PACIFIC JOURNAL OF MATHEMATICS
Vol. 41, No. 2, 1972

# IRREDUCIBLE CHARACTERS AND SOLVABILITY OF FINITE GROUPS

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**The relationship between the degree of an irreducible character $\zeta$ on a finite group $G$ induced from a nilpotent normal subgroup and the structure of the group $G$ are studied when the degree of $\zeta$ is large. In particular if the square of the degree of $\zeta$ is the index of the center of $G$ in $G$ then $G$ is solvable.**

Let $\zeta$ be an irreducible (complex) character on the finite group $G$. What conditions on $\zeta$ insure that $G$ is solvable? Of course, if $\zeta$ is a faithful linear character then $G$ is cyclic. We are interested in the other extreme when the degree of $\zeta$ is large, in part because of the relationship to the theory of projective representations and the Schur multiplier. Let $H$ be a nilpotent normal subgroup of $G$, assume $\zeta = \phi^G$ for some character $\phi$ on $H$, and assume for each Sylow $p$-subgroup $S$ of $G$ that $\zeta|_S = m\lambda$ for some irreducible character $\lambda$ on $S$ where $(m, p) = 1$, then $G$ is solvable. If $Z$ is the center of $G$ the last condition always holds if the degree of $\zeta$ is $[G: Z]^{1/2}$, that is, if $G$ is a "group of central type" [2]. It is easy to see that no irreducible character on $G$ can have degree larger than $[G: Z]^{1/2}$. Another upper bound for the degree of an irreducible character on $G$ is $d[G: H]$ where $d = \max\{\text{degree } \rho \mid \rho \text{ is an irreducible character on } H\}$ ([3] 17.9 p. 570). If $[G, G]$ is the commutator subgroup of $G$ and $Z \cap [G, G]$ contains an element of order $d[G: H]$ then there is an irreducible character $\zeta$ of degree $d[G: H]$ on $G$. Moreover, $\zeta = \phi^G$ for some character $\phi$ on $H$, and for each Sylow $p$-subgroup $S$ of $G$, $\zeta|_S = \sum_{j=1}^{n} \lambda_j$ where the $\lambda_j$ are irreducible characters on $S$ with $\lambda_j(1)$ equal to the $p$-part of $\zeta(1)$ $(j = 1, \cdots, n)$. If $n = 1$ for each prime $p$ dividing $[G: 1]$ then $G$ is solvable. An example showing the necessity of the hypothesis on $n$ is given. The conditions on the character $\zeta$ with respect to the Sylow subgroups $S$ of $G$ restrict the action of $G$ on $S$. To illustrate this we show $G$ is nilpotent if and only if for every Sylow subgroup $S$ of $G$ and every irreducible character $\chi$ on $G$, $\chi|_S = m\lambda$ for some irreducible character $\lambda$ on $S$.

In what follows all groups are finite and all characters and representations are taken over the complex numbers. If $n$ is an integer and $p$ is a prime integer we let $n_p$ denote the largest factor of $n$ which is a power of the prime $p$. Our standard reference is [3] and all unexplained terminology and notation coincides with [3].

THEOREM 1. *Let $\zeta$ be an irreducible character on the group $G$ and let $H$ be a nilpotent normal subgroup of $G$. Assume*

1. *$\zeta = \phi^G$ for some character $\phi$ on $H$.*
2. *For each Sylow $p$-subgroup $S$ of $G$, $\zeta|_S = m\lambda$ for some irreducible character $\lambda$ on $S$ where $(p, m) = 1$. Then $G$ is solvable.*

*Proof.* A theorem of P. Hall ([3] 1.10 p. 662) asserts that a group is solvable if every Sylow subgroup has a complement, this theorem will be applied to $G/H$. Let $p$ be a prime dividing $[G: 1]$, let $P$ be the Sylow $p$-subgroup of $H$ and $S$ a Sylow $p$-subgroup of $G$. Since $P$ is a characteristic subgroup of $H$, $P$ is a normal subgroup of $G$ and $P \subseteqq S$. By Clifford's Theorem ([3] 17.3 p. 565)

