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THE TRANSLATIONAL HULL OF AN N-SEMIGROUP

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An N-semigroup is a commutative, cancellative, archimedean semigroup having no idempotents. In the first section of this paper the Tamura representation of an N-semigroup is used to determine the translational hull. The maximal semilattice decomposition of the translational hull is then investigated resulting in a complete determination of the classes of this decomposition in the case that the N-semigroup is power joined. These results are used in the second section which deals with ideal extensions of an N-semigroup by an abelian group, and ideal extensions of an abelian group by an N-semigroup. These extensions arise naturally in the maximal semilattice decomposition of a commutative separative semigroup. The latter part of this section contains results on cancellative extensions of N-semigroups, and a structure theorem of the class of weakly power joined, commutative, cancellative semigroups.

Notation and Preliminaries. Let S be a semigroup. We will write left [right] translations as operators on the left [right]; $\Lambda(S)$ [$P(S)$] denotes the semigroup of all left [right] translations of S under the multiplication $(\lambda\lambda')x = \lambda(\lambda' x)$ [$x(\rho\rho') = (x\rho)\rho'$] for all $x \in S$. The translations $\lambda \in \Lambda(S)$ and $\rho \in P(S)$ are linked if $x(\lambda y) = (x\rho)y$ for all $x, y \in S$; $\Omega(S)$ denotes the translational hull of S, that is, the subsemigroup of $\Lambda(S) \times P(S)$ consisting of all pairs of linked translations. The subsemigroup of $\Omega(S)$ consisting of all pairs of the form (λ_a, ρ_a) is denoted by $\Pi(S)$. (Recall that $\lambda_a x = ax$ and $x\rho_a = xa$ for all $x \in S$).

We will need the following results concerning $\Omega(S)$ when S is a commutative cancellative semigroup. Proofs may be found in [6].

(1) $\Omega(S)$ is commutative and cancellative.

(2) $(\lambda, \rho) \in \Omega(S)$ if and only if $\lambda x = x\rho$ for all $x \in S$, and hence $\Omega(S) \cong \Lambda(S)$.

(3) $S \cong \Gamma(S)$ where $\Gamma(S) = \{\lambda_a \in \Lambda(S) \mid a \in S\}$.

We next describe the Tamura representation of an N-semigroup. Let N denote the positive integers and N_0 the nonnegative integers. Let G be an abelian group and $I\colon G \times G \to N_0$ be a function satisfying:

(i) $I(a, b) = I(b, a)$ $\qquad (a, b \in G)$,

(ii) $I(a, b) + I(ab, c) = I(a, bc) + (b, c)$ $\qquad (a, b, c \in G)$,

(iii) For each $a \in G$ there is an $m \in N$ such that $I(a^m, a) > 0$.

(iv) $I(e, e) = 1$ where e is the identity of G.

On the set $N_0 \times G$ define a multiplication by:

$$(m, a)(n, b) = (m + n + I(a, b), ab) .$$

With this multiplication $N_0 \times G$ becomes an N-semigroup which will be denoted by (G, I). The abelian group G is called the structure group and I an index function. We have the following fundamental result due to Tamura [7].

THEOREM. *Let S be an N-semigroup. Then $S \cong (G, I)$ for some abelian group G and some index function I on G.*

DEFINITION. A semigroup S is power joined if for each $a, b \in S$ there are $m, n \in N$ such that $a^m = b^n$.

The next theorem, due to Chrislock [1], points out the role the structure group plays in an N-semigroup.

THEOREM. *$S = (G, I)$ is power joined if and only if G is periodic. $S = (G, I)$ is finitely generated if and only if G is finite.*

For all concepts and notation not defined in this paper, the reader is referred to [2].

1. The translational hull. It follows from our preliminary remarks that to determine $\Omega(S)$, where S is an N-semigroup, we need only consider left translations.

