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OSCILLATION CRITERIA FOR SELFADJOINT ELLIPTIC EQUATIONS

KURT KREITH AND CURTIS C. TRAVIS

This paper extends several of the classical oscillation criteria for the Sturm-Liouville equation $(au')' + cu = 0$ to selfadjoint elliptic equations of the form

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + c(x)u = 0 .$$

Oscillation criteria for selfadjoint second order elliptic equations have been established by several authors [6], [3], and [4], generalizing the classical theory for Sturm-Liouville equations. The specific oscillation criteria of these studies have been stated in terms of functions which are *pointwise* majorants and minorants of the coefficients of an elliptic equation which arise naturally by separation of variables in various coordinate systems. Thus, for example, while Swanson and Headley [3] establish oscillation criteria of "limit type" and "integral type", the limit and integral tests must be applied to pointwise majorants and minorants which may not accurately reflect the limit or integral behavior of the coefficients of the equation under study.

In the present paper we establish oscillation criteria for elliptic equations in terms of majorants and minorants obtained by an averaging process. Specifically, in §2 an ordinary differential equation is derived which, if oscillatory at ∞, implies oscillatory behavior for $\Delta u + c(x)u = 0$ in E^n. Applying an integral oscillation criterion to this ordinary differential equation leads to results such as the following

(THEOREM 3.1): *if* $\iint_{E^2} c(x, y)dx\, dy = \infty$, *then* $\Delta u + cu = 0$ *is nodally oscillatory in* E^2.

The elliptic equations under study will be of the form

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right) + c(x)u = 0 \tag{1.1}$$

where $c(x)$ is continuous and the $a_{ij}(x)$ are of class C^1 in E^n. Points in E^n are denoted by $x = (x_1, \cdots, x_n)$, and $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$. We define $E_R = \{x \in E^n: |x| > R\}$, and if G is a domain in E^n, then $G_R = G \cap E_R$.

The elliptic equation (1.1) is said to be *nodally oscillatory* in E^n if for every $R > 0$ there is a domain $G \subset E_R$ such that G is a nodal domain for a solution of (1.1). By the Sturmian comparison theorem for elliptic equations (see for example [7]), if (1.1) is nodally oscillatory in E^n, then any global solution of (1.1) has a zero in every E_R.

Our basic tool, Theorem 2.2, is established in §2 below. In §3 we apply this Theorem to $\Delta u + cu = 0$ in the case $n = 2$ by exploiting known oscillation criteria for Sturm-Liouville equations of the form $(pv')' + qv = 0$ where $\int^{\infty} 1/(p(x))dx = \infty$. In case $n > 2$ we are led to Sturm-Liouville equations for which $\int^{\infty} 1/(p(x))dx < \infty$, and this case is considered in §4. More general elliptic equations of the form (1.1) and (1.2) are considered in §5, while §6 is devoted to oscillation criteria for $\Delta u + cu = 0$ in unbounded domains.

2. The related ordinary differential equation. Our principal tool is based on a special case of a theorem due to D. O. Banks [1]. Considering a nonnegative real valued function $\varphi(x)$ which is defined and measurable on a domain $G \subset E^n$, we define

$$G(y) = \{x: \varphi(x) \geqq y\} .$$

Theorem 1 of [1] then implies the following

THEOREM (Banks) 2.1. *If $f(x)$ is continuous in G and satisfies*

$$\int_{G(y)} f(x)\, dx \geqq 0$$

for all $y \in [0, \infty)$, then

$$\int_G f(x)\varphi(x)\, dx \geqq 0 .$$

REMARK. While Banks proved Theorem 1 of [1] in E^2, his method of proof works equally well in E^n.