$$\zeta|_P = e(\rho_1 + \cdots + \rho_n)$$

where the $\rho_i$ are inequivalent irreducible characters on $P$ conjugate in $G$. We determine the number $n$. By hypothesis 2, $\zeta|_P = m\lambda|_P$ so $\lambda|_P = e/m(\rho_1 + \cdots + \rho_n)$ and the $\rho_i$ are conjugate in $S$ by Clifford's Theorem. Now $(\phi, \zeta|_H) \geqq 1$ so by relabeling we can say $(\rho_1, \phi|_P) \geqq 1$. We claim $\rho_1^S = \lambda$ so $e/m = 1$ and $n = [S: P]$. To verify the claim hypothesis 2 says the $p$-part of $\zeta(1)$ is $\lambda(1)$. Also, $\phi|_P = q\rho_1$ (since $H$ is nilpotent) where $\rho_1(1)$ is a power of $p$ so $\rho_1^S(1)$ divides the $p$-part of $\phi^G(1) = \zeta(1)$. Since $\lambda$ is contained in $\rho_1^S$ this implies $\lambda = \rho_1^S$ verifying the claim.

Now $G$ acts on $\rho_1 \cdots \rho_n$ by conjugation and the inertia group $H^*$ of the action of $G$ on $\rho_1$ has index $n = \lambda(1)/\rho_1(1) = [S: P]$. Also $H^*$ contains $H$ since $H$ is nilpotent so $H^*/H$ is a $p$-complement in $G/H$. The Theorem of P. Hall completes the proof.

We next give a sufficient condition that $\zeta$ satisfy condition 2 of Theorem 1. (See [2] Theorem 2).

THEOREM 2. *Let $\zeta$ be an irreducible character on $G$ and let $Z$ be the center of $G$. If $\zeta(1)^2 = [G: Z]$ then for each Sylow $p$-subgroup $S$ of $G$, $\zeta|_S = m\lambda$ for some irreducible character $\lambda$ on $S$ and $(p, m) = 1$.*

*Proof.* By Schur's lemma $\zeta|_Z = \zeta(1)\psi$ where $\psi$ is a linear character on $Z$. Then by reciprocity $(\zeta, \psi^G) = (\zeta|_Z, \psi) = \zeta(1)$ so by counting degrees, $\zeta(1)\zeta = \psi^G$. Let $S$ be a Sylow $p$-subgroup of $G$ and let $R$ be the subgroup of $G$ generated by $Z$ and $S$. Let $\lambda$ be an irreducible character of $R$ contained in $\psi^R$. By Schur's lemma $\lambda|_S$ remains irreducible because the elements of $Z$ are represented by Sclars. Since $\lambda$ is contained in $\psi^R$, $\lambda^G = m\zeta$ for some integer $m$. By counting degrees

$$m = [G: R]\lambda(1)/\zeta(1) .$$

Since $\lambda$ is irreducible on $S$, $\lambda(1) = p^a$ for some $a$, $[G: R]$ is prime to $p$ since $R$ contains $S$. The $p$-part of $\zeta(1)^2$ is $[S: S \cap Z]$. Thus $\lambda(1)^2 = [S: S \cap Z]$ and $(\zeta, \lambda^G) = (\zeta|_R, \lambda) = (\zeta|_S, \lambda|_S) = [G: Z]/\lambda(1)^2$. Thus $\zeta|_S = m\lambda$ where $m$ is the largest divisor of $\zeta(1)$ prime to $p$. We combine the first two results to obtain.

COROLLARY 1. *Let $\zeta$ be an irreducible character on the group $G$, and let $H$ be a nilpotent normal subgroup of $G$. Assume $\zeta = \phi^G$ for some character $\phi$ on $H$ and $\zeta(1)^2 = [G: Z]$ where $Z$ is the center of $G$. Then $G$ is solvable.*

The principal theorem of [1] is now an easy consequence of Corollary 1.

COROLLARY 2. *Let $\zeta$ be an irreducible character on the finite group $G$, and let $A$ be an abelian normal subgroup of $G$. If $\zeta(1)^2 = [G: A]^2 = [G. Z]$ where $Z$ is the center of $G$ then $G$ is solvable.*

*Proof.* Let $\phi$ be a linear constitutent of $\zeta|_A$. Then by reciprocity, $\zeta$ is a constitutent of $\phi^G$. But $\zeta(1) = \phi^G(1) = [G: A]$ so $\phi^G = \zeta$. By Corollary 1, $G$ is solvable.