LEMMA 1.1. *Let $S = (G, I)$. Then $I(x, e) = 1$ for all $x \in G$.*

Proof. By setting $a = x$ and $b = c = e$ in property (ii) of an index function we have

$$I(x, e) + I(x, e) = I(x, e) + I(e, e) \text{ , whence } I(x, e) = I(e, e) = 1 .$$

THEOREM 1.1. *Let $S = (G, I)$. Then $\Lambda(S) = \{[m, g] \mid g \in G, m \in N_0,$ and $m + I(g, h) - 1 \geqq 0$ for all $h \in G\}$, where $[m, g]$ operates on elements of S as follows:*

$$[m, g](n, a) = (m + n + I(g, a) - 1, ga) .$$

Proof. The condition, $m + I(g, h) - 1 \geqq 0$ for all $h \in G$, insures that $[m, g]$ maps S into S. Let $(n, a), (p, b) \in S$. Then

$$\begin{aligned}[m, g]\{(n, a)(p, b)\} &= [m, g](n + p + I(a, b), ab) \\ &= (m + n + p + I(a, b) + I(g, ab) - 1, gab) \\ &= (m + n + p + I(g, a) + I(ga, b) - 1, gab) \\ &= (m + n + I(g, a) - 1, ga)(p, b) \\ &= \{[m, g](n, a)\}(p, b) ,\end{aligned}$$

where the third equality follows from the fact that $I(a, b) + I(g, ab) = I(ga, b) + I(g, a)$, and the others directly from the definition of $[m, g]$ and multiplication in S. Hence, $[m, g] \in \Lambda(S)$.

Conversely, let λ be any left translation of S. Then $\lambda = [m, g]$ where $\lambda(0, e) = (m, g)$. We first show that $[m, g]$ satisfies the condition $m + I(g, h) - 1 \geqq 0$ for all $h \in G$. This is clear if $m \geqq 1$ since I assumes only nonnegative values. Hence, assume $m = 0$. Let $h \in G$ and let $\lambda(0, h) = (n, b)$. Then

$$\lambda(1, h) = \lambda\{(0, e)(0, h)\} = \{\lambda(0, e)\}\ (0, h) = (0, g)(0, h) = (I(g, h), gh) ,$$

and

$$\lambda(1, h) = \lambda\{(0, h)(0, e)\} = \{\lambda(0, h)\}\ (0, e) = (n, b)\ (0, e) = (n + 1, b) .$$

A comparison of the first coordinates of the above expressions for $\lambda(1, h)$ shows that $I(g, h) = n+1$ which certainly implies that $I(g, h) \geqq 1$. It follows that $m + I(g, h) - 1 \geqq 0$ in any case. It remains to show that $\lambda = [m, g]$. Let $(n, a) \in S$ where $n \geqq 1$. Then

$$\begin{aligned}\lambda(n, a) &= \lambda\{(0, e)(n - 1, a)\} = \{\lambda(0, e)\}(n - 1, a) \\ &= (m, g)(n - 1, a) = (m + n + I(g, a) - 1, ga) = [m, g](n, a) .\end{aligned}$$

If $n = 0$ let $\lambda(0, a) = (p, b)$. Then

$$\begin{aligned}(p + 1, b) &= (p, b)(0, e) = \{\lambda(0, a)\}(0, e) = \lambda\{(0, a)(0, e)\} \\ &= \lambda\{(0, e)\}(0, a) = (m, g)(0, a) = (m + I(g, a), ga) .\end{aligned}$$

Hence, we have $p = m + I(g, a) - 1$ and $b = ga$, whence

$$\lambda(0, a) = (p, b) = (m + I(g, a) - 1, ga) = [m, g](0, a) .$$

Therefore, $\lambda = [m, g]$ and the proof is complete.

REMARK 1.1. The condition on $[m, g]$ in order that it be a left translation is always satisfied when $m \geqq 1$. Hence, it is relevant only when we want to determine whether pairs of the form $[0, g]$ are left translations. A necessary and sufficient condition for $[0, g]$ to be a left translation is that $I(g, h) \geqq 1$ for all $h \in G$. In particular, $[0, e]$ is always a left translation since $I(e, h) = 1$ for all $h \in G$. In fact, $[0, e]$ is the identity function on S. As we will establish later, the elements of $\Lambda(S)$ with a zero in the first coordinate are precisely those left translations that are not inner.

PROPOSITION 1.1. *Let* $S = (G, I)$ *and* (m, g), $(n, h) \in \Lambda(S)$. *Then*
(i) $[m, g] = [n, h]$ *if and only if* $m = n$ *and* $g = h$,
(ii) $[m, g][n, h] = [m + n + I(g, h) - 1, gh]$.

