $[a, t_0]$ with

$$v_n(a) > v_{n+1}(a) > x(a) , \qquad \text{for all } n .$$

Then define

$$\bar{x}_n(t) = \begin{cases} v_n(t) & \text{on } [a, t_0] \\ u_n(t) & \text{on } [t_0, b] . \end{cases}$$

$\{\bar{x}_n(t)\}$ is the required sequence.

Case 2. $t_1 \geqq t_2$. We claim in this case that there exist solutions $\{u_n(t)\}$ and $\{v_n(t)\}$ as given in Lemma 2.4 with

$$u_n(t_0) = v_n(t_0) \qquad \text{for some } t_0 \in [t_1, t_2], \qquad \text{for all } n$$

sufficiently large. To see this, given $\varepsilon' > 0$, let δ', C_0, and C_0' be the constants assured by Lemma 2.2. Choose $\varepsilon > 0$ so small that

(i) $$\varepsilon < \varepsilon'$$

and

(ii) $$\delta \equiv \frac{2\varepsilon}{\varepsilon'} < \varepsilon' .$$

Thus there exists an N such that

$$\| u_N(t) - x_0(t) \|_{C^1} < \varepsilon \qquad \text{on } [a, t_0]$$

Since $t_1 \geqq t_2$ we may assume $u_N(t_0) > x_0(t_0)$ and hence

$$x_0(t_0) < v_n(t_0) < u_N(t_0) \quad \text{for all } n \text{ sufficiently large} ,$$

say $n \geqq N'$. By [3], Theorem 2.5, there exists a solution $x_n(t)$ to BVP

$$x'' = F(t, x, x'; \mu_0)$$

$$x(a) = \alpha \qquad x(t_0) = v_n(t_0) \qquad \text{for all } n \geqq N' ,$$

where $F(t, x, x'; \mu_0)$ is the modification of $f(t, x, x'; \mu_0)$ with respect to $x_0(t)$, $u_N(t)$ and C_0. Furthermore

$$x_0(t) \leqq x_n(t) \leqq u_N(t) \quad \text{on } [a, t_0] .$$

Hence

$$|x'_n(t) - x'_0(t)| \ngeqq \varepsilon'$$

on any subinterval of $[a, t_0]$ of length $\delta = 2\varepsilon/\varepsilon'$. But this in turn implies that

$$|x'_n(t)| \leqq C_0$$

on any subinterval of length δ and hence

$$|x'_n(t)| \leqq C_0 \quad \text{on } [a, t_0] .$$

By definition of $F(t, x, x'; \mu_0)$, $x_n(t)$ is a solution to (3). Now for n sufficiently large we may extend the solution $x_n(t)$ to $[a, b]$ and use them as the solutions $u_n(t)$ assured by Lemma 2.4.

Let $\varepsilon > 0$ be arbitrary, but fixed, and let δ, C_0, and C'_0 be the constants assured by Lemma 2.2. There exists an $N > 0$ such that for all $n \geqq N$

$$\| u_n(t) - x_0(t) \|_{C^1} < \varepsilon$$

and

$$\| v_n(t) - x_0(t) \|_{C^1} < \varepsilon .$$

Suppose for definiteness that $u'_N(t_0) \geqq v'_N(t_0)$. Define

$$C_1^N = \{(x, x'): v_N(t_0) = u_N(t_0) = x , \quad v'_N(t_0) \leqq x' \leqq u'_N(t_0)\} .$$

C_1^N is a compact connected set in R^2. Let C_2^N be the set of all (β, β') such that there exists a solution $x(t; \alpha, \alpha')$ to (3) with $x(t_0) = \alpha$, $x'(t_0) = \alpha'$, where

$$(\alpha, \alpha') \in C_1^N, \ x(t_0 - \delta; \alpha, \alpha') = \beta$$

and

$$x'(t_0 - \delta; \alpha, \alpha') = \beta' .$$

By an extension of Knesser's Theorem, ([6], page 386), C_2^N is a compact connected set in R^2 containing the two points $(u_N(t_0 - \delta), u'_N(t_0 - \delta))$ and $(v_N(t_0 - \delta), v'_N(t_0 - \delta))$.

