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NON-OPENNESS AND NON-EQUIDIMENSIONALITY IN ALGEBRAIC QUOTIENTS

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Suppose given an equivalence relation R on an algebraic variety V and the associated fibering of V by a family of subvarieties. This paper treats the question of the existence of a quotient structure for this situation when the fibering is non-equidimensional. For this purpose a general definition of quotient variety for algebraic equivalence relations is used which contains no topological requirements.

The results are of two types. In §1 it is shown that certain maps into nonsingular varieties are quotient maps for the induced equivalence relation whenever the union of the excessive orbits has codimension $\geqq 2$. This theorem yields many examples of non-equidimensional quotients. Section 2 contains a converse showing that no excessive orbit containing a normal hypersurface can be fitted into a quotient. This theorem depends on a stronger and less conceptual field-theoretic result which fails without the normality hypothesis. Section 3 contains a counterexample.

These results are a first attempt at answering the question: when can a good algebraic structure be assigned to a non-equidimensional fibering of V? Analogously, one might ask whether there is a good sense in which an equidimensional family of subvarieties of V can degenerate into subvarieties of different dimension. It is a standard result that no fibers of a morphism have dimension less than that of the generic fiber, so that we must concern ourselves only with degeneration into higher-dimensional subvarieties. Thus if the equivalence relation R comes from action of an algebraic group G, in which case any exceptional orbits are *smaller* than the generic ones, there can be no non-equidimensional quotients. In this classical situation, then, existence of a quotient map $p\colon V \rightarrow V/G$ depends on equidimensionality of R. Moreover, such a quotient map is always open, whence topological. Since the substantial volume of recent work on the quotient problem mostly focuses on group actions, the typical definition of quotient map includes the requirement of openness, or even universal openness (cf. Mumford's book [2], p. 4, for example). Now a non-equidimensional morphism is never open (cf. Proposition 1), so we must abandon the topological point of view here.

A number of questions arise that we do not consider in this paper. Is there any relationship between exceptional fibers of a quotient and complete (or noncomplete) intersections? What local data at an exceptional orbit relate to the existence locally of a quotient? For example, flatness is out. Finally, is there any further use for the concept of level field introduced in §2 and 3 for the purpose of proving Theorem 2?

0. Preliminaries. We use the following definitions: An *algebraic equivalence relation* or *ER* R on a variety V (assumed hereafter to be *irreducible*, for simplicity) is an algebraic subset of $V \times V$ which, as a set, is an equivalence relation. An *R-invariant function* f on V is one whose values are constant on the intersection of domain (f) with any R-equivalence class. The R-invariant functions thus constitute a subsheaf of the structure sheaf $\mathscr{O}_V$ of V. *A quotient* for the pair (V, R) is a morphism of varieties p: $V \rightarrow W$ which as a map on sets is a quotient, and which satisfies the local condition

LC: the sheaf of functions on W is identified via p with the sheaf of R-invariant functions on V.

These general quotients have many expected properties, for which my paper [1] provides an elementary exposition. In particular we have

PROPOSITION 1. *Let R be an ER on the irreducible variety V with quotient p: $V \rightarrow W$. Then p is an open map if and only if it is equidimensional.*

Proof. If p is equidimensional, then p is open by a result of Rosenlicht ([4], Lemma 2) on algebraic group quotients. His proof goes through in our more general setting word for word.

For the converse, we apply the following general result.

PROPOSITION 2. *An open morphism of irreducible varieties is equidimensional.*

We give an elementary proof of this to conclude the proof of Proposition 1. Let f: $V \rightarrow W$ be the morphism. There is an open dense U of W such that for all $u \in U$, $f^{-1}(u)$ is equidimensional and $\dim f^{-1}(u) = \dim V\text{-}\dim W$. Let $w \in S = W - U$ be a point such that $\dim f^{-1}(w) > \dim V\text{-}\dim W$. We get a contradiction as follows.

Let C be an irreducible curve in W through w but not contained in S. Then $f^{-1}(C) \not\subset f^{-1}(S)$. Let X be the union of the components of $f^{-1}(C)$ not contained in $f^{-1}(S)$. X is the closure of $f^{-1}(C - S)$ and $\dim X = \dim V\text{-}\dim W + 1$. Now

$$\dim (X - f^{-1}(C - S)) < \dim X \leqq \dim f^{-1}(w)$$

and $f^{-1}(w) \cap f^{-1}(C - S)$ is empty, so not every component of $f^{-1}(w)$ is in X. Let $v \in f^{-1}(w) - X$. Then $V - X$ is open and contains v but $f(V - X)$ can't be open—it contains w but misses a dense of C.

