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HOMOMORPHISMS OF BANACH ALGEBRAS WITH MINIMAL IDEALS

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Let A be a semi-simple Banach algebra with socle F, and let ν be a homomorphism of A into a Banach algebra. It is shown that if I is a minimal one-sided ideal of A, then the restriction of ν to I is continuous. This is then used to deduce continuity properties of the restriction of ν to F. In particular, if F has a bounded left or right approximate identity, then ν is continuous on F.

In [1] and [2] we deduced continuity properties of $\nu|F$ in case A was a semi-simple annihilator Banach algebra. In this paper we obtain essentially the same results, but without the hypothesis that A be an annihilator algebra.

We first show that the restriction of ν to any minimal one-sided ideal is continuous. The proof is almost purely algebraic. We then show that there exists a constant K such that

$$\|\nu(xy)\| \leqq K\|x\|\|y\|, \quad x \in F, \quad y \in \bar{F}.$$

As a corollary we obtain that $\nu|F$ is continuous if F has a bounded left or right approximate identity.

1. Preliminaries. Throughout this section we assume that A is a complex semi-simple Banach algebra. The *socle*, F, is defined to be the sum of the minimal right ideals. An idempotent e is called *minimal* if eA is a minimal right ideal. We use without reference the basic facts about the socle of a Banach algebra (see e.g. [7, pp. 45–47]).

The following two lemmas, together with the "Main Boundedness Theorem" of Bade and Curtis ([3, Thm. 2.1], [2, Thm. 4.1]) are the basic ingredients in the proofs that follow. The first lemma is due essentially to Barnes.

LEMMA 1.1. *Let $\{x_1, \cdots, x_n\} \subset F$. Then there exist idempotents e and f in F such that $\{x_1, \cdots, x_n\} \subset eAf$ and eAf is finite-dimensional.*

Proof. By hypothesis, there exist minimal right ideals, $I_1, \cdots, I_m$, whose sum contains $\{x_1, \cdots, x_n\}$. By [4, Thm. 2.2], there exists an idempotent $e \in F$ such that $eA = I_1 + \cdots + I_m$. Thus $x_k \in eA$, $1 \leqq k \leqq n$.

Similarly, there exists an idempotent $f \in F$ such that $x_k \in Af$, $1 \leqq k \leqq n$. Hence $x_k = ex_k f \in eAf$, $1 \leqq k \leqq n$.

If u and v are minimal idempotents, and $uAv \neq (0)$, then uAv is one-dimensional [9, Lemma 5.1]. Since e and f are each the sum of minimal idempotents, eAf is finite-dimensional.

LEMMA 1.2. *Let e be a minimal idempotent and suppose that eA is infinite-dimensional. Then there exists a sequence of minimal idempotents $\{g_n\}$ such that $g_n g_m = 0$, $n \neq m$, and $eAg_n \neq (0)$ for all n.*

Proof. Let $g_1 = e$. Assume that $g_1, \cdots, g_n$ have been chosen with the desired properties. Let $f = g_1 + \cdots + g_n$. Then $f = f^2$ and eAf is finite-dimensional. Thus there exists $x \in eA$ such that $x(1-f) \neq 0$. Since eA is a minimal right ideal, there exists $w \in A$ such that $x(1-f)w = e$. Let $g_{n+1} = (1-f)\, wex\, (1-f)$. Then $fg_{n+1} = g_{n+1}f = 0$, so $g_k g_{n+1} = g_{n+1} g_k = 0$, $1 \leqq k \leqq n$. Also

$$\begin{aligned} g_{n+1}^2 &= (1-f)wex(1-f)\,(1-f)wex(1-f) \\ &= (1-f)wex(1-f)wex(1-f) \\ &= (1-f)wex(1-f) \\ &= g_{n+1}\,. \end{aligned}$$

Since e is minimal, g_{n+1} is as well. Since

$$exg_{n+1} = ex(1-f) = x(1-f) \neq 0\,,$$

$eAg_{n+1} \neq (0)$. The conclusion follows by induction.

NOTE. Lemma 1.2 above takes the place of [2, Lemma 2.2] in what follows. Evidently the latter does not hold in this more general situation, since the norm induced in eA as a subset of $(Ae)^*$ (the set of bounded linear functionals on Ae) need not be equivalent to the given norm on eA (see Remark 2.5).

2. The main results. Throught this section we assume that A is a complex semi-simple Banach algebra with socle F and that ν is a homomorphism of A into a Banach algebra. We first show the following.

THEOREM 2.1. *If I is a minimal one-sided ideal, then $\nu \mid I$ is continuous.*

Proof. Suppose that I is a minimal right ideal. Then there exists a minimal idempotent e such that $I = eA$.

Let $J = \{x \in A \mid y \to \nu(xy)$ is continuous on $A\}$. Then one verifies that J is a two-sided ideal in A [8, p. 153], and that an idempotent g is in J if and only if $\nu \mid gA$ is continuous.

