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TENSOR PRODUCTS OF PARTIALLY ORDERED GROUPS

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All groups considered in this paper are abelian. It is concerned for the most part with defining suitable tensor products on categories of partially ordered groups. There is introduced the purely auxiliary notion of a partial vector space for the purpose of leading to a reasonable construction of a "vector lattice cover". The so-called o-tensor product from the category of p.o. groups into the category of lattice-ordered groups (l-groups) yields some surprising and surely disappointing results, such as that the functor $G \otimes_o (.)$ preserves monics if and only if G is trivially ordered. This follows from the fact that if G is trivially ordered then $G \otimes_o H$ is independent of the order on H and in fact l-isomorphic to the free l-group on the ordinary tensor product $G \otimes H$. It should be observed that the latter applies to torsion free groups only.

Section 2 is devoted to a discussion of vector lattice coverings. We give a categorical "construction" of such a cover, and establish the connection with the o-tensor product. More specifically, if the l-group G is free over some p.o. group H then the cover of G is obtained by o-tensoring the reals with H. Once again, the results or rather the lack of them, is surprising and perhaps revealing.

Finally we define the l-tensor product of l-groups, which leads us to the vector lattice cover via an alternate route.

The basic material on partially ordered algebraic structure may be found in [2] or [8]. For a discussion of free l-groups we refer the reader to [11], [12], [1], [6] and [7], listed more or less chronologically. Viswanathan has given a construction of a tensor product of partially ordered modules in [10] which reduces to the o-tensor product when restricted to groups. For the categorical background necessary in this context we suggest [9].

NOTATION. Z, Q and R will denote the totally ordered groups of integers, rationals and reals respectively, with the usual orderings. If $\{G_i \mid i \in I\}$ is an arbitrary family of l-groups, the cardinal sum, denoted by $\boxplus \{G_i \mid i \in I\}$ is the restricted direct sum with coordinatewise ordering. If A and B are subsets of a set X, proper containment of A in B is denoted by $A \subset B$. If $C \subseteqq X$ then $A \backslash C$ is the complement of C in A.

1. *o*-Tensor products of p.o. groups. A *partial (real) vector space* is a group G with an operation in G defined on a subset of $R \times G$ such that

(i) ng is defined for all $n \in Z$ and $g \in G$ and

$$ng = g + g + \cdots + g \ (n \text{ times});$$

(ii) if rg and rh are defined then so is $r(g + h)$ and $r(g + h) = rg + rh$;

(iii) if rg and sg are defined then so is $(r + s)g$ and $(r + s)g = rg + sg$;

(iv) if rg and $s(rg)$ are defined then so is $(sr)g$ and $(sr)g = s(rg)$;

(v) $r0 = 0$ whenever defined.

A *p.o. partial vector space* G is a p.o. group which is also a partial vector space, and $rg \geqq 0$ whenever $r \geqq 0$, $g \geqq 0$ and rg is defined. Of course any group is a trivial partial vector space, and every p.o. group is a trivial p.o. partial vector space. We emphasize here, and it will be apparent soon that the partial scalar multiplication is intended only as an auxiliary device.

Let G and H be p.o. partial vector spaces. An *o-bilinear map* θ: $G \times H \rightarrow L$ into the l-group L is a bilinear form such that $(g, \cdot)\theta$ is an o-homomorphism for each $0 \leqq g \in G$ and $(\cdot, h)\theta$ is an o-homomorphism for each $0 \leqq h \in H$, and in addition we specify that

$$(rg, h)\theta = (g, rh)\theta \ ,$$

whenever both rg and rh are defined.

Let G and H be p.o. partial vector spaces. The l-group T is the *o-tensor product* of G and H (notation: $G \otimes_o H$) if

(1) there is an o-bilinear map τ: $G \times H \rightarrow T$ having the property that

(2) whenever θ: $G \times H \rightarrow L$ is an o-bilinear map into the l-group L, then there is a unique l-homomorphism θ^*: $T \rightarrow L$ such that $\tau\theta^* = \theta$. It is clear that T is unique up to an l-isomorphism.

A p.o. group G is *semiclosed* if $ng \geqq 0$ for some positive integer n, implies that $g \geqq 0$; this is equivalent to the fact that the cone of G is the meet of total orders. [8]

THEOREM 1.1. *Given two p.o. partial vector spaces G and H the o-tensor product $G \otimes_o H$ exists.*

Proof. Let F be the free group over the set $G \times H$, and ι: $G \times H \rightarrow F$ be the natural embedding. An element of F can be expressed uniquely by $\sum_{i=1}^t n_i(x_i, y_i)\iota$; accordingly, we define $0 \neq x = \sum_{i=1}^t n_i(x_i, y_i)\iota$ to be positive if each $0 \leqq x_i \in G$, $0 \leqq y_i \in H$ and all the integers n_i are positive. F becomes a semiclosed p.o. group with respect to this order. Let $\boldsymbol{F}(F)$ be the free l-group on F, and N be the l-ideal generated by all the elements of the form:

$$\begin{aligned}&(g_1 + g_2, h)\iota\sigma - (g_1, h)\iota\sigma - (g_2, h)\iota\sigma ,\\&(g, h_1 + h_2)\iota\sigma - (g, h_1)\iota\sigma - (g, h_2)\iota\sigma ,\\&(rg, h)\iota\sigma - (g, rh)\iota\sigma ,\end{aligned}$$

for all $g, g_1, g_2 \in G$, $h, h_1, h_2 \in H$, and whenever rg and rh are defined. (Note: $\sigma\colon F \rightarrow \boldsymbol{F}(F)$ is the natural embedding.) Let $T = \boldsymbol{F}(F)/N$ and $\tau\colon G \times H \rightarrow T$ be defined by $(x, y)\tau = (x, y)\iota\sigma + N$; then (τ, T) is the pair we want.

It is clear that τ is o-bilinear. Now let $\theta\colon G \times H \rightarrow L$ be an o-bilinear map into the l-group L. Let $\alpha\colon F \rightarrow L$ be the induced homomorphism. Then

$$\left(\sum_{i=1}^{t} n_i(x_i, y_i)\iota\right)\alpha = \sum_{i=1}^{t} n_i(x_i, y_i)\theta ,$$

so that α is in fact order preserving. So let $\boldsymbol{F}(\alpha)\colon \boldsymbol{F}(F) \rightarrow L$ be the unique l-homomorphism such that $\sigma\boldsymbol{F}(\alpha) = \alpha$. For each $x, x_1, x_2 \in G$ and $y, y_1, y_2 \in H$ we have

$$\begin{aligned}&[(x_1 + x_2, y)\iota\sigma - (x_1, y)\iota\sigma - (x_2, y)\iota\sigma]\boldsymbol{F}(\alpha)\\&= (x_1 + x_2, y)\iota\alpha - (x_1, y)\iota\alpha - (x_2, y)\iota\alpha\\&= (x_1 + x_2, y)\theta - (x_1, y)\theta - (x_2, y)\theta = 0 .\end{aligned}$$

Similarly, $[(x, y_1 + y_2)\iota\sigma - (x, y_1)\iota\sigma - (x, y_2)\iota\sigma]\boldsymbol{F}(\alpha) = 0$ and $[(rx, y)\iota\sigma - (x, ry)\iota\sigma]\boldsymbol{F}(\alpha) = 0$ whenever the two scalar multiplications are defined. Hence $(N)\boldsymbol{F}(\alpha) = 0$ and so $\boldsymbol{F}(\alpha)$ factors uniquely through T; that is, only one l-homomorphism θ^* exists such that $(z + N)\theta^* = z\boldsymbol{F}(\alpha)$, for each $z \in \boldsymbol{F}(F)$. Obviously $\tau\theta^* = \theta$, and it is routine to verify that θ determines θ^* uniquely in this sense. Our theorem is therefore proved.