(i) $$C_2^N \subset \overline{N_\varepsilon(x_0(t_0 - \delta), x'_0(t_0 - \delta))} .$$

In this case $|\beta - x_0(t_0 - \delta)| \leqq \varepsilon$ and $|\beta' - x'_0(t_0 - \delta)| \leqq \varepsilon$ for all $(\beta, \beta') \in C_2^N$. Hence the solutions $x(t; \alpha, \alpha')$ may be continued to $[t_0 - 2\delta, t_0]$. Similarly, we may continue these solutions to the left by δ intervals as long as the end points of each solution lie inside the ε-tube about $x_0(t)$. Let n_0 be such that

$$t_0 - (n_0 - 1)\,\delta \geqq a > t_0 - n_0\delta$$

and suppose that

$$C_n^N \subset \overline{N_\varepsilon(x_0(t_0 - (n-1)\delta),\, x_0'(t_0 - (n-1)\delta))}$$

for all $n \leqq n_0$. Thus all solutions $x(t; \alpha, \alpha')$ may be continued to $[a, t_0]$.

We now perform a similar procedure to the right of t_0. Define $D_1^N = C_1^N$ and D_n^N analogously to C_n^N. Assume also in (i) that

$$D_n^N \subset \overline{N_\varepsilon(x_0(t_0 + (n-1)\delta),\ x_0'(t_0 + (n-1)\delta))} \quad \text{for all } n \leqq n_0' ,$$

where

$$t_0 + (n_0' - 1)\delta < b \leqq t_0 + n_0'\delta \ .$$

Define C_a^N to be the set of all (γ, γ') such that there exists a solution $x(t)$ to IVP

$$x'' = f(t, x, x'; \mu_0)$$

$$x(t_0 - (n_{-1})\delta) = \beta,\ x'(t_0 - (n_{-1})\delta) = \beta'$$

with $(\beta, \beta') \in C_{n_0-1}^N$ and $x(a) = \gamma$, $x'(a) = \gamma'$. Analogously, define D_b^N.

Now for each $(\gamma, \gamma') \in C_a^N$ with $\gamma > x_0(a)$, there exists a solution $x(t; \gamma, \gamma')$ on $[a, b]$ such that $x(a; \gamma, \gamma') = \gamma$, $x'(a; \gamma, \gamma') = \gamma'$, $x(t_0; \gamma, \gamma') = u_N(t_0) = v_N(t_0)$ and $(x(b; \gamma, \gamma'),\ x'(b; \gamma, \gamma')) \in D_b^N$. If $x(b; \gamma, \gamma') > x_0(b)$, we've found an $\bar{x}(t)$. Hence suppose $x(b; \gamma, \gamma') \leqq x_0(b)$ for all $(\gamma, \gamma') \in C_a^N$ with $\gamma > x_0(a)$. Pick $\{\gamma_n\} \to x_0(a)$, $\gamma_n > x_0(a)$ for all n. By Lemma 2.2, we have

$$|\,x(t; \gamma_n, \gamma_n')\,| < C_0'$$

and

$$|\,x'(t; \gamma_n, \gamma_n')\,| < C_0 \qquad \text{for all } t \in [a, b] \ .$$

By Ascoli's Theorem there exists a subsequence $\{x^*(t; \gamma_n, \gamma_n')\}$ which converges in C^1-norm to a solution $z(t)$ on $[a, b]$. But $z(a) = x_0(a)$, $z(b) \leqq x_0(b)$, and $z(t_0) = u_N(t_0) > x_0(t_0)$ contradicting our uniqueness assumption. We may thus assume that we have an $\bar{x}_1(t)$. Also by II and the definition of t_0, $\bar{x}_1(t) > x_0(t)$ on $[a, b]$. Now let

$$\varepsilon' = \min_{[a,b]}\ (\bar{x}_1(t) - x_0(t)) > 0 \ .$$

Using this ε' in place of ε we may procede to find an $\bar{x}_2(t)$ as long as (i) occurs.