Proof. To prove (i) note that $(m, g) = [m, g](o, e) = [n, h](0, e) = (n, h)$. Since equality in S is defined coordinatewise we have $m = n$ and $g = h$. If $m = n$ and $g = h$, clearly $[m, g] = [n, h]$.

To prove (ii) let $(p, a) \in S$. Then

$$\begin{aligned}\{[m, g][n, h]\}(p, a) &= [m, g]\{[n, h](p, a)\}\\ &= [m, g](n + p + I(a, h) - 1, ha)\\ &= (m + n + p + I(a, h) + I(g, ha) - 2, gha)\\ &= (m + n + p + I(g, h) + I(gh, a) - 2, gha)\\ &= [m + n + I(g, h) - 1, gh](p, a)\ ,\end{aligned}$$

where the fourth equality follows from the fact that $I(a, h) + I(g, ha) = I(gh, a) + I(g, h)$. Hence, $[m, g][n, h] = [m + n + I(g, h) - 1, gh]$.

THEOREM 1.2. *Let* $S = (G, I)$. *Then* $\Gamma(S) = \{[m, g] \mid g \in G, m \geqq 1\}$. *Moreover*, $\lambda_{(n,a)} = [n + 1, a]$.

Proof. In the proof of Theorem 1.1 it was established that $\lambda = [m, g]$ where $\lambda(0, e) = (m, g)$ for each $\lambda \in \Lambda(S)$. Since $\lambda_{(n,a)}(0, e) = (n, a)(0, e) = (n + 1, a)$, it follows that $\lambda_{(n,a)} = [n + 1, a]$.

We next determine the group of units of $\Lambda(S)$. This group will be denoted by $\Sigma(S)$.

THEOREM 1.3. *Let* $S = (G, I)$. *Then*
(i) $\Sigma(S) = \{[0, g] \mid I(g, h) > 0$ *and* $I(g^{-1}, h) > 0$ *for all* $h \in G$, $I(g, g^{-1}) = 1\}$.
(ii) *If* G *is periodic, then* $\Sigma(S) = \{[0, g] \mid I(g, h) > 0$ *for all* $h \in G$, $I(g, g^m) = 1$ *for all* $m \in N\}$.

Proof. (i) Let $[0, g]$ be an element of the set on the right. The conditions $I(g, h) > 0$ and $I(g^{-1}, h) > 0$ for all $h \in G$ guarantee that $[0, g]$ and $[0, g^{-1}]$ are left translations. Since $I(g, g^{-1}) = 1$, we have $[0, g][0, g^{-1}] = [0, e]$, and hence, $[0, g] \in \Sigma(S)$. Conversely, let $[m, g] \in \Sigma(S)$, and let $[n, h] = [m, g]^{-1}$. Then $[m, g][n, h] = [0, e]$, whence, $m + n + I(g, h) - 1 = 0$ and $gh = e$. Since $[n, h] \in \Lambda(S)$, we have $n + I(g, h) - 1 \geqq 0$. Hence, $m = 0$ and $[0, g] \in \Lambda(S)$. From this it follows that $I(g, k) > 0$ for all $k \in G$, and in particular, $I(g, h) =$

$I(g, g^{-1}) > 0$. Then $n=0$ and $[0, h]=[0, g^{-1}] \in \Lambda(S)$, whence, $I(g^{-1}, k) > 0$ for all $k \in G$. Since $m = n = 0$, it follows from $m + n + I(g, h) - 1 = 0$ that $I(g, h) = I(g, g^{-1}) = 1$ and the proof of (i) is complete.