We now verify some of the hypothesis of Theorem 1 in another situation. We begin by summarizing basic results relating ordinary representations, projective representions, and the Schur Multiplier. Our nontrivial assertions are the contents of 23.3, p. 629 of [3]. Let $G$ be a finite group with center $Z$, assume $n$ is the exponent of $[G, G] \cap Z$ and let $\bar{G} = G/Z$. Write

$$G = \bigcup_{g \in \bar{G}} ZR(g)$$

where $R(g)$ is an element in $G$ corresponding to $g$. Then $R(g_1)R(g_2) = A(g_1, g_2)R(g_1g_2)$ where $A(g_1, g_2) \in Z$. Let $a \in [G, G] \cap Z$ order $n$ and let $\theta$ be a linear character on $Z$ which is faithful on the cyclic group generated by $a$. Define a 2-cycle $\alpha$ on $\bar{G}$ by

$$\alpha(g_1, g_2) = \theta(A(g_1, g_2)) .$$

Let $K^*$ be the multiplicative group of the complex numbers. The element $\alpha$ represents in the Schur multipler $H^2(\bar{G}, K^*)$ has order $n$.

Form the projective group algebra $K\bar{G}_\alpha$ and let $M$ be a left $K\bar{G}_\alpha$-module. For each $g \in \bar{G}$, left multiplication by $g$ on $M$ induces a $K$-linear transformation $T(g)$ of $M$ and

$$T(g_1)T(g_2) = \alpha(g_1, g_2)T(g_1g_2) .$$

If $x \in G$ then $x = z_1R(g_1)$ where $z_1 \in Z$ and $g_1 \in \bar{G}$. Let left multiplication by $x$ on $M$ be the linear transformation $T^*(x) = \theta(z_1)T(g_1)$. If $y = z_2R(g_2) \in G$ *then*

$$xy = z_1z_2A(g_1, g_2)R(g_1g_2)$$

and

$$T^*(x)T^*(y) = \theta(z_1)T(g_1)\theta(z_2)T(g_2) = \theta(z_1z_2)\theta(A(g_1, g_2))T(g_1g_2) = T^*(xy) .$$

Thus $M$ can be viewed as a $KG$-module. Notice that $M$ is irreducible over $KG$ if and only if $M$ is irreducible over $K\bar{G}_\alpha$. Also, note that $T^*|_Z = T^*(1)$. This process can be reversed when $M$ is a $KG$-module giving the $G$ representation $T^*$ if $T^*|_Z = T^*(1)\theta$ for the given linear character $\theta$ on $Z$. Define a linear character $\psi$ on $G$ by the equation $\psi(x) = \det(T^*(x))$. Since $a \in [G, G]$, $\psi(a) = 1$. But $\psi(a) = \theta(a)^m$ where $m = T^*(1)$ so $n$ divides $T^*(1)$.

Let $S$ be a Sylow $p$-subgroup of $G$ and $\bar{S}$ the natural image of $S$ in $\bar{G}$. The element the restriction of $\alpha$ to $\bar{S}$ represents in $H^2(\bar{S}, K^*)$ is realized by the equation $\alpha(y_1, y_2) = \theta(A(y_1, y_2))$ in the group $SZ$. By ([3] 16.21, p. 118) $\alpha$ represents an element whose order is $n_p$ in $H^2(\bar{S}, K^*)$. In the correspondence of ([3] 23.3, p. 629) this implies $\theta$ is faithful on a cyclic group of order $n_p$ in $[S, S] \cap Z$. Form the projective group algebra $K\bar{S}_\alpha$. Now $M$ can be viewed as a $K\bar{S}_\alpha$-module, let $M = M_1 \oplus \cdots \oplus M_k$ where the $M_i$ are irreducible $K\bar{S}_\alpha$ modules. As above, each $M_i$ affords an ordinary representation $T_i^*$ on $SZ$ which is irreducible. The restriction of $T_i^*$ to $S$ is also irreducible since each $T_i^*$ restricted to $Z$ is $T_i^*(1)\theta$. Also, $\theta$ is faithful on a cyclic group of order $n_p$ in $[S, S] \cap Z$ so arguing as before $n_p$ must divide the degree of $T_i^*$.