We shall discuss various categories of p.o. groups: $\boldsymbol{P}$, $\boldsymbol{L}$ and $\boldsymbol{A}$ will denote respectively, the category of all p.o. groups and o-homomorphisms, l-groups and l-homomorphisms, and finally all trivially ordered groups and all group homomorphisms; (i.e. the category of abelian groups.) $G \otimes_o (\cdot)$ is a functor from $\boldsymbol{P}$ to $\boldsymbol{L}$. If H_1 and H_2 are p.o. groups and $\psi\colon H_1 \rightarrow H_2$ is an o-homomorphism, there is a unique l-homomorphism $G \otimes_o \psi\colon G \otimes_o H_1 \rightarrow G \otimes_o H_2$ having the property that the diagram below is commutative,

$$\begin{tikzcd} H_1 \arrow[r, "\tau(g, H_1)"] \arrow[d, "\psi"'] & G \otimes_o H_1 \arrow[d, "G \otimes_o \psi"] \\ H_2 \arrow[r, "\tau(g, H_2)"] & G \otimes_o H_2 \end{tikzcd}$$

where $x\tau(g, H_1) = (g, x)\tau$ for all $g \in G$, $x \in H_1$. If $\theta\colon H_2 \to H_3$ is an o-homomorphism of p.o. group then $(G \otimes_o \psi)(G \otimes_o \theta) = G \otimes_o \psi\theta$. Of course, if 1_H is the identity map on the p.o. group H then $G \otimes_o 1_H$ is the identity on $G \otimes_o H$.

It would appear that the functor $G \otimes_o (\cdot)$ is co-adjoint to some functor from $\boldsymbol{L}$ to $\boldsymbol{P}$; ([9], pp. 66–67, 117–119.) It is also suggestive that the latter functor should be Hom $(G, \cdot)$ where Hom $(G, H) = \{$*all* homomorphisms of G into $H\}$. Heuristically, we would expect Hom (G, H) to be a p.o. group whose positive cone is precisely the set of o-homomorphisms of G into H. Unfortunately, a nonzero homomorphism can both preserve and invert order; to see this let $G = R \times R$ and put $(x, y) \geqq 0$ if $x \geqq 0$ and $y = 0$. The projection on the second component is an example of such a mapping. However, if G is taken to be a directed p.o. group then the above pathology disappears and Hom (G, H) is a p.o. group as indicated. (Since we wish to consider Hom $(G, \cdot)$ as a functor of $\boldsymbol{L}$ into $\boldsymbol{P}$, we let H be an l-group. In this case Hom (G, H) is also semiclosed.) We summarize as follows:

THEOREM 1.2. *If G is a directed p.o. group then Hom $(G, \cdot)$ is the adjoint of the functor $G \otimes_o (\cdot)$.*

On the other hand if we just consider o-tensor products of trivially ordered groups, the problem in the above discussion does not arise, and so $G \otimes_0 (\cdot)$ is co-adjoint to Hom $(G, \cdot)\colon \boldsymbol{L} \to \boldsymbol{A}$. In fact, if G and H are torsion free groups we can say much more about $G \otimes_o H$.

THEOREM 1.3. *Let G be a trivially ordered group, H be any p.o. group. Then $G \otimes_o H \cong G \otimes_o H_t$, where $H_t = H$, but H_t has the trivial order. Moreover, if G and H are torsion free then $G \otimes_o H$ is the free l-group on the trivially ordered group $G \otimes H$, the usual group tensor product of G and H.*

Proof. Let θ be a bilinear map from $G \times H$ into the l-group L; we show θ is o-bilinear. That will imply immediately that $G \otimes_o H \cong G \otimes_o H_t$. If we fix $0 \leqq x \in H$ then $(\cdot, x)\theta$ is clearly an o-homomorphism since G is trivially ordered. 0 is the only positive element of G, and $(0, \cdot)\theta$ is trivial, so it certainly preserves order. θ is indeed o-bilinear.

Next suppose G and H are torsion free groups; then $G \otimes H$ is torsion free, and so the free l-group on $G \otimes H$ exists. Let T denote this l-group, and $\bar{\tau}\colon G \times H \to G \otimes H$ and $\sigma\colon G \otimes H \to T$ be the appropriate canonical maps; let $\tau = \bar{\tau}\sigma$. The fact that (τ, T) is the desired pair is straightforward to prove and will be left to the reader.

As a consequence of a result of Bernau ([1], theorem 4.3) one obtains

COROLLARY 1.3.1. *If G is a trivially ordered, torsion free group, then for any semiclosed p.o. group H $G \otimes_o H$ is an archimedean l-group.*

Proof. A semiclosed p.o. group is torsion free.

We can generalize Theorem 1.3 provided we assume G is divisible and both p.o. groups are semiclosed.

THEOREM 1.4. *Let G and H be semiclosed p.o. groups, and assume G is divisible. Then $G \otimes_o H$ is l-isomorphic to the free l-group on $G \otimes H$. More precisely, if we set $x \in G \otimes H$ positive when $x = \sum_{i=1}^{t} a_i \otimes b_i$ with $0 \leqq a_i \in G$ and $0 \leqq b_i \in H (i = 1, \cdots, t)$, then $G \otimes H$ becomes a semiclosed p.o. group. $G \otimes_o H$ is free over this p.o. group.*

Proof. Once again let F be the free group on the set $G \times H$. Put on F the partial order described in the proof Theorem 1.1. $G \otimes H$ can then be obtained by factoring out the subgroup M generated by the elements of the form

$$(a_1 + a_2, b)\iota - (a_1, b)\iota - (a_2, b)\iota \text{ and}$$
$$(a, b_1 + b_2)\iota - (a, b_1)\iota - (a, b_2)\iota \,.$$

Viswanathan [10] proved M is convex in F. The order on F then induces the desired order on $G \otimes H$. We must still verify that $G \otimes H$ is semiclosed. Let $x \in G \otimes H$ and suppose $nx \geqq 0$, for some positive integer n. Then $nx = \sum_{i=1}^{t} a_i \otimes b_i$ where $0 < a_i \in G$ and $0 < b_i \in H$ $(1 \leqq i \leqq t)$. For each such i let $c_i \in G$ be the (unique) solution to the equation $nz = a_i$; necessarily $c_i > 0$ since G is semiclosed. Also,

$$nx = \sum_{i=1}^{t} a_i \otimes b_i = \sum_{i=1}^{t} nc_i \otimes b_i = n\Big(\sum_{i=1}^{t} c_i \otimes b_i\Big) \,;$$

$G \otimes H$ is torsion free, so that $x = \sum_{t=1}^{i} c_i \otimes b_i \geqq 0$.