(ii) Let $[0, g]$ be an element of the set on the right. To see that $[0, g] \in \Sigma(S)$ it will be sufficient, using (i), to show that $I(g^{-1}, h) > 0$ for all $h \in G$, and that $I(g, g^{-1}) = 1$. Since G is periodic, $g^{-1} = g^r$ for some $r \in N$ and hence, $I(g, g^{-1}) = I(g, g^r) = 1$ by hypothesis. We use induction to show that $I(g^m, h) > 0$ for all $h \in G$ and $m \in N$. It is true by hypothesis for $m = 1$, so assume $I(g^{m-1}, h) > 0$ for all $h \in G$. By setting $a = g$, $b = g^{m-1}$ and $c = h$ in property (ii) of an index function, we have

$$I(g, g^{m-1}) + I(g^m, h) = I(g, g^{m-1}h) + I(g^{m-1}, h) .$$

By hypothesis $I(g, g^{m-1}) = 1$. Also, both terms on the right are positive, $I(g, g^{m-1}h)$ by hypothesis, and $I(g^{m-1}, h)$ by the induction hypothesis. Hence, $I(g^m, h) > 0$ and the induction is complete. Since $g^{-1} = g^r$, we have $I(g^{-1}, h) = I(g^r, h) > 0$ for all $h \in G$. Hence, by (i) $[0, g] \in \Sigma(S)$. Conversely, let $[0, g] \in \Sigma(S)$. Then $I(g, h) > 0$ for all $h \in G$. Also, every power of $[0, g]$ must fall in $\Sigma(S)$, and hence, must have a zero in the first coordinate. Therefore, $I(g, g^m) = 1$ for all $m \in N$.

COROLLARY 1.1. *Let $S = (G, I)$. Then $\Sigma(S)$ is isomorphic to a subgroup of G.*

Proof. The mapping $\theta\colon \Sigma(S) \to G$ defined by $\theta[0, g] = g$ is clearly an injective homomorphism by (i) of the preceding theorem.

REMARK 1.2. An abstract N-semigroup may have many representations in the form (G, I). In each of these representations we may have a different structure group. Corollary 1.1 shows, however, that each structure group contains a subgroup isomorphic to $\Sigma(S)$.

We next investigate the maximal semilattice decomposition of $\Lambda(S)$. Recall that the minimal semilattice congruence σ on a commutative semigroup S is defined by $x\sigma y$ if x and y divide a power of each other.

THEOREM 1.4. *Let $S = (G, I)$, and let σ be the minimal semilattice congruence on $\Lambda(S)$. Then $\Gamma(S)$ is contained in a congruence class of σ, and this class, which we will denote by $[\Gamma(S)]$, is $\Gamma(S) \cup \{[0, g] \mid I(g, h) > 0$ for all $h \in G$, $I(g, g^m) > 1$ for some $m \in N\}$.*

Proof. Since S is archimedean and $\Gamma(S)$ is isomorphic to S, it is clear that $\Gamma(S)$ is contained in a single σ-class. The remaining elements of $\Lambda(S)$ have a zero in the first coordinate. If $I(g, g^m) > 1$, then $[0, g]^{m+1}$ has a nonzero first coordinate, and hence, is in $\Gamma(S)$. Therefore, $[0, g]^{m+1}\sigma[1, e]$ and it follows immediately from the definition of σ that $[0, g]\sigma[1, e]$. Thus, $[0, g]$ is in the same σ class as $\Gamma(S)$. Conversely, if $[0, g] \in [\Gamma(S)]$ there exist $r \in N$ and $[p, k] \in \Lambda(S)$ such that $[1, e][p, k] = [0, g]^r$. Hence,

$$[p + 1, k] = [I(g, g) + \cdots + I(g, g^{r-1}) - (r - 1), g^r] .$$

A comparison of the first coordinates shows that $I(g, g^m) > 1$ for some $m \leqq r - 1$.

THEOREM 1.5. *Let $S = (G, I)$ where G is periodic, and let σ be the minimal semilattice congruence on $\Lambda(S)$. There are exactly two congruence classes of σ, namely $\Sigma(S)$ and $[\Gamma(S)]$.*

Proof. This follows immediately from Theorems 1.3 and 1.4.

COROLLARY 1.3. *Let $S = (G, I)$ where G is finite. Then the maximal semilattice decomposition of S consists of two classes, $\Sigma(S)$ and $[\Gamma(S)]$.*

Proof. A finite group is clearly periodic so the result follows immediately from Theorem 1.5.

REMARK 1.3. In the case that $S = (G, I)$ is power joined (G is periodic) or the case that S is finitely generated (G is finite) precise descriptions of the congruence classes of the minimal semilattice congruence are given by Theorems 1.3 and 1.4.