LEMMA 1. *Let $G$ be a finite group with center $Z$, let $a \in [G, G] \cap Z$ of order $n$, and let $\theta$ be a linear character on $Z$ faithful on the cyclic group generated by $a$. Then*

(1) *$\theta^G = \sum_{i=1}^{s} \zeta_i(1)\zeta_i$ where $n \mid \zeta_i(1)$ and the $\zeta_i$ are inequivalent irreducible characters of $G$.*

(2) *If $\zeta$ is an irreducible character on $G$ with $(\theta^G, \zeta) \geqq 1$ and $S$ is a Sylow $p$-subgroup of $G$ then $\zeta|_S = \sum_{j=1}^{l} b_j\lambda_j$ where $n_p \mid \lambda_j(1)$, the $\lambda_j$ are inequivalent irreducible characters on $S$, and the $b_j$ are positive integers.*

*Proof.* Let $\zeta$ be an irreducible character on $G$. By Schur's lemma $\zeta|_Z = \zeta(1)\psi$ for a linear character $\psi$ on $Z$. Now $(\zeta, \psi^G) = (\zeta|_Z, \psi) = \zeta(1)$. This shows $\theta^G = \sum_{i=1}^{s} \zeta_i(1)\zeta_i$ where the $\zeta_i$ are inequivalent irreducible characters of $G$. If $T_i$ is the representation affording $\zeta_i$ then $\det T_i$ is a linear character on $G$. Since $a \in [G, G]$, $1 =$

$\det(T_i(a)) = \det[\theta(a)T_i(1)] = \theta(a)^{\zeta_i(1)}$. Therefore $n \mid \zeta_i(1)$.

To prove (2) we need the analysis which preceded the lemma. Let $T^*$ be the ordinary representation on $G$ which affords $\zeta$ and $T$ the corresponding projective representation on $\bar{G}$. In this situaion we showed $T^*|_S = T_1^* + \cdots + T_k^*$ where the $T_i^*$ are irreducible and their degree are divisible by $n_p$. Let $\lambda_1, \cdots, \lambda_l$ be a full set of inequivalent characters afforded by the $T_1^*, \cdots, T_k^*$. Then $\zeta|_S = \sum_{i=1}^{l} b_j\lambda_j$ where $b_j$ is the multiplicity of $\lambda_j$ in $\zeta|_S$ and $\lambda_j(1)$ is the degree of some $T_i^*$ and so is divisible by $n_p$. We can now prove

THEOREM 3. *Let $G$ be group with center $Z$. Let $H$ be a normal nilpotent subgroup of $G$ and let $d = \max\{\rho(1) \mid \rho$ is an irreducible character of $H\}$. If $[G, G] \cap Z$ contains an element of order $d[G: H]$ then there is an irreducible character $\zeta$ on $G$ so that $\zeta = \phi^G$ for some character $\phi$ on $H$, and for each Sylow $p$-subgroup $S$ of $G$, $\zeta|_S = \sum_{i=1}^{n} b_i\lambda_i$ where $\lambda_i(1) = \zeta(1)_p$. If $n=1$ for each $p$ then $G$ is solvable.*

*Proof.* Let $n = d[G: H]$ and let $a \in [G, G] \cap Z$ of order $n$. Let $\theta$ be a linear character on $Z$ which is faithful on the cyclic group generated by $a$. By the first part of LEMMA 1

$$\theta^G = \sum_{i=1}^{s} \zeta_i(1)\zeta_i$$

where $n \mid \zeta_i(1)$ and the $\zeta_i$ are inequivalent irreducible characters of $G$. We will show each of the $\zeta_i$ satisfy the conclusion of the Theorem. By 17.9 p. 570 of [3], $n$ is the largest possible degree of an irreducible character on $G$ so $n = \zeta_i(1) (i = 1, \cdots, s)$ and $H$ is a maximal nilpotent normal subgroup of $G$ so $Z \subseteqq H$. Now $\theta^G(1) = [G: Z]$ so $[G: Z] = sn^2$ where $s$ is the number of inequivalent $\zeta_i$ in $\theta^G$. By Clifford's Theorem (17.3 p. 565, [3])

$$\zeta_i|_H = e(\phi_1^i + \cdots + \phi_m^i)$$

where the $\phi_i^i (j = 1, \cdots, m)$ are inequivalent irreducible characters on $H$ conjugate in $G$. Now $\zeta_i$ is a constitutent of $(\phi_j^i)^G$ and $(\phi_j^i)^G(1) \leqq d[G: H] = \zeta_i(1)$ so for each $j$, $\phi_j^i(1) = d$ and $(\phi_j^i)^G = \zeta_i$. This verifies the first conclusion of Theorem 3 for each $i (i = 1, 2, \cdots, s)$.