We may assume eA is infinite-dimensional, since otherwise the conclusion trivially follows. Choose $\{g_n\}$ as given by Lemma 1.2. If $g_n \notin J$ then there exists $x_n \in g_nA$ such that $\| x_n \| = 1$ and $\| \nu(x_n) \| > n \| g_n \|$. Since $g_n x_n = x_n$ and $g_m x_n = g_m g_n x_n = 0$, $m \neq n$, the Main Boundedness Theorem [2, Thm. 4.1] implies that $g_n \in J$ for some n. Since $eAg_n \neq (0)$ and J is a left ideal, we have that $eA \cap J \neq (0)$. But J is a right ideal and eA is a minimal right ideal. Thus $e \in eA \subset J$, and $\nu \mid eA$ is continuous.

REMARK 2.2. (cf. [3, p. 597]) If $I = eA$ is an infinite-dimensional minimal right ideal, then it is always possible to construct a discontinuous homomorphism ν of eA into a Banach algebra. For let ϕ be a discontinuous linear functional on eA, and define

$$\| x \|_1 = \| x \| + | \phi(x) | , \qquad x \in eA .$$

If $x \in eA$, then $xe = \lambda e$, λ complex. Thus

$$x^n = (xe)^{n-1}x = \lambda^{n-1}x ,$$

so

$$\| x^n \|^{1/n} = | \lambda |^{(n-1)/n} \| x \|^{1/n} .$$

Hence $| \lambda | = \rho(x)$, the spectral radius of x.

If $x, y \in A$, then

$$\begin{aligned} \| xy \|_1 &= \| xy \| + | \phi(xy) | \\ &\leqq \| x \| \| y \| + | \phi(xey) | \\ &\leqq \| x \| \| y \| + \rho(x) | \phi(y) | \\ &\leqq \| x \| (\| y \| + | \phi(y) |) \\ &\leqq \| x \|_1 \| y \|_1 , \end{aligned}$$

so $\| \cdot \|_1$ is a normed algebra norm on eA.

Now let B be the completion of eA in this norm and define $\nu: eA \to B$ by $\nu(x) = x$. Then ν is a discontinuous homomorphism of eA. By the above theorem, ν does not extend to a homomorphism of A.

We now have:

THEOREM 2.3. *Let A be a semi-simple Banach algebra with socle F and let ν be a homomorphism of A into a Banach algebra. Then there exists a constant K such that*

$$\| \nu(xy) \| \leqq K \| x \| \| y \| \qquad x \in F , \quad y \in \bar{F}$$

Proof. Since F is the sum of the minimal right ideals and also the sum of the minimal left ideals, it follows from Theorem 2.1 that, for any $x \in F$, the mappings $y \to \nu(xy)$ and $y \to \nu(yx)$ are continuous on A. Thus it suffices to show that

$$\sup_{x,y \in F} \frac{\| \nu(xy) \|}{\| x \| \| y \|} < \infty .$$

In addition, if $e = e^2 \in F$, then $\nu \mid eA$ and $\nu \mid Ae$ are continuous. With these observations, the proof is virtually the same as that of [2, Thm. 4.5], with [2, Lemma 2.1] replaced by Lemma 1.1.

COROLLARY 2.4. *If F has a bounded left or right approximate identity, then ν is continuous on F.*

Proof. If F has a bounded left or right approximate identity, then of course so does $\bar{F}$. The proof now follows as that of [2, Cor. 4.9], with [1, Cor. 4.9] replaced by Lemma 1.1.

REMARK 2.5. Let X and Y be Banach spaces with $Y \subset X$ and such that the inclusion map i: $Y \to X$ is continuous and $i(Y)^- = X$. Let $B(X, Y)$ denote the bounded operators from X to Y and let $B'(X, Y)$ denote the compact operators from X to Y. Then $B(X, Y)$ is a semi-simple Banach algebra with $B'(X, Y)$ as a closed two-sided ideal. If A is a closed two-sided ideal of $B(X, Y)$ containing $B'(X, Y)$, then A is semi-simple with socle F consisting of those bounded operators from X to Y with finite-dimensional range. Each minimal right ideal of A is linearly homeomorphic to X^* and each minimal left ideal is linearly homeomorphic to Y. Now i^*: $Y^* \to X^*$ is one-to-one and continuous, but not bi-continuous if $Y \neq X$.

If $X = Y$ and satisfies the metric approximation property [5, p. 178], then F has a bounded left approximate identity, so the above corollary applies to A. If in addition X has a continued bisection, then Johnson [6, Thm. 3.5] has shown that any homomorphism of A into a Banach algebra is continuous on $B'(X, X)$ $(= \bar{F})$. He has also shown that any homomorphism of $B(X, X)$ into a Banach algebra is continuous if X has a continued bisection [6, Thm. 3.3].

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