We see that for our initial problem to have any meaning, we must abandon the obvious topological restriction on quotients, realizing also that the quotients of the new type will probably fail to be topological.

Our notations are essentially standard. Ω stands for a universal domain. $\mathscr{O}_{X,Y}$ is the local ring of a subvariety X of the variety Y. Tr. deg. A/Ω denotes the transcendence degree of A over Ω.

1. An existence theorem. We can restate the local condition LC for quotients as follows: We are considering the set-wise quotient map $p\colon V \to W$ for the ER R on V. Let Ω be a universal domain and let $g \in \Omega(V)$ be an R-invariant function, so that within its domain, g is constant on orbits. Let $p(v) = w$ for $v \in V$.

From LC we know

> LC_1: given an R-invariant $g \in \Omega(V)$ there exists $f \in \Omega(W)$ such that $f \circ p = g$.

Furthermore, LC then implies

> LC_2: for all $f \in \Omega(W)$, $f \circ p \in \mathscr{O}_{v,V}$ if and only if $f \in \mathscr{O}_{w,W}$.

Now, conversely, LC_1 and LC_2 yield LC. In fact it is well-known that $g \in \Omega(V)$ is constant (or generically constant) on the fibers of $p\colon V \to W$ if and only if g is a purely inseparable element in the induced field extension $\Omega(V)/\Omega(W)$. Thus a set-wise quotient satisfying LC_2 is a quotient if and only if $\Omega(V)/\Omega(W)$ is a field extension with no pure inseparability. Since our first result is unrelated to any field-theoretic problem, we define a *quotient within inseparability, QWI*, to be a set-wise quotient satisfying LC_2.

THEOREM 1. *Let V be an irreducible variety, let W be a nonsingular variety, and let $p\colon V \to W$ be a surjective morphism. Let $E = \{w \in W\colon \dim p^{-1}(w) > \dim V - \dim W\}$. If codimension $p^{-1}(E) > 1$, then p is a QWI for the induced ER, i.e. the one whose orbits are the fibers of p.*

Proof. We must show LC_2. Let $f \in \Omega(W)$ be a function for which $f \circ p \in \mathscr{O}_{v,V}$. Replacing V by an open subvariety, if necessary, we may express f as g/h in lowest terms over the UFD $\mathscr{O}_{p(v),W}$. If

$f \notin \mathcal{O}_{p(v),W}$, then f has a pole at $p(v)$, so $h(p(v)) = 0$. From $f \circ p \in \mathcal{O}_{v,V}$, we see $g(p(v)) = 0$ also. Denote by $\mathscr{Z}(\cdot)$ the zeros of $(\cdot)$. Then $\mathscr{Z}(h \circ p) \subset \mathscr{Z}(g \circ p)$, so that in a suitable neighborhood of $p(v)$ we have $p(\mathscr{Z}(h \circ p)) \subset \mathscr{Z}(h)$. Now $\mathscr{Z}(g) \cap \mathscr{Z}(h)$ has codimension 2 near $p(v)$ by the relative primeness of g and h, whence $p^{-1}(E) \supset \mathscr{Z}(h \circ p)$. This contradicts codimension $p^{-1}(E) > 1$.

COROLLARY. *If p as above is birational, then either p is an isomorphism or codimension $p^{-1}(E) = 1$.*

Proof. Birational quotients are isomorphisms.

NOTE: The corollary is essentially *ZMT* for nonsingular varieties, and I have found precisely the same proof of it in this case in Mumford ([3], pp. 415–416).

The theorem also shows that with obvious meanings the product of two quotients is the quotient of a product *ER* when the quotients are of the type indicated in the theorem. I don't know if this standard categorical property is true in all non-equidimensional cases.

EXAMPLE. Using Theorem 1 we can easily construct non-equidimensional quotients.

(a) A general technique is to use subvarieties of $\Omega^n \times \Omega^m$ that project onto one factor. For example, let $V = \langle x_1x_2 + x_1^2x_3 + x_1^3x_4 = x_2 \rangle$ in $\Omega \times \Omega^3$, with $p = pr_2 \colon V \to \Omega^3$. Then $E = \{0, 0, 0\}$, and codimension $p^{-1}(E) = 2$. Thus p is a *QWI* and, by the separability of p, even a quotient. By the next theorem this is the lowest dimensional example for a normal variety of a non-equidimensional quotient—$\dim V = \dim p(V) = 3$.