Let $S$ be a Sylow $p$-subgroup of $G$. By the second part of LEMMA 1,

$$\zeta_i|_S = \sum_{j=1}^{l} b_j\lambda_j^i$$

where the $\lambda_j^i$ are inequivalent irreducible characters on $S$ and $n_p$ divides $\lambda_j^i(1)$. Since $H$ is nilpotent, $d_p = \max\{\gamma(1) \mid \gamma$ is an irreducible character on $P\}$. If $\lambda$ is an irreducible constitutent of $\lambda_j^i|_P$ then

$\gamma^S(1) \leqq \lambda_j^i(1)$ so $\gamma^S = \lambda_j^i$ and $\lambda_j^i(1) = d_p[S: P] = n_p$. This verifies the second conclusion of Theorem 3. If $n = 1$ for each $p$ then $\zeta|_S = b_1\lambda_1$ and $\zeta(1) = b_1\lambda_1(1)$. But $\zeta(1)_p = \lambda_1(1)_p$ so $(p, b_1) = 1$ and by Theorem 1, $G$ is solvable. This completes the proof.

For an example to show the necessity of Condition 2 in Theorem 1 let $H$ be any group of order $n$ and $J_n(H)$ the group algebra of $H$ over the ring $J_n$ of integers modulo $n$. Let $A = J_n(H)$ viewed as an additive group and let $H$ act as a group of automorphisms of $A$ by

$$h(ax) = ahx \text{ (regular representation) } x, h \in H, a \in J_n .$$

Let $G$ be the semi-direct product of $A$ by $H$ with respect to this action. Let $\phi$ be the linear character defined on $A$ by $\phi(\sum_{h \in H} a_h h) = \xi^a$ where $\xi$ is a primitive $n^{\text{th}}$ root of 1 and $a$ is an integer representing the coefficient in $J_n$ of the identity $e$ of $H$. One checks that $[G, A] \cap Z = Z$ where $Z$, the center of $G$, is $\{\sum a_h h \mid a_h = a_k \text{ all } h, k \in H\}$ and has exponent $n$. Also $\phi$ is distinct from all its conjugates so $\phi^G = \zeta$ is irreducible. Yet $G$ need not be solvable. The problem is that the restriction of $\zeta$ to a Sylow subgroup does not behave properly. For example, if $H = A_5$ (the simple group of order 60), and $S$ is the Sylow 5-subgroup of $G$ then $\zeta|_S = \sum_{i=1}^{12} \lambda_i$ where the $\lambda_i$ are 12 distinct irreducible characters on $S$ of degree 5.

If $G$ is a finite group with center $Z$ and $\zeta$ is a faithful irreducible character on $G$ with $\zeta|_S = m\lambda$ for some Sylow subgroup $S$ and irreducible character $\lambda$ on $S$ then the center of $S$ is $Z \cap S$. The proof of this observation also proves

THEOREM 4. *The group $G$ is nilpotent if and only if for each irreducible character $\zeta$ on $G$ and each Sylow subgroup $S$ of $G$, $\zeta|_S = m\lambda$ for some irreducible character $\lambda$ on $S$.*

*Proof.* Assume $G$ is nilpotent, let $\zeta$ be an irreducible character on $G$ and $S$ a Sylow subgroup. Then $S$ is normal in $G$ so by Clifford's Theorem

$$\zeta|_S = e(\phi_1 + \cdots + \phi_m)$$

with the $\phi_i$ distinct conjugate irreducible characters on $S$. If $g \in G$ then $g = g_1 g_2$ where $g_1$ centralizes $S$ and $g_2 \in S$. Then, $\phi_1^g = \phi_1^{g_1 g_2} = \phi^{g_2} = \phi$. So $m = 1$.

Conversely, let $S$ be a Sylow subgroup of $G$ and let $a$ be an element of the center of $S$. Let $\zeta$ be an irreducible character on $G$, then $\zeta|_S = m\lambda$ where $\lambda$ is an irreducible character on $S$. Let $Z(S)$ be the center of $S$. Then by Schur's lemma, $\lambda|_{Z(S)} = \lambda(1)\theta$ for some linear character on $Z(S)$. Thus $\zeta(a) = \zeta(1)\theta(a)$ so $a$ is an element of the center of $G/\ker \zeta$. Since this is true for all irreducible characters on $G$, a is an element of the center of $G$. If $\langle a \rangle$ is the central subgroup of $G$ generated by $a$ then the irreducible characters of $G/\langle a \rangle$ correspond to the irreducible characters of $G$ with kernel $\langle a \rangle$. Thus $G/\langle a \rangle$ satisfies the same hypothesis $G$ does so by induction $G$ is nilpotent.

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Received March 19, 1971 and in revised form November 13, 1971.

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