2. Extensions. A semigroup is separative if for each pair $a, b \in S$ with $a^2 = ab = b^2$ we have $a = b$. In [5] Hewitt and Zuckerman proved that a commutative semigroup S is separative if and only if S is a semilattice of N-semigroups and abelian groups. This result suggests the construction of all commutative separative semigroups from the basic building blocks of semilattices, N-semigroups, and abelian groups. As a first step in this direction we consider the construction in the case where the semilattice has two elements, that is, the extension problem. We will be concerned with finding all extensions of an N-semigroup by an abelian group with zero adjoined, and all extensions of an abelian group by an N-semigroup with zero adjoined. In the latter part of this section we will consider cancellative extensions.

In view of Theorem 4.21 of [2] and our remarks in the preliminary section, every extension of a commutative concellative semigroup S by a commutative cancellative semigroup Q with zero adjoined is determined by a homomorphism of Q into $\Lambda(S)$. Hence, it will be sufficient to determine these homomorphisms in the cases in question.

THEOREM 2.1. *Let $S' = (G', I')$ and G be an abelian group. Let $\tau = G \to G'$ be a homomorphism satisfying*

(i) $I'(\tau a, g') > 0 \qquad (a \in G;\ g' \in G')$,

(ii) $I'(\tau a, \tau b) = 1 \qquad (a, b \in G)$.

Define $\theta_\tau\colon G \to \Lambda(S')$ as follows:

$$\theta_\tau(a) = [0, \tau a] \qquad (a \in G)\ .$$

Then θ_τ is a homomorphism of G into $\Lambda(S')$ and every homomorphism of G into $\Lambda(S')$ is of this form.

Proof. Condition (i) insures that θ_τ maps G into $\Lambda(S')$. Let $a, b \in G$. Then $\theta_\tau(ab) = [0, \tau(ab)] = [0, \tau a][0, \tau b] = \theta_\tau(a)\theta_\tau(b)$, where the second equality follows from (ii) and the fact that τ is a homomorphism.

Conversely, let θ be a homomorphism of G into $\Delta(S')$. Let $\tau\colon G \to G'$ be defined by:

$$\theta(a) = [0, \tau a] \qquad (a \in G)\ .$$

This definition is valid since θ must map G into $\Sigma(S')$, that is, there must be a zero in the first coordinate of $\theta(a)$. Let $a, b \in G$. Then $\theta(ab) = \theta(a)\theta(b)$, whence,

$$[0, \tau(ab)] = [0, \tau a][0, \tau b] = [I'(\tau a, \tau b) - 1, (\tau a)(\tau b)]\ .$$

Therefore, $I'(\tau a, \tau b) = 1$ and $\tau(ab) = (\tau a)(\tau b)$. That τ satisfies (ii) follows immediately from the fact that θ maps G into $\Lambda(S)$. Finally, let $a \in G$. Then $\theta(a) = [0, \tau a] = \theta_\tau(a)$ and $\theta = \theta_\tau$.

THEOREM 2.2. *Let $S = (G, I)$ and G' be an abelian group. Let $\tau = G \to G'$ be a function satisfying*

(i) $(\tau a)(\tau b) = (\tau e)^{I(a,b)}\tau(ab) \qquad (a, b \in G)$.

Define $\theta_\tau\colon S \to G'$ as follows:

$$\theta_\tau(m, a) = (\tau e)^m(\tau a) \qquad ((m, a) \in S)\ .$$

Then θ_τ is a homomorphism of S into G', and every homomorphism of S into G' is of this form.

Proof. Since $G' \cong \Lambda(G')$, it is sufficient to find all homomorphisms of S into G'. Let (m, a) and (n, b) be elements of S. Then

$$(1) \qquad \theta_\tau\{(m, a)(n, b)\} = \theta_\tau(m + n + I(a, b), ab) = (\tau e)^{m+n+I(a,b)}\tau(ab) ,$$

while

$$(2) \qquad \theta_\tau(m, a)\theta_\tau(n, b) = (\tau e)^m(\tau a)(\tau e)^n(\tau b) = (\tau e)^{m+n}(\tau a)(\tau b) .$$

That (1) and (2) are equal follows from condition (i). Hence, θ_τ is a homomorphism.