(ii) Suppose for some n,

$$C_n^N \not\subset \overline{N_\varepsilon(x_0(t_0 - (n-1)\delta), x_0'(t_0(n-1)\delta))} .$$

Then all solutions $x(t; \alpha, \alpha')$ for $(\alpha, \alpha') \in C_1^N$ may not be extendable to $[a, t_0]$. However, if we let ${}^uC_n^N$ and ${}^vC_n^N$ be the components of

$$C_n^N \cap \overline{N_\varepsilon(x_0(t_0 - (n-1)\delta), x_0'(t_0 - (n-1)\delta))}$$

which contain $(u_N(t_0 - (n-1)\delta), u_N'(t_0 - (n-1)\delta))$ and $(v_N(t_0 - (n-1)\delta), v_N'(t_0 - (n-1)\delta))$ respectively, then we may continue the solutions ending in ${}^uC_n^N$ or ${}^vC_n^N$ as before.

Let $C_1^{N'} = \{(\alpha, \alpha'): (\alpha, \alpha') \in C_1^N$ and $x(t; \alpha, \alpha')$ exists on $[a, t_0]$ by the extension procedure above$\}$. It is easy to see that $C_1^{N'}$ is a compact interval. Analogously we define the compact set $D_1^{N'}$.

(a) $C_1^{N'} = C_1^N = D_1^N = D_1^{N'}$. Thus $x(t; \alpha, \alpha')$ exists on $[a, b]$ for all $(\alpha, \alpha') \in C_1 = D_1$. Suppose no $x(t; \alpha, \alpha')$ satisfies $x(a; \alpha, \alpha') > x_0(a)$ and $x(b; \alpha, \alpha') > x_0(b)$.

Let $\alpha_0' = \sup\{\alpha': u_N'(t_0) \geqq \alpha' \geqq v_N'(t_0), x(a; \alpha, \alpha') > x_0(a)$, and $(\alpha, \alpha') \in C_1^N\}$. By an application of Ascoli's Theorem, there exists a solution $z_1(t)$ with $z_1(b) \leqq x_0(b)$, $z_1(a) > x_0(a)$, $z_1(t_0) > x_0(t_0)$, and $z_1'(t_0) = \alpha_0'$. By uniqueness $\alpha_0' < u'(t_0)$ and hence there exists a sequence $\{\alpha_m'\} \to \alpha_0'$ with $\alpha_0' < \alpha_m' \leqq u_n'(t_0)$ and a sequence of solutions $x(t; \alpha, \alpha_m')$ such that $x(a; \alpha, \alpha_m') \leqq x_0(a)$. By Ascoli's Theorem again there exists a subsequence converging in C^1-norm to a solution $z_2(t)$ with $z_2'(t_0) = \alpha_0'$ and $z_2(a) \leqq x_0(a)$. The solution

$$z(t) = \begin{cases} z_2(t) & \text{on } [a, t_0] \\ z_1(t) & \text{on } [t_0, b] \end{cases}$$

contradicts our uniqueness assumption. We must conclude that there exists an $x(t; \alpha, \alpha')$ such that $x(a; \alpha, \alpha') > x_0(a)$ and $x(b; \alpha, \alpha') > x_0(b)$. We then define $\bar{x}(t) \equiv x(t; \alpha, \alpha')$ and continue as before.

(b) $C_1^{N'} \neq C_1^N$, $D_1^N = D_1^{N'}$. Let $\alpha_0' = \sup\{\alpha': v_N'(t_0) \leqq \alpha' \leqq u_N'(t_0)$ and such that there exists a solution $x^r(t; \alpha, \alpha')$ to IVP

$$x'' = f(t, x, x'; \mu_0) \tag{3}$$

$$x(t_0) = u_N(t_0) \qquad x'(t_0) = \alpha' \tag{15}$$

which exists on $[t_0, b]$ and satisfies $x^r(b; \alpha, \alpha') \leqq x_0(b)\}$. Then $\alpha_0' < u_N'(t_0)$ by uniqueness. Let $\alpha_m' \to \alpha_0'$, $\alpha_0' < \alpha_m' \leqq u_N'(t_0)$ for all m. Let $x^1(t; \alpha, \alpha_m')$ be a solution to IVP (3) – (15) on some interval $[\lambda^-, t_0]$. By Kamke's convergence theorem a subsequence of $\{x^1(t; \alpha, \alpha_m')\}$ converges to a solution $x_N(t; \alpha, \alpha_0')$ at least on $[t - \delta, t_0]$. If $x_N(t; \alpha, \alpha_0')$ exists on

$[a, b]$, then $x^1(t; \alpha, \alpha'_m)$ exists on $[a, b]$ for m sufficiently large and $x^1(a; \alpha, \alpha'_m) > x_0(a)$ since $x_N(a; \alpha, \alpha'_0) > x_0(a)$. But $x(b; \alpha, \alpha'_m) > x_0(b)$ by definition of α'_0 and we have our $\bar{x}_1(t)$.