Next let T be the free l-group over $G \otimes H$ as a p.o. group, and let $\tau\colon G \times H \to T$ be defined by $(x, y)\tau = (x \otimes y)\sigma$ where σ is the natural embedding of $G \otimes H$ in T; τ is clearly o-bilinear. Now let α be an o-bilinear map of $G \times H$ into the l-group L; let $\bar{\alpha}\colon G \otimes H \to L$ be the unique homomorphism determined by α. The effect of the partial order on $G \otimes H$ is to make $\bar{\alpha}$ an o-homomorphism. It therefore extends uniquely to an l-homomorphism of T into L, say α^*, and $\sigma\alpha^* = \bar{\alpha}$; hence $(x, y)\tau\alpha^* = (x \otimes y)\sigma\alpha^* = (x \otimes y)\bar{\alpha} = (x, y)\alpha$. Once more, it is straightforward to show α^* is the unique l-homomorphism making $\tau\alpha^* = \alpha$. Consequently, $T \cong G \otimes_o H$ with τ as the natural o-bilinear map.

For the remainder of this section all p.o. groups will be semiclosed, unless otherwise specified. Let G be a p.o. group; since it is torsion free its divisible hull can be realized as $Q \otimes G = \bar{G}$. In this case the order as defined above becomes: $x = r \otimes g > 0$ if and only if r is a positive rational and $0 < g \in G$. Putting it differently, $x \in \bar{G}$ is positive if and only if some positive integer multiple of x is positive in G. (We think of G as a subgroup of $\bar{G}$ under the embedding $g \mapsto 1 \otimes g$.) It is well known that if G is an l-group then so is $\bar{G}$, and G is an l-subgroup of $\bar{G}$; ([3], p. 147.) Also, G is totally ordered if and only if $\bar{G}$ is; (we shall refer to a totally ordered group as an o-group.)

COROLLARY 1.4.1. *Let G be a p.o. group; then $Q \boldsymbol{\otimes}_o G$ is the free l-group on the divisible hull $\bar{G}$ of G. Thus $\bar{G} = Q \boldsymbol{\otimes}_o G$ if and only if G is an o-group.*

Proof. The first statement follows immediately from the theorem. As for the second, it is a consequence of the definition of free l-group; (see 3.10 [6].)

COROLLARY 1.4.2. *Let G and H be p.o. groups with G divisible; then $G \otimes H \subseteqq G \boldsymbol{\otimes}_o H$, and in fact $G \boldsymbol{\otimes}_o H$ is the l-subgroup generated by $G \otimes H$.*

It is perhaps a good place here to give some kind of description of the elements of $G \boldsymbol{\otimes}_o H$ for arbitrary (not necessarily semiclosed) p.o. partial vector spaces G and H. Let τ be the canonical o-bilinear map, and write $x \otimes y = (x, y)\tau$, for $x \in G$ and $y \in H$. Then according to the construction in Theorem 1.1 and Weinberg's representation for free l-groups [11] we can write a typical element of $G \boldsymbol{\otimes}_o H$ as

$$\bigvee_A \bigwedge_B \sum_{i=1}^{t} x(\alpha, \beta, i) \otimes y(\alpha, \beta, i) ,$$

where $\alpha \in A$, $\beta \in B$ and A, B are finite sets. Whenever it is clear from the context we shall omit reference to the index-sets A and B; it should be understood then that the joins and meets are taken over finite sets.

It $G = Z$ (with the usual order) we can simplify this to $\bigvee_\alpha \bigwedge_\beta 1 \otimes y(\alpha, \beta)$. It is easy to prove the analogue for Z of Corollary 1.4.1.

PROPOSITION 1.5. *Let G be a p.o. group; then $Z \boldsymbol{\otimes}_o G$ is the free l-group on G. Consequently, $Z \boldsymbol{\otimes}_o G = G$ if and only if G is an o-group.*

Proof. Let T be the l-group on G and $\sigma: G \to T$ be the canonical embedding. Define $\tau: Z \times G \to T$ by $(n, g)\tau = (ng)\sigma = n(g\sigma)$; τ is obviously o-bilinear. Next if ξ is an o-bilinear map of $Z \times G$ into the l-group L, the map $\psi: G \to L$ defined by $g\psi = (1, g)\xi$ is an o-homomorphism. Let $\xi^*: T \to L$ be the unique l-homomorphism such that $\sigma\xi^* = \psi$; then $(n, g)\tau\xi^* = (ng)\sigma\xi^* = (ng)\psi = (n, g)\xi$, for all $n \in Z$ and $g \in G$, so that $\xi = \tau\xi^*$. Hence $T \cong Z \otimes_o G$.

The other statement in the proposition holds once again by 3.10 in [6].

We now turn to some "preservation" questions about the functor $G \otimes_o (\cdot)$. It is obvious that if the o-homomorphism θ is onto then so is $G \otimes_o \theta$. The situation with monomorphisms is interesting; categorically speaking, the monomorphisms in $\boldsymbol{P}_s$, the category of semiclosed p.o. groups and o-homomorphisms, are simply the 1–1 homomorphisms. As for groups we call a p.o. group *flat* if the funtor $G \otimes_o (\cdot)$ preserves all monomorphisms. We note that in $\boldsymbol{L}$ the monomorphisms are precisely the l-embeddings.

THEOREM 1.6. *G is flat if and only if it is trivially ordered.*

Proof. *Necessity*; suppose G is flat. Let $j: Z_t \to Z$ be the identity map on the integers with trivially ordered domain. Since j is a monomorphism, $G \otimes_o j$. is an l-embedding; but j is also onto and hence $G \otimes_o j$ is an l-isomorphism. However, $G \otimes_o Z_t$ is the free l-group on G with the trivial order (Theorem 1.3), whereas $G \otimes_o Z$ is the free l-group on G with the given partial order. Therefore G must have been trivially ordered in the first place.

Sufficiency; suppose G is trivially ordered. According to Theorem 1.3 it suffices to consider 1–1 homomorphisms of trivially ordered groups. Also by 1.3 $G \otimes_o (\cdot)$ is the composite of two functors: $G \otimes (\cdot)$ and the free l-group functor. Both of these preserve monomorphisms; (in fact the free l-group functor preserves all o-embeddings [11].) Thus G is flat, and the proof is complete.