Disturbingly enough, though this quotient apparently has the quotient topology (though of course the map is not open), there are many open denses of V for which the restricted quotient fails to have the quotient topology! For example, let $V' = V - \{1, 0, 0, 0)\}$. Then the line $\langle x_3 = x_4 = 0 \rangle$ in Ω^3 has an inverse image in V' consisting of two disjoint $R_{V'}$-saturated closeds, $X_1 = \langle x_1 = 1, x_3 = x_4 = 0 \rangle$ and $X_2 = \langle x_2 = x_3 = x_4 = 0 \rangle$. Now $p(V' - X_1)$ is not open, but $V' - X_1$ is open saturated.

(b) An example with messier fibers, but for which V and W are nonsingular, is the following: $p \colon \Omega^4 \to \Omega^3 \colon (x, y, z, w) \to (xy - zw, y^2 - zw, x^2 - yz)$. p is surjective, with $E = \{(0, 0, 0)\}$ and $p^{-1}(E) =$ cone of the twisted cubic $= \{(u^2t, ut^2, t^3, u^3) \mid u, t \in \Omega\}$.

(c) Examples similar to (a) but without the condition on codimension $p^{-1}(E)$ do not seem to yield quotients, even when the fibers over E are not hypersurfaces. Let $V = \langle x_1x_3 + x_1^2x_4 = x_2x_3 \rangle$ in $\Omega \times \Omega^3$

with $p = pr_2\colon V \to \Omega^3$. Then $E = \langle x_3 = x_4 = 0\rangle$ and $p^{-1}(E)$ is a hypersurface, while the exceptional fibers all have codim 2. Here p is not a quotient: for example, $x_3/x_4 = x_1^2/x_2 - x_1$ is not defined at (0, 0, 0), but composing with p gives a function defined along $p^{-1}((0, 0, 0)) = \{(x, 0, 0, 0) \mid x \in \Omega\}$.

Since $pr_2\colon \Omega \times \Omega^3 \to \Omega^3$ is a quotient, another consequence of the example is that restricting quotients to arbitrary subvarieties needn't even yield QWI's. (Of course, there are simpler examples of this phenomenon involving loss of normality. Consider $pr_2\colon \Omega \times \Omega^2 \to \Omega^2$ restricted to $\{(x, x^2, x^3) \mid x \in \Omega\}$.)

2. A non-existence theorem. The following partial converse to Theorem 1 limits unknown cases to ones like that of Example (c) above.

THEOREM 2. *If $p\colon V \to W$ is a quotient of irreducible varieties, then no fiber of excessive dimension contains a normal hypersurface.*

Proof. Identify $\Omega(W)$ with a subfield of $\Omega(V)$ via cohom p. Now if F is a fiber of p, the elements of $\Omega(W)$ are either constant on F or nowhere defined on F. Call any subfield K of $\Omega(V)$ with this property for a subvariety F of V a *level field* for F.

Exceptional orbits can exist only if tr. deg. $\Omega(W)/\Omega > 1$. Note also that we can delete all but one component of any orbit without destroying the quotient property. Our theorem can thus be restated in slightly stronger form:

THEOREM 3. *If H is a normal irreducible hypersurface and if K is a level field for H, then* tr. deg. $K/\Omega \leqq 1$.

By example (a), Theorem 3 is false for codimension $H > 1$. We proceed now in two steps to prove Theorem 3 and hence Theorem 2.

Step 1. Here we do not use the level field concept. Let f be any minimal prime in the discrete valuation ring $\mathcal{O}_H$. Let L denote the residue field $\mathcal{O}_H/f\mathcal{O}_H$, and let $\pi\colon \mathcal{O}_H \to L$ be the residue map. Although we do not introduce the technique of completion for the following lemmas, one can see that from one point of view we are considering the first nonzero coefficients in the f-power series expansions of various functions.

LEMMA 1. *Let $x_1, \cdots, x_n$* in *$\mathcal{O}_H$ form a transcendence base for $\Omega(V)/\Omega$. Then the image $\pi(R)$ in L of $R = \mathcal{O}_H \cap \Omega[x_1, \cdots, x_n, f^{-1}]$ contains a transcendence base of L/Ω.*

Proof of Lemma 1. Suppose the contrary and let $c \in L$ be algebraically independent of $\pi(R)$. Let $\pi(y) = c$ for $y \in \mathcal{O}_H$. There is an equation

$$(*) \qquad y^m + P_{m-1}(x)y^{m-1} + \cdots + P_0(x) = 0$$

of algebraic dependence of y on $\Omega(x)$. We want to have the coefficients of (*) in R and to guarantee this we adjust y.