At this point let us do a similar analysis with $u_{N+K}(t)$ and $v_{N+K}(t)$ for $K = 1, 2, \cdots$. We wish to show that $x_{N+K}(t; \alpha_K, \alpha'_{K_0})$ fails to exist on $[a, t_0]$ for at most a finite number of indices K. Suppose for contradiction $x_{N+K}(t; \alpha_K, \alpha'_{K_0})$ does not exist on $[a, t_0]$ for $K = 1, 2, \cdots$, relabeling if necessary. Since $(\alpha_K, \alpha'_{K_0}) \to (x_0(t_0), x'_0(t_0))$, by Kamke's convergence theorem a subsequence of $\{x_{N+K}(t; \alpha_K, \alpha'_{K_0})\}_{K=1}^{\infty}$ converges to a solution $z(t)$ of the IVP (3) $-$ (14) in C^1-norm on some interval $(\omega^-, t_0]$. By definition of t_1, $z(t) \leqq x_0(t)$ on $(\omega^-, t_0]$ and by uniqueness of $x_0(t)$, $z(t) \geqq x_0(t)$. Hence $z(t) \equiv x_0(t)$ and thus for K sufficiently large $x_{N+K}(t; \alpha_K, \alpha'_{K_0})$ exists on $[a, t_0]$, which is a contradiction. Thus for all but perhaps a finite number of cases in which (b) occurs we obtain an $\bar{x}_1(t)$.

(c) $C_1^{N'} \neq C_1^N$ and $D_1^{N'} \neq D_1^N$ with

$$[C_1^N - ({}^uC_1^N \cup {}^vC_1^N)] \cap [D_1^N - ({}^uD_1^N \cup {}^vD_1^N)] \neq \phi .$$

Again we claim (c) can happen for at most a finite number of indices K without obtaining a suitable $\bar{x}_1(t)$. Suppose for contradiction that (c) occurs for $K = 1, 2, \cdots$. Pick α'_K such that

$$v'_{N+K}(t_0) < \alpha'_K < u'_{N+K}(t_0)$$

with $\alpha'_K \in [C_1^{N+K} - ({}^uC_1^{N+K} \cup {}^uC_1^{N+K})] \cap [D_1^{N+K} - ({}^uD_1^{N+K} \cup {}^vD_1^{N+K})]$. Then if we assume $x(t; \alpha_K, \alpha'_K)$ does not exist on $[a, b]$ and argue as in (b) we arrive at a contradiction.

(d) $C_1^{N'} \neq C_1^N$, $D_1^{N'} \neq D_1^N$ with

$$[C_1^N - ({}^uC_1^N \cup {}^vC_1^N)] \cap [D_1^N - ({}^uD_1^N \cup {}^vD_1^N)] = \phi .$$

In this case $C_1^N - ({}^uC_1^N \cup {}^vC_1^N)$ is a subset of ${}^uD_1^N \cup {}^vD_1^N$ and we may proceed as in (b) to show that this can occur without obtaining a suitable $\bar{x}(t)$ for at most a finite number of indices K.

Hence (b), (c), and (d) are inconclusive for at most a finite number of indices. We may thus conclude that an $\bar{x}_1(t)$ exists, and as before $\bar{x}_1(t) > x_0(t)$ on $[a, b]$. Letting $\varepsilon' = \min_{[a,b]}(\bar{x}_1(t) - x_0(t)) > 0$ and using ε' in place of ε, we may repeat the above procedure to obtain an $\bar{x}_2(t)$ with $x_0(t) < \bar{x}_2(t) < \bar{x}_1(t)$ on $[a, b]$ and so on.

Similarly we may construct $\{\underline{x}(t)\}$.

Theorem 2.5 generalizes a result of Klaasen, [4], Corollary 7, where he assumes uniqueness to two-point BVP'S (3) $-$ (2). The following example shows that Condition II does not imply uniqueness to two-point BVP'S in any neighborhood of $x_0(t)$.