It is still worth considering whether $G \otimes_o (\cdot)$ preserves o-embeddings. In particular, suppose an o-embedding $\theta: H_1 \to H_2$ is given; we assume that G is divisible for the present. The induced o-homomorphism $G \otimes \theta$ is certainly 1–1, since all the groups are torsion free. If it is an o-embedding then the induced l-homomorphism $G \otimes_o \theta$ is an l-embedding; (recall $G \otimes_o H_i$ is free over $G \otimes H_i$, $i = 1, 2$.) Therefore, if G is divisible then $G \otimes_o \theta$ is an l-embedding if and only if $G \otimes \theta$

is an o-embedding. The lack of a more satisfactory answer here will haunt us in the next section.

We mention (without proof) the following with regard to exactness of $G \otimes_o (\cdot)$. Suppose $0 \to H_1 \xrightarrow{\theta} H_2 \xrightarrow{\psi} H_3 \to 0$ is an exact sequence (in the group sense) of o-homomorphisms. Then at worst the convex hull of $\operatorname{Im}(G \otimes_o 0)$ in $G \otimes_o H_2$ is equal to $\operatorname{Ker}(G \otimes_o \psi)$.

We close this section with a result on decomposability.

LEMMA 1.7. (Bernau [1]) *Let G be a p.o. group. The free l-group $\boldsymbol{F}(G)$ over G is decomposable (as a cardinal sum of two non-trivial l-ideals) if and only if G is trivially ordered, and then the rank of G is* 1.

PROPOSITION 1.8. *Let G be a divisible p.o. group; then $G \otimes_o H$ is decomposable if and only if G or H is trivially ordered and then both of them have rank* 1.

Proof. By Theorem 1.4 $G \otimes_o H$ is the free l-group on $G \otimes H$, and hence by Lemma 1.7 $G \otimes_o H$ is decomposable if and only if $G \otimes H$ is trivially ordered and of rank 1. Now $G \otimes H$ is trivially ordered if and only if one of the components is, and since $\operatorname{rank}(G \otimes H) = \operatorname{rank}(G). \operatorname{rank}(H)$, it follows that $G \otimes H$ will have rank 1 if and only if both G and H have rank 1.

Note that for the sufficiency here the divisibility of G may be dropped in view of Theorem 1.3.

2. The vector lattice cover of an l-group. Again in this section all p.o. groups will be semiclosed. We record the following general result.

PROPOSITION 2.1. *Let V be a p.o. vector space and G be a p.o. group. Then $V \otimes_o G$ can be made into a vector lattice by defining scalar multiplication for $0 \leqq r \in R$*

$$r[\bigvee_\alpha \bigwedge_\beta \sum_{i=1}^{t} v(\alpha, \beta, i) \otimes g(\alpha, \beta, i)] = \bigvee_\alpha \bigwedge_\beta \sum_{i=1}^{t} [rv(\alpha, \beta, i)] \otimes g(\alpha, \beta, i) .$$

Proof. Let $0 < r \in R$; the mapping ρ_r: $V \times G \to V \otimes_o G$ defined by $(v, g)\rho_r = rv \otimes g$ is o-bilinear. Let r° be the induced l-endomorphism of $V \otimes_o G$; we have

$$\left[\bigvee_\alpha \bigwedge_\beta \sum_{i=1}^{t} v(\alpha, \beta, i) \otimes g(\alpha, \beta, i)\right] r^\circ = \bigvee_\alpha \bigwedge_\beta \sum_{i=1}^{t} [rv(\alpha, \beta, i)] \otimes g(\alpha, \beta, i) .$$

One can easily verify all of the following:

(1) $(r+s)^\circ = r^\circ + s^\circ$, for all $0 < r, s \in R$;

(2) 1° is the identity map on $V \otimes_o G$;

(3) $(rs)^\circ = r^\circ s^\circ$, for all $0 < r, s \in R$;

(4) $r^\circ (r^{-1})^\circ = (r^{-1})^\circ r^\circ = 1^\circ$, hence $(r^\circ)^{-1} = (r^{-1})^\circ$, for all $0 < r \in R$;

(5) the map $r \mapsto r^\circ$ is 1-1.

We can also define this operation for negative real numbers; just let $r^\circ = -(-r)^\circ$; set $0^\circ = 0$. Clearly then the map $r \mapsto r^\circ$, from R into the group of automorphisms of $V \otimes_o G$ is an o-isomorphism. We let $rx = xr^\circ$, for all $x \in V \otimes_o G$ and $r \in R$; this of course is the desired scalar multiplication. It should be obvious from the preceding remarks that $V \otimes_o G$ is indeed a vector lattice with respect to this operation.

COROLLARY 2.1.1. *Let G be a p.o. group; then $R \otimes_o G$ is a vector lattice. If $\phi: G \to V$ is an o-homomorphism into the vector lattice V there is a unique lattice preserving linear transformation* (henceforth: *l-linear transformation*) *$\phi^*: R \otimes_o G \to V$ such that $(1 \otimes g)\phi^* = g\phi$, for all $g \in G$.*

Proof. Define $\sigma: G \to R \otimes_o G$ *by* $g\sigma = 1 \otimes g$; σ is an o-homomorphism. If ϕ is an o-homomorphism of G into the vector lattice V then the mapping given by $(r, g)\bar{\phi} = r(g\phi)$ is an o-bilinear map. Let ϕ^* be the induced l-homomorphism; then $g\sigma\phi^* = (1 \otimes g)\phi^* = (1, g)\bar{\phi} = g\phi$; it should be clear that ϕ^* is a linear transformation. And if $\theta: R \otimes_o G \to V$ is any l-linear transformation such that $\sigma\theta = \phi$ then $(r \otimes g)\theta = [r(1 \otimes g)]\theta = r(1 \otimes g)\theta = r(g\phi) = (r, g)\bar{\phi}$; this implies that $\theta = \phi^*$ and we're done.

COROLLARY 2.1.2. *Let G be a p.o. vector space. Then $R \otimes_o G$ is the free vector lattice over G. Consequently $R \otimes_o G \cong G$ if and only if G is an o-group.* (Remark: *This is where we first use the transfer property of scalar multiples in the tensor product.*)

We now turn to the main definition in this section. Let G be an l-group; the *vector lattice cover* of G is a vector lattice $V(G)$ together with an l-homomorphism $\mu_G: G \to V(G)$ having the property that if ϕ is any l-homomorphism of G into the vector lattice W, there is a unique l-linear transformation $\phi^*: V(G) \to W$ such that $\mu_G\phi^* = \phi$. As in the beginning of §1 we need the notion of a partial vector space to insure that objects do not become too large. Thus a *partial vector lattice* is an l-group which is at the same time a p.o. partial vector space. An l-homomorphism $\alpha: G_1 \to G_2$ of partial vector lattices is called an *l-linear transformation* if whenever rg exists in G_1 then $r(g\alpha)$ exists in G_2 and $(rg)\alpha = r(g\alpha)$. Thus the vector lattice cover $V(G)$ of a partial vector lattice G is a pair $(V(G), \mu_G)$ where μ_G is an *l-linear transformation*, and of course the appropriate universal mapping diagram is valid. This device of partial vector lattices guarantees that $V(G) = G$ if G is already a full vector lattice. The existence of the vector lattice cover is easily settled.