First we need an element $u \neq 0$ in $R \cap f\mathcal{O}_H$. There exists a nonzero polynomial Q such that $Q(\pi(x_1), \cdots, \pi(x_n)) = 0$ since the transcendence degree of L/Ω is less than that of $\Omega(V)/\Omega$. Now $Q(x_1, \cdots, x_n) \neq 0$ by algebraic independence of the x_i, so this is our u.

For sufficiently large N, multiplying (*) through by u^{mN} gives us an equation of integral dependence of $u^N y$ on R. Thus $u^N y \in \mathcal{O}_H$. For some M, $f^{-M}u^N y$ is in $\mathcal{O}_H - f\mathcal{O}_H$. Moreover its image in L is still algebraically independent of $\pi(R)$. Thus replacing y by $u^N y$, we may take (*) to be an equation of integral dependence of $y \in f^M\mathcal{O}_H$ on R such that $\pi(f^{-M}y) = c$. Let n_r be the order of $P_r(x)$ in the valuation ring $\mathcal{O}_H$. Then $rM + n_r$ is the order of $y^r P_r(x)$. Let p be the minimal order among the $rM + n_r$. Dividing out f^p from (*) and mapping down to L, we get 0 on the right and a sum of nonzero terms

$$\sum_{\text{various } r} c^r \pi(f^{-n_r} P_r(x))$$

on the left. The algebraic independence of c over the set of coefficients gives us a contradiction.

LEMMA 2. *For any elements* $x_1, \cdots, x_r \in \mathcal{O}_H$,

$$\text{tr. deg. } \pi(\mathcal{O}_H \cap \Omega[x_1, \cdots, x_r, f^{-1}])/\Omega \leqq r\,.$$

Proof of Lemma 2. We may assume tr. deg. $\Omega[x_1, \cdots, x_r, f^{-1}]/\Omega = r + 1$. Let $P_1, \cdots, P_s$ be polynomials in the x_i and f^{-1} with algebraically independent images under π. For simplicity, we can take the P_i to be monomials in f^{-1}. Since the case of no x_i's is immediate we also assume inductively that at most $r - 1$ P_i's do not involve x_r and that the π-images of these form a transcendence base for $\pi(\mathcal{O}_H \cap \Omega[x_1, \cdots, x_{r-1}, f^{-1}])$. If $s \geqq r$, then we have $P_r = (f^{-ab}x_r^b)(f^{-c}M)$ for some polynomial M in $x_1, \cdots, x_{r-1}$ and with neither factor in parentheses divisible by f. We conclude that $\pi(f^{-ab}x_r^b)$ is algebraically independent of $\{\pi(P_1), \cdots, \pi(P_{r-1})\}$ and algebraically dependent on $\{\pi(P_1), \cdots, \pi(P_r)\}$; i.e. we can exchange $f^{-ab}x_r^b$ for P_r. Since no two polynomials of the form $f^{-ab}x_r^b$ are algebraically independent, we obtain $s \leqq r$.

NOTE. A less elementary, but quicker, proof of Lemma 2 results from considering the embedding of $\mathcal{O}_H/f\mathcal{O}_H$ in the completion of $\mathcal{O}_H$

with respect to $f\mathcal{O}_H$. In that case it is easy to check that f is transcendental over the embedded image of $\pi(\mathcal{O}_H \cap \Omega[x_1, \cdots, x_r, f^{-1}])$.

Step 2. We now introduce the level field property of K into the situation. We have by definition $\pi(K \cap \mathcal{O}_H) = \Omega$. We use this fact and some easy manipulations to contradict the lemmas of Step 1 if tr. deg. $K/\Omega \geqq 2$.

We may always choose a transcendence base for K from $f\mathcal{O}_H$ by subtracting suitable constants if necessary. Let $g \in K \cap f\mathcal{O}_H$ have minimal order r, so that $g = f^r u$ with u a unit. Let $h \in K \cap f\mathcal{O}_H$ have order m. By minimality of r, $m = qr$ for an integer q, and hg^{-q} takes a constant value, say $c_0 \neq 0$, on H. Thus $hg^{-q} - c_0 \in K \cap f^r\mathcal{O}_H$, whence $h = c_0 g^q + h_1$, $h_1 \in K \cap f^{qr}\mathcal{O}_H$. Continuing by induction, h has a finite expansion

$$(*) \qquad h = \sum c_i g^{qi} + h_s,\ h_s \in K \cap f^{qrs}\mathcal{O}_H .$$

Suppose now that for $\alpha \in \Omega[g, h, x_3, \cdots, x_n, f^{-1}]$, $\pi(\alpha) \notin \Omega$. Let s be such that $qrs >$ highest power of f^{-1} appearing in α. Then making the substitution (*) for h in α, we see that $\pi(\alpha) \in \pi(\Omega[g, x_3, \cdots, x_n, f^{-1}])$. Thus (using Lemma 2 also)

$$\text{tr. deg. } \pi(\Omega[g, h, x, \cdots, x_n, f^{-1}]) \leqq n - 1 .$$

Since we may take $x_3, \cdots, x_n \in \mathcal{O}_H$ as independent transcendentals independent of $\Omega(g, h)$, we conclude from Lemma 1 that h is algebraic over $\Omega(g)$.