Conversely, let θ be a homomorphism of S into G'. Define $\tau: G \to G'$ as follows:

$$\tau a = \theta(0, a) \qquad (a \in G) .$$

Let $(m, a) \in S$. Then

$$\theta(m, a) = \{\theta(0, e)^m(0, a)\} = (\tau e)^m(\tau a) = \theta_\tau(m, a) .$$

Thus, $\theta = \theta_\tau$ and θ_τ is a homomorphism. Therefore, we have the equality of (1) and (2) above. That τ satisfies (i) follows immediately from this equality.

REMARK 2.1. Note that any homomorphism of G into G' satisfies condition (i) of Theorem 2.2. Hence, the functions satisfying this condition may be considered "generalized" homomorphisms.

A commutative cancellative semigroup is clearly separative and hence is a semilattice of N-semigroups and abelian groups. It is of interest then to investigate cancellative extensions of N-semigroups and abelian groups. The reader is referred to [4] for an investigation of extensions of nonpotent cancellative semigroups by groups. Our investigation will be based on the following results which are due to Grillet and Petrich.

THEOREM. (Grillet and Petrich [3]). *Let V be an extension of a semigroup S. For each $a \in V$ let*

$$\lambda^a x = ax \qquad x\rho^a = xa \qquad (x \in S) ,$$

and $\tau = \tau(V: S): a \to (\lambda^a, \rho^a)$. Then $\tau(V: S)$ is a (canonical) *homomorphism of V into $\Omega(S)$.*

THEOREM 2.3. (Petrich [6]). *An extension V of a semigroup S is cancellative if and only if S is cancellative and $\tau(V: S)$ is injective.*

REMARK 2.2. When the semigroups under consideration are commutative and cancellative we may take $\tau(V: S)$ to be a homomorphism of V into $\Lambda(S)$.

PROPOSITION 2.1. *An extension V of a cancellative semigroup S with identity cannot be cancellative. In particular there exists no cancellative extension of a group.*

Proof. Since S is assumed to have an identity, it follows that $S \cong \Pi(S) = \Omega(S)$. The canonical homomorphism $\tau(V\colon S)$ maps S onto $\Pi(S)$, and hence, cannot be injective. By Theorem 2.3, the extension V cannot be cancellative.

PROPOSITION 2.2. *An extension V of an N-semigroup is cancellative if and only if V is isomorphic to a subsemigroup of $\Lambda(S)$ that contains $\Gamma(S)$.*

Proof. This follows immediately from Theorem 2.3.

If S is a semigroup with zero let S^* denote the set $S \backslash \{0\}$.

PROPOSITION 2.3. *Let Q be a semigroup with zero such that Q^* is a semigroup. Then there exists a cancellative extension V of a power joined N-semigroup $S = (G, I)$ by Q if and only if Q^* is isomorphic to a subgroup of $\Sigma(S)$. In particular, Q^* must be a periodic abelian group that is isomorphic to a subgroup of G.*

Proof. If Q is isomorphic to a subgroup H of $\Sigma(S)$, then $H \cup \Gamma(S)$ is a cancellative extension of S by Q. Conversely, if V is a cancellative extension of S by Q, it follows from Proposition 2.2 and Theorem 1.5 that Q^* is isomorphic to a subsemigroup of $\Sigma(S)$. We complete the proof by noting that Corollary 1.1 implies that any subsemigroup of $\Sigma(S)$ is in fact a subgroup.

PROPOSITION 2.4. *Let Q be a semigroup with zero such that Q^* is a semigroup. Then there exists a cancellative extension V of a finitely generated N-semigroup $S = (G, I)$ by Q if and only if Q^* is isomorphic to a subgroup of $\Sigma(S)$. In particular, Q^* must be a finite abelian group that is isomorphic to a subgroup of G.*

Proof. The proof is analogous to that of Proposition 2.3 taking into consideration the fact that G is finite.

We next characterize abstractly the extensions that may occur in Propositions 2.3 and 2.4.

DEFINITION. A semigroup S is weakly power joined if any two elements of infinite order are power joined.

LEMMA 2.1. *A group G is weakly power joined if and only if*

G is periodic.