THEOREM 2.2. *Given a partial vector lattice G, the vector lattice cover $V(G)$ exists and is unique up to an l-isomorphism of vector lattices. If G is simply an l-group the canonical map μ_G is an l-embedding.*

Proof. Let G be a partial vector lattice; let $V(G)$ be the free vector lattice over the set G. Let N_G be the l-ideal (and hence subspace) generated by all elements of the form:

(a) $g\sigma + h\sigma - (g + h)\sigma$,

(b) $(g\sigma \vee h\sigma) - (g \vee h)\sigma$,

(c) $r(g\sigma) - (rg)$, whenever rg is defined.

(Note: $\sigma\colon G \to V(G)$ is the natural embedding.) Let $V(G) = V(G)/N_G$, and $\mu_G\colon G \to V(G)$ be defined by $g\mu_G = g\sigma + N_G$. The effect of factoring out N_G is that μ_G is an l-linear transformation.

So let ϕ be an l-linear transformation of G into the vector lattice W. Consider the induced l-linear transformation ϕ' from $V(G)$ into W; recall that any element of $V(G)$ may be put in the form

$$\bigvee_{\alpha} \bigwedge_{\beta} \sum_{i=1}^{t} r(\alpha, \beta, i)(g(\alpha, \beta, i))\sigma \ ,$$

where again the joins and meets are over finite sets. Then ϕ' is defined by $[\bigvee_\alpha \bigwedge_\beta \sum_{i=1}^t r(\alpha, \beta, i)\, (g(\alpha, \beta, i))\sigma]\phi' = \bigvee_\alpha \bigwedge_\beta \sum_{i=1}^t r(\alpha, \beta, i)$ $(g(\alpha, \beta, i))\phi$. Therefore, $[g\sigma + h\sigma - (g + h)\sigma]\phi' = g\phi + h\phi - (g + h)\phi = 0$, for all $g, h \in G$. Similarly, $[g\sigma \vee h\sigma - (g \vee h)\sigma]\phi' = 0$ and $[(rg)\sigma - r(g\sigma)]\phi' = 0$ whenever rg is defined. Hence $N_G\phi' = 0$ and so ϕ' factors uniquely through $V(G)$ by $\phi^*\colon V(G) \to W$. It is evident that $\mu_G\phi^* = \phi$, and also that ϕ determines ϕ^* uniquely in this sense.

If G is just an l-group then there is an l-embedding θ of G into a vector lattice W ([3], theorem 4.2). This l-embedding must extend to $\theta^*\colon V(G) \to W$; then $\mu_G\theta^* = \theta$, and in particular μ_G is 1–1. (In fact, for an arbitrary partial vector lattice H, μ_H is 1–1 if and only if H is embeddable in a vector lattice.)

This "cover" functor V is defined from the category V^p of partial vector lattices into the category V of vector lattices. V is co-adjoint to the embedding functor $E\colon V \to V^p$. This is a full embedding functor since V is a full subcategory of V^p ([9], p. 3, pp. 66–67 and pp. 117–119).

It is obvious that there is some connection between the o-tensor functor $R \otimes_o (\cdot)$ and the cover functor V. Our next result points out this connection and also generalizes (in part) a result of Conrad in [7] which we shall derive as a corollary. An l-group G is l-free if there exists a (semiclosed) p.o. group H such that $G = \boldsymbol{F}(H)$, the free l-group over H.

THEOREM 2.3. *Let G be an l-free l-group, say $G = \boldsymbol{F}(H)$; then $V(G) = R \otimes_o H$.*

Proof. Let $\sigma: H \rightarrow G$ be the canonical embedding, and $\sigma_1: H \rightarrow R \otimes_0 H$ the o-homomorphism given by $x\sigma_1 = 1 \otimes x$. Since G is free over H σ_1 extends uniquely to $\mu: G \rightarrow R \otimes_o H$; we have $\sigma\mu = \sigma_1$ Next let $\phi: G \rightarrow W$ be an l-homomorphism into the vector lattice W. Consider $\sigma\phi$; by Corollary 2.1.1 there is exactly one l-linear transformation ϕ^*: $R \otimes_o H \rightarrow W$ such that $\sigma\phi = \sigma_1\phi^* = \sigma\mu\phi^*$, and hence $\phi = \mu\phi^*$. That ϕ determines ϕ^* uniquely is straightforward to check, and is left to the reader. Consequently, $V(G) = R \otimes_o H$.

COROLLARY. 2.3.1. *If G is an o-group then $V(G) = R \otimes_o G$.*

Proof. An o-group is free over itself; (see 3.10 in [6]).

COROLLARY 2.3.2. (Conrad) *Let S be a set of generators and $\boldsymbol{F}(S)$* resp. *$V(S)$) be the free l-group* (resp, *free vector lattice*) *over S. Then $V(\boldsymbol{F}(S)) = V(S)$.*

Proof. $\boldsymbol{F}(S)$ is free over G_S, the free group over S; (G_S is taken as a trivially ordered group.) According to Theorem 1.3 $R \otimes_o G_S$ is the free l-group over $R \otimes G_S \cong V_S$, the vector space with S as basis. Quite clearly then the canonical mapping of V_S into $R \otimes_o G_S$ is a linear transformation, so that $R \otimes_o G_S$ is the free vector lattice over V_S, i.e. precisely $V(S)$. Hence by 2.3 $V(\boldsymbol{F}(S)) = V(S)$.

Conrad proved more; he also showed that $\boldsymbol{F}(S)$ is *dense* in $V(S)$; that is each strictly positive element of $V(S)$ exceeds a strictly positive element of $\boldsymbol{F}(S)$. [7]

Let G be an l-group; recall from the proof of Theorem 2.2 that $V(G)$ may be viewed as a suitable quotient of the free vector lattice on G as a set. Consequently, each element of $V(G)$ is of the form

$$\bigvee_{\alpha} \bigwedge_{\beta} \sum_{i=1}^{t} r(\alpha, \beta, i)[g(\alpha, \beta, i)\mu_G] ,$$

where of course the indexing of the joins and meets are over finite sets, and the $r(\alpha, \beta, i) \in R$ and $g(\alpha, \beta, i) \in G$, and finally μ_G is the canonical embedding of G in $V(G)$. We shall omit mention of μ_G except when confusion might arise from doing so.

We want to know when the cover of a group G is the unique minimal vector lattice cover. (In such a case we say that G is *v-covered.*) More precisely, when does an l-embedding ϕ of G into a vector lattice W induce an l-embedding ϕ^*: $V(G) \to W$. We have the following lemma.