An immediate corollary of the above reasoning is the

COROLLARY. *With notation as in Lemma* 1, *there is at most one* x_i *such that*

$$\text{tr. deg. } \pi(\mathcal{O}_H \cap \Omega[x_i, f^{-1}])/\Omega = 0 .$$

In terms of power series in a minimal prime, this implies that any function in $\mathcal{O}_H$ *whose f-power series has constant coefficients is algebraic over* $\Omega(f)$.

We remark that in general for subvarieties of higher codimension one may write down level fields of excessive transcendence degree without reference to quotient maps. For example, consider $\Omega(x, y, xu + yv)$ as a level field for the plane $x = y = 0$ in 4-space.

We also note that it is not clear what can happen if the union of excessive orbits is a normal hypersurface, a situation intermediate between Theorems 1 and 2. A much more delicate problem is here not considered at all, namely to give a decent local condition on an excessive orbit which would guarantee existence of a quotient nearby.

3. A counterexample. We give an example of a level field of transcendence degree 2 for a singular curve Z on a surface V in characteristic 0. This counterexample to a generalization of Theorem 3 suggests that if Theorem 2 is true without normality hypotheses, a more geometric approach will be needed to prove it.

THEOREM 4. *Let V be the product of the line Ω with the curve with a cusp $\{(x^2, x^3)\} \subset \Omega^2$, so that V embeds as $\{(x^2, x^3, y)\} \subset \Omega^3$. Let Z be the singular line $\{(0, 0, y)\}$. Then the field $K = \Omega(x^2(x+y), x^2y^2)$ is a level field for Z in characteristic* 0.

Proof. Set $u = x^2(x+y)$, $v = x^2y^2$. One checks that the map $(x^2, x^3, y) \mapsto (u, v)$ is a surjection from V onto Ω^2 whose only excessive fiber is Z. (The map is not quite a quotient, however, since for example $v/(v+u^2)$ is defined along Z but not at its image (0, 0).) Thus relatively prime polynomials f and g in u, v, when viewed as polynomials in x, y, can have only powers of x as a common factor.

Let $f(u, v) = H_1(u, v) + F$, $g(u, v) = H_2(u, v) + G$, where H_1, H_2 are homogeneous in u, v, and F, G consist of higher powers. By the above remark, a typical element f/g of $\Omega(u, v)$ is in lowest terms when viewed as a quotient of polynomials in x, y unless deg. $H_1 > 0$, deg. $H_2 > 0$. Cancelling x^{2d} where $d = \min(\deg. H_1, \deg. H_2)$ to yield a rational function in x, y in lowest terms, we see that f/g has a pole along Y unless deg. $H_1 \geqq$ deg. H_2 and has a zero along Z if deg. $H_1 >$ deg. H_2. Let deg. $H_1 =$ deg. $H_2 = d$ and set $H_1 = \sum a_i u^i v^{d-i}$, $H_2 = \sum b_i u^i v^{d-i}$. We have $f/g = (Y_1 + A + R)/(Y_2 + B + S)$ where

$$Y_1 = \sum a_i y^{2d-i},\ Y_2 = \sum b_i y^{2d-i}\ ,$$

$$A = x \sum i a_i y^{2d-i-1} = x\left(2dy^{-1}Y_1 - \frac{d}{dy}(Y_1)\right),$$

$$B = x \sum i b_i y^{2d-i-1} = x\left(2dy^{-1}Y_2 - \frac{d}{dy}(Y_2)\right),$$

and both R and S are divisible by x^2. Multiplying top and bottom by $B - (Y_2 + S)$, we see that f/g is defined on Z if and only if

$$\begin{aligned}
&(Y_1 + A + R)(B - Y_2 - S) \in \Omega[x^2, x^3, y] \\
&\Leftrightarrow Y_1 B = Y_2 A \\
&\Leftrightarrow Y_2 \frac{d}{dy}(Y_1) = Y_1 \frac{d}{dy}(Y_2) \\
&\Leftrightarrow Y_1 = cY_2,\ c \in \Omega\ (\text{char. } \Omega = 0)\ .
\end{aligned}$$

But then $(f/g)|_Z = c$, a constant.

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