EXAMPLE 2.6 Define

$$f(t, x, x'; \mu) = \begin{cases} 2 \cdot 3^{3/4} \cdot \mu \cdot |x'|^{1/4} |x|^{1/6} & \text{if } x \geqq 0 \\ 0 \quad \text{if } x \leqq 0 . \end{cases}$$

Then $f(t, x, x'; \mu)$ is continuous on $[-1, 1] \times R^2$, and

$$x_c(t) = \begin{cases} (t-c)^3 & \text{if } t \geqq c \\ (c-t)^3 & \text{if } t \leqq c \end{cases} \quad c \in [-1, 1]$$

is a solution to the differential equation

$$x''(t) = f(t, x, x'; 1)$$

on $[-1, 1]$ for any $c \in [-1, 1]$. Also $x_0(t) \equiv 0$ is a solution satisfying Condition II. However, $x_\varepsilon(t) \equiv \varepsilon$ is a solution for any $\varepsilon > 0$ and $x_c(t)$ intersects $x_\varepsilon(t)$ twice if $c \in [-1+\varepsilon^{1/3}, 1-\varepsilon^{1/3}]$.

DEFINITION 2.7. Let $x(t; \mu)$ be a solution to (1). We will say that solutions to (1) are unique with respect to $x(t; \mu)$ on $[a, b]$ if $x(t)$ is any other solution to (1) with $x(t_i) = x(t_i; \mu)$, $i = 1, 2$ for any t_1, t_2 satisfying $a \leqq t_1 < t_2 \leqq b$, then

$$x(t) \equiv x(t; \mu) \quad \text{on } [t_1, t_2] .$$

THEOREM 2.8. *Assume, in addition to I and II, that solutions to BVP's* (3) – (2) *are unique if they exist. By Theorem* 2.3, *there exist a* $\delta > 0$ *such that for all* $|\mu - \mu_0| < \delta$, *a solution* $x(t; \mu)$ *to BVP* (1) – (4) *exists. Assume also that solutions to* (1) *are unique with respect to* $x(t; \mu)$ *for all* $|\mu - \mu_0| < \delta$. *Let* $\varepsilon > 0$ *be given sufficiently small. Then there exists a* $\delta' \leqq \delta$ *such that for all* $|\mu - \mu_0| < \delta'$, *there exists a solution* $x(t; \mu; \alpha, \beta)$ *to BVP* (1) – (2) *where*

$$|\alpha - \alpha_0| < \varepsilon , \quad |\beta - \beta_0| < \varepsilon .$$

Furthermore, $x(t; \mu; \alpha, \beta) \to x_0(t)$ *in* C^1-norm on $[a, b]$ as $\mu \to \mu_0$, $\alpha \to \alpha_0$, and $\beta \to \beta_0$.

Proof. If suffices to show that if $\{\mu_n\}$ is any sequence converging to μ_0, then there is a subsequence, relabeled the same, such that for all n, there exists a solution $x(t; \mu_n; \alpha, \beta)$ to BVP (11) – (2) for any $|\alpha - \alpha_0| < \varepsilon$ and $|\beta - \beta_0| < \varepsilon$, and that $x(t; \mu_n; \alpha, \beta) \to x_0(t)$ as $n \to \infty$, $\alpha \to \alpha_0$ and $\beta \to \beta_0$

Since solutions to BVP's (3) – (2) are unique if they exist, by Theorem 2.5 there exist sequences $\{\bar{x}_m(t; \mu_0)\}_{m=1}^{\infty}$ and $\{\underline{x}_m(t; \mu_0)\}_{m=1}^{\infty}$ which converge to $x_0(t)$ in the C^1-norm on $[a, b]$ with

$$\bar{x}_m(a;\ \mu_0) > x_0(a) > \underline{x}_m(a;\ \mu_0)$$

and

$$\bar{x}_m(b;\ \mu_0) > x_0(b) > \underline{x}_m(b;\ \mu_0)\ .$$

Let δ'' and C_0 be the constant assured by Lemma (2.2) for $\varepsilon = 1$, and let $\varepsilon' = \min(1/2, \delta''/4)$. Then there exists an M such that for $m \geqq M$

$$\| \bar{x}_m(t;\ \mu_0) - x_0(t) \|_{C^1} < \varepsilon'$$

and

$$\| \underline{x}_m(t;\ \mu_0) - x_0(t) \|_{C^1} < \varepsilon'\ .$$

Fix $m \geqq M$ and let

$$0 < \varepsilon < \min(\varepsilon',\ 1/2(\bar{x}_m(a) - x_0(a)),\ 1/2(\bar{x}_m(b) - x_0(b))\ .$$
$$1/2(x_0(a) - \underline{x}_m(a)),\ 1/2(x_0(b) - \underline{x}_m(b))\ .$$