LEMMA 2.4. *For an l-group G the following are equivalent*:
(1) *G is v-covered.*
(2) *V preserves monomorphisms* (i.e. *l-embeddings*) *out of G.*
(3) *Every nonzero l-ideal of V(G) has a nonzero meet with G.*

Proof. (1) $\to$ (2) Let H be an l-group and β: $G \to H$ be an l-embedding. Then $\beta\mu_H$: $G \to V(H)$ is an l-embedding, and so by (1) $(\beta\mu_H)^* = V(\beta)$ is 1-1.

(2) $\to$ (3) Let K be a nonzero l-ideal of $V(G)$; it is also a subspace and $V(G)/K$ is a vector lattice. Therefore $V(V(G)/K) = V(G)/K$; let ρ: $V(G) \to V(G)/K$ denote the canonical l-homomorphism; if $K \cap G = 0$ then $\mu_G\rho$ is 1-1, and so ρ itself must be 1-1 by (2), a contradiction.

(3) $\to$ (1) Let ϕ be an l-embedding of G into a vector lattice W. Let ϕ^* be the unique extension to $V(G)$, and let $K = \operatorname{Ker}(\phi^*)$; since $\operatorname{Ker}(\phi) = K \cap G = 0$ it follows that $K = 0$, i.e. ϕ^* is 1-1.

If G is an l-subgroup of the l-group H we say H is an *essential extension* of G (or that G is *large* in H) if every nonzero l-ideal of H has a nontrivial meet with G. It is easy to see that G is large in H if and only if for each $0 < x \in H$ there is a $g \in G$ and a positive integer n such that $0 < g \leqq nx$. In particular, if G is dense in H it is also large in H; in view of the remark after 2.3.2 $\boldsymbol{F}(S)$ is v-covered (by $V(S)$); this was shown by Conrad directly in [7].

It must be noted here that the equivalence of (2) and (3) were proved by Conrad for archimedean l-groups.

Corollary 2.3.1 gives a description of $V(G)$ when G is an o-group. If we knew that $R \otimes_o (\cdot)$ preserved o-embeddings we could show using the Hahn-embedding theorem for o-groups, that $V(G)$ is indeed totally ordered and G is large in $V(G)$. The author is not aware of any counter examples but suspects some exist. We hasten to add however, that if an o-group G is large in $V(G)$ then $V(G)$ is necessarily totally ordered; the proof of this fact is trivial.

If G is an archimedean o-group (or a subgroup of R, in view of Hölder's theorem [8]) then we can show that $V(G) = R$, which is both an o-group and an essential extension of G. To see this merely observe that the o-homomorphisms of an archimedean o-group form a multiplicative subgroup of positive real numbers, so that an o-homomorphism ϕ of G into a vector lattice is entirely determined by 1ϕ; (we assume without loss of generality that $1 \in G$.) It is obvious then that ϕ has a unique extension to R which must be 1-1.

Let H be an l-group and G be an l-subgroup; H is an *archimedean extension* of G (or G is *archimedean in* H) if for each $0 < x \in H$ there is a $g \in G$ and a pair of positive integers m and n such that $0 < g \leqq mx$ and $0 < x \leqq ng$. Clearly an archimedean extension is essential, and in the case of the previous paragraph G is archimedean in $V(G) = R$.

THEOREM 2.5. *Let $\{G_i \mid i \in I\}$ be a family of l-groups and $G = \boxplus \{G_i \mid i \in I\}$; then $V(G) = \boxplus \{V(G_i) \mid i \in I\}$. If in addition each G_i is large (archimedean) in $V(G_i)$ then G is large (archimedean) in $V(G)$.*

Proof. We shall derive this result as a corollary of a more general result in the next section. At any rate, the second statement of the theorem is easy to prove and is left to the reader.

To the expert in category theory a proof for the above may seem unnecessary, since V, having an adjoint, must preserve co-limits; (see [9], p. 44 and p. 67). It can be shown (the author will prove this fact elsewhere) that the cardinal sum is in fact not a co-limit, and the functor we shall investigate in §3 does not preserve co-limits yet respects cardinal sums!

Let K be an l-ideal of the l-group G; we say that G is a *lex(icographic) extension* of K, notation: $G = \operatorname{lex}(K)$, if (1) G/K is an o-group, and (2) for each $0 < g \in G\backslash K$ and $x \in K$, $g > x$. If $K \neq 0$ then condition (1) is superfluous ([4], p. 214). G is a *direct* lex-extension of K if $G = \operatorname{lex}(K)$ and K splits as a direct summand; then G is l-isomorphic to $H \times K$, where $H = G/K$, and in $H \times K$ $(h, k) \geqq 0$ if $h > 0$, or $h = 0$ and $k \geqq 0$. We write $G = H \overrightarrow{\times} K$.

PROPOSITION 2.6. *Let $G = H \overrightarrow{\times} K$ where H is an o-group which is large in $V(H)$. Then $V(H)$ is an o-group and $V(G) = V(H) \overrightarrow{\times} V(K)$; moreover if K is large in $V(K)$ then G is large in $V(G)$.*

Proof. We already know $V(H)$ must be totally ordered. Let ϕ: $G \to W$ be an l-homomorphism into the vector lattice W. Let ϕ_H and ϕ_K be the restrictions to H and K respectively; both are l-homomorphisms. Let θ_H and θ_K be their respective extensions to $V(H)$ and $V(K)$. We define θ to be the induced linear transformation from

$V(H) \overrightarrow{\times} V(K)$ into W; to show θ preserves the lattice structure, it suffices to show it preserves disjointness. So suppose $x \wedge y = 0$ in $V(H) \overrightarrow{\times} V(K)$; then $x = (0, x')$ and $y = (0, y')$ with $x', y' \in V(K)$. Therefore $x\theta \wedge y\theta = x'\theta_K \wedge y'\theta_K = 0$ since $x' \wedge y' = 0$ in K.

The remainder of the proof is straightforward and is left to the reader.

The above result certainly throws out the possible conjecture that if G is large in $V(G)$ then G must be archimedean. It should be pointed out that in the statement of 2.6 the largeness of H in $V(H)$ is necessary only to insure that $V(H)$ be totally ordered; we may assume the latter hypothesis outright. We can generalize 2.6 to arbitrary lex-extensions; we need the following result concerning divisibility.

LEMMA 2.7. *Let $\bar{G}$ be the divisible hull of the l-group G; then $V(\bar{G}) = V(G)$.*

A proof may be given for 2.7 using the appropriate universal mapping diagram definition, but we shall give a much simpler proof in §3.

COROLLARY 2.7.1. *Let K be an l-ideal of G such that $G = \mathrm{lex}\,(K)$. If $V(G/K)$ is totally ordered then $V(G) = V(G/K) \overrightarrow{\times} V(K)$. If K is large in $V(K)$ then G is large in $V(G)$.*

Proof. Follows easily from the lemma and Theorem 2.6, for if $G = \mathrm{lex}\,(K)$ then $\bar{G} = \overline{G/K} \overrightarrow{\times} \bar{K}$.