Proof. It is clear that a periodic group is power joined, and hence, certainly weakly power joined. Let G be a weakly power joined group. If $a \in G$ has infinite order, then a^{-1} has infinite order, whence there exists $m, n \in N$ such that $a^m = (a^{-1})^n$. Hence $a^{m+n} = e$ contradicting our assumption that has a infinite order. Thus, every element of G has finite order and G is periodic.

The class of weakly power joined commutative semigroups contains at least the periodic abelian groups and power joined N-semigroups. The next theorem shows that the remaining semigroups in this class are the cancellative extensions of Proposition 2.3.

THEOREM 2.4. *Let S be a weakly power joined, commutative, cancellative semigroup that is neither a periodic abelian group nor a power joined N-semigroup. Then there is a power joined N-semigroup $T = (G, I)$ such that S is isomorphic to a subsemigroup of $\Lambda(T)$ of the form $H \cup \Gamma(T)$ there H is a subgroup of $\Sigma(T)$. Conversely every such subset of $\Lambda(T)$ is a weakly power joined, commutative, cancellative semigroup.*

Proof. Let S be as in the statement of the theorem. Then S is clearly separative, and hence, is a semilattice Y of semigroups S_α, where for each $\alpha \in Y$, S_α is either an abelian group or an N-semigroup. Since S is weakly power joined, each abelian group must be periodic and each N-semigroup must be power joined. (Recall that every element in an N-semigroup has infinite order.) The assumption that any two elements of infinite order are power joined implies that we can have at most one $\gamma \in Y$ with S_γ being an N-semigroup. Also, since a commutative cancellative semigroup can have at most one idempotent, we can have at most one $\beta \in Y$ with S_β being an abelian group. Thus Y has at most two elements. The assumption that S is neither a power joined N-semigroup nor a periodic abelian group rules out the possibility that Y has exactly one element. By Proposition 2.1 there can be no cancellative extension of a group. Hence, S is an extension of the weakly power joined N-semigroup $T = S_\gamma$ by the periodic abelian group S_β. By Proposition 2.3 such an extension exists if and only if S_β is isomorphic to a subgroup H of $\Sigma(T)$. Hence, the canonical homomorphism $\tau(S\colon T)$ provides an isomorphism of S onto a subsemigroup of $\Lambda(T)$ of the desired type.

To establish the converse statement first note that any subset of $\Lambda(T)$ where $T = (G, I)$ is a power joined N-semigroup is commutative and cancellative. Let $R = H \cup \Gamma(T)$ where H is a subgroup of $\Sigma(T)$. It follows easily from our results in § 1 that R is a subsemigroup. Since $T \cong \Gamma(T)$, we have that $\Gamma(T)$ is power joined. Also, $\Sigma(T)$, and hence, H is periodic. Thus any two elements of infinite order that occur in R must be in $\Gamma(T)$, and hence, are power joined. Therefore, R is weakly power joined.

REMARK 2.3. A similar argument shows that the class of finitely generated, weakly power joined, commutative cancellative semigroups consists of

(i) the finite abelian groups,

(ii) the finitely generated N-semigroups,

(iii) subsemigroups of $\Lambda(T)$ of the form $H \cup \Gamma(T)$ where T is a finitely generated N-semigroup and H is a subgroup of $\Sigma(T)$.

REFERENCES

1. J. R. Chrislock, *The structure of archimedean semigroups*, Dissertation, University of California, Davis, 1966.

2. A. H. Clifford and G. B. Preston, *The Algebraic Theory of Semigroups*, Vol. I, Amer. Math. Soc., Providence, R. I., 1961.

3. P. Grillet and M. Petrich, *Ideal extensions of semigroups*, Pacific J. Math., **26** (1968), 493-508.

4. C. V. Heuer, and D. W. Miller, *An extension problem for cancellative semigroups*, Trans. Amer. Math. Soc., **83** (1966), 499-515.

5. E. Hewitt and H. Zuckerman, *The L_1 algebra of a commutative semigroup*, Trans. Amer. Math. Soc., **83** (1956), 70-97.

6. M. Petrich, *Topics in Semigroups*, Lecture Notes, The Pennsylvania State University, 1967-1968.

7. T. Tamura, *Commutative nonpotent archimedean semigroups with cancellation law I*, J. Gakugei Tokushima Univ., **8** (1957), 5-11.

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