(This puts an upper bound on the possible choices of ε). By Theorem 2.3 there exists a subsequence of $\{\mu_n\}$, relabeled the same, and an $N > 0$ such that for $n \geqq N$ there exist solutions $\bar{x}_m(t;\ \mu_n)$ and $\underline{x}_m(t;\ \mu_n)$ to equation (11) satisfying

$$x(a) = \bar{x}_m(a;\ \mu_0) \qquad x(b) = \bar{x}_m(b; \mu_0)$$

and

$$x(a) = \underline{x}_m(a;\ \mu_0) \qquad x(b) = \underline{x}(b;\ \mu_0)\ , \qquad \text{respectively}\ .$$

with

$$\| \bar{x}_m(t;\ \mu_n) - \bar{x}_m(t;\ \mu_0) \|_{C^1} < \varepsilon$$

$$\| \underline{x}_m(t;\ \mu_n) - \underline{x}_m(t;\ \mu_0) \|_{C^1} < \varepsilon\ .$$

Let $F(t, x, x';\ \mu_n)$ be the modification of $f(t, x, x';\ \mu_n)$ with respect to $\bar{x}(t;\ \mu_n)$, $\underline{x}(t;\ \mu_n)$ and C_0. By Theorem 2.5, [3], there exists a solution $x(t;\ \mu_n;\ \alpha, \beta)$ to $x'' = F(t, x, x';\ \mu_n)$ satisfying (2) for all $n \geqq N$, provided $| \alpha - x_0(a) | < \varepsilon$ and $| \beta - x_0(b) | < \varepsilon$, with

$$\underline{x}_m(t;\ \mu_n) \leqq x(t;\ \mu_n;\ \alpha, \beta) \leqq \bar{x}_m(t;\ \mu_n) \qquad \text{on } [a, b]\ .$$

Again it is easy to show that $x(t;\ \mu_n;\ \alpha, \beta)$ is a solution to (11).

It remains to show that $x(t;\ \mu_n;\ \alpha, \beta) \to x_0(t)$ as $n \to \infty$, $\alpha \to \alpha_0$ and $\beta \to \beta_0$. Let $\{\alpha_m\}$ be any sequence converging to α_0 and $\{\beta_m\}$ any sequence converging to β_0. By construction the sequences $\{x(t; \mu_n; \alpha_m, \beta_m)\}$ and $\{x'(t;\ \mu_n;\ \alpha_m, \beta_m)\}$ are uniformly bounded and equicontinuous on $[a, b]$. Hence there exists a subsequence which converges in C^1-norm on $[a, b]$ to $x_0(t)$ by the uniqueness of $x_0(t)$. This implies that $x(t;\ \mu;\ \alpha, \beta) \to x_0(t)$ as $\mu \to \mu_0$, $\alpha \to \alpha_0$, and $\beta \to \beta_0$.

References

1. R. Gaines, *Continuous dependence for two-point boundary value problems*, Pacific J. Math., **28** (1969), 327-336.
2. P. Hartman, *Ordinary Differential Equations*, John Wiley and Sons, New York (1964).
3. L. K. Jackson, *Subfunctions and second order ordinary differential inequalities*, Advances in Math., **2** (1968) 307-363.
4. G. Klaasen, *Dependence of solutions on boundary conditions for second order ordinary differential equations*, J. Diff. Eqs., **7** (1970), 24-33.
5. ———, *Boundary Value Problems for Second Order Ordinary Differential Equations*, a thesis presented to Univ. of Nebraska, (1968).
6. G. A. Sell, *On the fundamental theory of ordinary differential equations*, J. Diff. Eqs., **1** (1965) 370-392.

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