PROPOSITION 2.8. *Let G be an l-group and suppose $\{H_\alpha \mid \alpha \in A\}$ is a directed system of l-subgroups of G such that each H_α is large in $V(H_\alpha)$ and $G = \cup\{H_\alpha \mid \alpha \in A\}$. Then $V(G) = \cup\{V(H_\alpha) \mid \alpha) \mid \alpha \in A\}$ and G is large in $V(G)$.*

Proof. There is no harm in assuming A to be directed such that $\alpha \leqq \beta$ implies that $H_\alpha \subseteqq H_\beta$; we let $V_\alpha = V(H_\alpha)$. If $\alpha \leqq \beta$ the induced map $V_\alpha \rightarrow V_\beta$ is 1–1 since H_α is large in V_α. The induced map $V_\beta \rightarrow V(G)$ is also 1–1 for each β. So we may think of the V_α as a directed family of l-subspaces of $V(G)$. If $x \in V(G)$ we may write $x = \bigvee_\gamma \bigwedge_\delta \sum_{i=1}^{t} r(\gamma, \delta, i) g(\gamma, \delta, i)$ with the $g(\gamma, \delta, i) \in G$. Pick an $\alpha \in A$ such that each $g(\gamma, \delta, i) \in H_\alpha$; this can be done since there are only finitely many $g(\gamma, \delta, i)$, and the H_β are directed. Then $x \in V_\alpha$, and hence $V(G) = \cup V_\alpha$.

Now suppose K is l-ideal of $V(G)$ such that $K \cap G = 0$. Then $(K \cap V_\alpha) \cap H_\alpha = K \cap H_\alpha = 0$, and so $K \cap V_\alpha = 0$ since H_α is large in V_α $(\alpha \in A)$. Therefore $K = K \cap V(G) = 0$, and hence G is large in $V(G)$.

COROLLARY 2.8.1. *Let G be an l-group; if each principal l-ideal of G is v-covered then G itself is v-covered.*

The last two results in this section are stated without proof because we shall not have an occasion to use them in the sequel. The first is very easy to prove; the second requires that we set down some definitions and discuss material we would rather avoid at this time. Let G be large in $V(G)$.

2.9. If G is an archimedean l-group then so is $V(G)$. (The big question is left unanswered here: is the conclusion true without the assumption that G is large in $V(G)$?)

2.10. G is completely distributive if and only if $V(G)$ is. (A lattice is completely distributive when the most general distributive law holds for the lattice operations. For background material on complete distributivity in l-groups we suggest [5].)

3. The l-tensor product. Let G and H be l-groups (resp. partial vector lattices) A bilinear mapping of $G \times H$ into the l-group L is said to be l-bilinear if $(\cdot, h)\theta$ is an l-homomorphism for each $0 \leqq h \in H$, and $(g, \cdot)\theta$ is an l-homomorphism for each $0 \leqq g \in G$. (In the case of partial vector lattices we also require that $(rg, h)\theta = (g, rh)\theta$, whenever both rg and rh are defined.)

The *l-tensor product* of G and H, denoted by $G \otimes_l H$, is an l-group together with an l-bilinear map $\tau: G \times H \rightarrow G \otimes_l H$ having the property that whenever θ is an l-bilinear map of $G \times H$ into the l-group L then there is a unique l-homomorphism $\theta^*: G \otimes H_l \rightarrow L$ such that $\tau\theta^* = \theta$.

The existence and uniqueness of $G \otimes_l H$ is settled in a routine way.

THEOREM 3.1. *Given l-groups G and H $G \otimes_l H$ exists and is unique up to an l-isomorphism.*

Proof. Let $\boldsymbol{F}$ be the free l-group over $G \times H$ as a set. Let M be the l-ideal of $\boldsymbol{F}$ generated by all elements of the form:

$$(a_1 + a_2, b)\sigma - (a_1, b)\sigma - (a_2, b)\sigma ,$$
$$(a_1 \vee a_2, b)\sigma - [(a_1, b)\sigma \vee (a_2, b)\sigma] \text{ for } b \geqq 0 ,$$

together with their left-right duals. (If G and H are partial vector lattices we require also that all elements

$$(rg, h)\sigma - (g, rh)\sigma ,$$

be factored.)

Let $\tau\colon G \times H \to G \otimes_l H \equiv \boldsymbol{F}/M$ be defined by setting $(g, h)\tau = (g, h)\sigma + M$. It is evident that τ is l-bilinear. The completion of this proof is by now straightforward, and we will not bore the reader with it.

By way of contrast with the tensor product in §1, and in view of what lies ahead in connection with §2, we shall represent an image $(g, h)\tau$ under the canonical l-bilinear mapping τ by $g \cdot h$, or simply gh. Thus a typical element of $G \otimes_l H$ can be put in the form

$$\bigvee_{\alpha} \bigwedge_{\beta} \sum_{i=1}^{t} g(\alpha, \beta, i)h(\alpha, \beta, i) ,$$

where as before the α and β range over finite indexing sets.

The following results have, for the most part, simple proofs. Where appropriate we shall indicate the crucial arguments.

3.2. $G \otimes_l (\cdot)$ is a functor from the category $\boldsymbol{V}^p$ of partial vector lattices into the category $\boldsymbol{L}$ of l-groups.

3.3. $Z \otimes_l H = H$, for each l-group H.

Proof. For any positive integer n, $n(g \vee h) = ng \vee nh$. Thus the mapping $(n, g) \mapsto ng$ is l-bilinear.

3.4. $Q \otimes_l H = \bar{H}$, the divisible hull of H ordered as in §1.

3.5. Given the l-groups G and H with $g \in G$ and $h \in H$ and $0 < gh \in G \otimes_l H$ then $gh = g_1h_1 + g_2h_2$, where $0 < g_i \in G$ and $0 < h_i \in H$ $(i = 1, 2)$.

Proof. Write g and h as the difference of their respective positive and negative parts: $g = g^+ - g^-$, $h = h^+ - h^-$. Then $gh = g^+h^+ + g^-h^- - (g^+h^- + g^-h^+)$, and $(g^+h^+ + g^-h^-) \wedge (g^+h^- + g^-h^+) = 0$ in $G \otimes_l H$. Hence $g^+h^- + g^-h^+ = 0$, which implies what we want.

3.6. Let $0 < x \in G \otimes_l H$; then there exist $0 < g \in G$ and $0 < h \in H$ such that $x \leqq gh$.

Proof. Write $x = \bigvee_{\alpha} \bigwedge_{\beta} \sum_{i=1}^{t} g(\alpha, \beta, i)h(\alpha, \beta, i)$; after decomposing into positive and negative parts we may suppose each $g(\alpha, \beta, i)$ and $h(\alpha, \beta, i)$ is comparable to 0. In fact, in view of bilinearity we may take all the $h(\alpha, \beta, i)$ to be positive; (since zeros do not contribute anything to the sum we assume that the $h(\alpha, \beta, i)$ are *strictly* positive.) Not all the $g(\alpha, \beta, i)$ can be negative since $x > 0$; thus $g = \bigvee_g (\alpha, \beta, i) \vee 0 > 0$. Clearly,

$$x = \bigvee_\alpha \bigwedge_\beta \sum_{i=1}^{t} g(\alpha, \beta, i)h(\alpha, \beta, i) \leqq \bigvee_\alpha \bigwedge_\beta \sum_{i=1}^{t} gh(\alpha, \beta, i)$$
$$\leqq g(\bigvee_\alpha \bigwedge_\beta \textstyle\sum_{i=1}^{t} h(\alpha, \beta, i) \leqq gh ,$$

where $h = \bigvee h(\alpha, \beta, i)$, the indicated join being taken over all α, β and i.

3.7. $R \otimes_l H$ is a vector lattice with respect to the scalar multiplication given by $r(\bigvee_\alpha \bigwedge_\beta \sum_{i=1}^{t} r(\alpha, \beta, i)h(\alpha, \beta, i)) = \bigvee_\alpha \bigwedge_\beta \sum_{i=1}^{t} [r \cdot r(\alpha, \beta, i)]h(\alpha, \beta, i)$, if $r \geqq 0$.

THEOREM 3.8. *For each partial vector lattice G $V(G) = R \otimes_l G$.*

Proof. The map μ: $G \rightarrow R \otimes_l G$ given by $g\mu = 1 \cdot g$ is an l-linear transformation. Furthermore, if ϕ: $G \rightarrow W$ is any l-linear transformation into the vector lattice W, then the map θ from $R \times G$ into W given by

$$(r, g)\theta = r(g\phi) ,$$

is l-bilinear. Let ϕ^* be the unique l-linear transformation such that $\tau\phi^* = \theta$; then $g\mu\phi^* = (1g)\phi^* = (1, g)\theta = 1(g\phi) = g\phi$, for each $g \in G$; that is $\mu\phi^* = \phi$. (We really don't know that ϕ^* preserves scalar multiplication right away, but it is easy enough to check.) Finally if ψ is any l-linear transformation such that $\mu\psi = \phi$ then $(rg)\psi = (r1g)\psi = [r(g\mu)]\psi = r(g\mu\psi) = r(g\phi) = (r, g)\theta$; this implies that $\psi = \phi^*$. This completes the proof of the theorem.

We point out here (taking into account the fact that the l-tensor product is clearly associative) that $V(\bar{G}) = R \otimes_l \bar{G} = R \otimes_l (Q \otimes_l G) = (R \otimes_l Q) \otimes_l G = R \otimes_l G$, where $\bar{G}$ is the divisible hull of G. This is precisely Lemma 2.7.

COROLLARY 3.8.1. *For any l-group G, the l-ideal generated by G in $V(G)$ is $V(G)$ itself.*

Proof. Follows immediately from 3.8 and 3.6, and the fact that every positive real number is exceeded by a positive integer.

THEOREM 3.9. *Let $\{H_\lambda \mid \lambda \in \Lambda\}$ be a collection of l-groups, and $H = \boxplus \{H_\lambda \mid \lambda \in \Lambda\}$. Then $G \otimes_l H = \boxplus\, G \otimes_l H_\lambda$ $(\lambda \in \Lambda)$.*

Proof. Consider the mapping $\psi\colon G \times H \to \boxplus\, G \otimes_l H$ defined by

$$(g, \Sigma h_\lambda)\psi = \Sigma g \cdot h_\lambda ;$$

ψ is clearly bilinear. In addition if $g \geqq 0$ then $g(h_\lambda \vee k_\lambda) = gh_\lambda \vee gk_\lambda$, $(\lambda \in \Lambda)$, and so $(g, \cdot)\psi$ is an l-homomorphisms. On the other hand if $h = \Sigma h_\lambda \geqq 0$ then each $(\cdot)\cdot h_\lambda$ is an l-homomorphism; hence so is $(\cdot, h)\psi$. Thus ψ is l-bilinear; let ψ^* be the l-homomorphism induced by ψ from $G \otimes_l H$ into $\boxplus\, G \otimes_l H_\lambda$.

Now let $\omega_\lambda = G \otimes_l u_\lambda\colon G \otimes_l H_\lambda \to G \otimes_l H$, where u_λ is the λth embedding of H_λ in H. There is an obvious homomorphism $\omega\colon \boxplus\, G \otimes_l H_l \to G \otimes_l H$ induced by the ω_λ; we must show it preserves the lattice structure. It suffices to show that whenever $0 < a, b \in G$, $0 < h_i \in H_{\lambda_i}$ $(i = 1, 2)$ with $\lambda_1 \neq \lambda_2$, then $ah_1 \wedge bh_2 = 0$ in $G \otimes_l H$. (This follows from 3.6). Let $c = a \vee b$; then $ch_1 \wedge ch_2 = 0$ in $G \otimes_l H$, and so $ah_1 \wedge bh_2 = 0$ as promised.

Consider the mappings ψ^* and ω; it can be easily shown that $\psi^*\omega$ is the identity on all gh, where $g \in G$ and $h \in H$. Consequently $\psi^*\omega = 1_{G\otimes_l H}$. Likewise $\omega\psi^* = 1_{\boxplus G\otimes_l H_\lambda}$. Our proof is therefore done.

It is immediate from 3.8 and 3.9 that the functor V preserves cardinal sums; this is the first part of Theorem 2.5. We think it's perhaps well to point out again the remarkable fact that although $G \otimes_l (\cdot)$ preserves all cardinal sums it need not preserve co-limits, and hence does not have an adjoint.

4. Open questions. Certainly a large number of important questions remain unanswered; some of these we have already mentioned. What follows is a list of such questions in which the author has a particular interest. The reader is invited to enlarge this list with some queries of his own.

I. If the semiclosed partially ordered groups G and H are given when is $G \otimes_o H$ an o-group? Is it the case for example that if $G \otimes_o H$ is an o-group, then both G and H must also be totally ordered?

II. When does the functor $G \otimes_l (\cdot)$ have an adjoint? We have already remarked that it does not always have one. If it has an adjoint can the latter be realized as a Hom-functor of some kind?

III. What l-groups have the property that $G \otimes_l (\cdot)$ preserves l-embeddings?

IV. If G and H are archimedean l-groups is $G \otimes_l H$ also archimedean?

V. When is the l-group G dense in $V(G)$?

VI. If both G and H are l-groups with at most a finite number of pairwise disjoint elements, is the same true of $G \otimes_l H$? If not, is it possible to find two such l-groups such that $G \otimes_l H$ has arbitrarily many pairwise disjoint elements?

Footnote on question VI. The author has found an example of two o-groups G and H such that $G \otimes_l H$ is not totally ordered. It is in fact possible to find o-groups such that $G \otimes_l H$ has an infinite subset of pairwise disjoint elements! This answers the first part of the question in the negative, and the second at least partially in the affirmative.

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