Consider now the equation

$$\Delta u + c(x)u = 0 \quad \text{in } E^n \tag{2.1}$$

where $c(x)$ is continuous in E^n. Letting (r, θ) denote hyperspherical coordinates for E^n, volume integrals over annular regions can be written in the form

$$\int_{r_1}^{r_2}\left[\int_{\Omega} f(r, \theta)d\theta\right]r^{n-1}dr$$

where Ω denotes the full range of the angular coordinates. Letting

$$\textstyle\sum_n = \int_{\Omega} d\theta$$

denote the area of the unit sphere in E^n, we define

$$\gamma(r) = \frac{1}{\sum_n} \int_\Omega c(r, \theta) d\theta \ .$$

It follows readily from this definition that

$$\int_\Omega [c(r, \theta) - \gamma(r)] \, d\theta = 0 \tag{2.2}$$

for $0 \leqq r < \infty$. Our principal tool is the following.

THEOREM 2.2. *If the ordinary differential equation*

$$\frac{d}{dr}\left(r^{n-1}\frac{dv}{dr}\right) + r^{n-1}\gamma(r)v = 0 \tag{2.3}$$

is oscillatory at $r = \infty$, *then* (2.1) *is nodally oscillatory in* E^n.

Proof. Consider the equation

$$\Delta v + \gamma(r)v = 0 \tag{2.4}$$

where $\gamma(r)$ is defined as above. Equation (2.4) has solutions $v(r)$ which are independent of θ, and these can be found by solving (2.3). If (2.3) is oscillatory, then there exists a function $v_1(r)$ which satisfies (2.4) and has a sequence of annular nodal domains determined by the zeros of $v_1(r)$, $r_1 < r_2 < \cdots$ where $\lim_{k\to\infty} r_k = \infty$. In each annular nodal domain $G_{(k)} \equiv \{(r, \theta): r_{k-1} < r < r_k\}$, the function $v_1(r)$ is the first eigenfunction of

$$\begin{aligned} \Delta v + \gamma(r)v + \mu v &= 0 \quad \text{in } G_{(k)} \\ v(r, \theta) &= 0 \quad \text{on } \partial G_{(k)} \end{aligned} \tag{2.5}$$

corresponding to $\mu_1 = 0$. Furthermore the level curves (surfaces) of v_1^2 are circles (spheres). Therefore if $\varphi(x) = v_1^2(x)$ in Theorem 2.1 and

$$G_{(k)}(y) = \{x: \varphi(x) \geqq y\}$$

then by (2.2)

$$\int_{G_{(k)}(y)} [c(r, \theta) - \gamma(r)] dx = 0$$

for all $y \geqq 0$. Therefore by Theorem 2.1

$$\int_{G_{(k)}} c v_1^2 dx = \int_{G_{(k)}} \gamma v_1^2 dx$$

for $k = 1, 2, \cdots$.

Consider now the eigenvalue problem

$$(2.6)\qquad \begin{aligned} \Delta u + c(x)u + \lambda u &= 0 \text{ in } G_{(k)} \\ u &= 0 \text{ on } \partial G_{(k)} . \end{aligned}$$

By the standard variational characterization of λ_1 we have

$$\lambda_1 = \min_{u \in \Phi} \frac{\int_{G_{(k)}} (|\nabla u|^2 - cu^2)dx}{\int_{G_{(k)}} u^2 dx} \leqq \frac{\int_{G_{(k)}} (|\nabla v_1|^2 - cv_1^2)dx}{\int_{G_{(k)}} v_1^2 dx}$$

$$= \frac{\int_{G_{(k)}} (|\nabla v_1|^2 - \gamma v_1^2)dx}{\int_{G_{(k)}} v_1^2 dx} = 0 ,$$

where Φ denotes the class of "admissible" functions for (2.6). Therefore $\lambda_1 \leqq 0$ and, by classical variational techniques, for each $G_{(k)}$ there exists a subdomain $G'_{(k)} \subseteqq G_{(k)}$ for which the eigenvalue problem

$$(2.6')\qquad \begin{aligned} \Delta u + cu + \lambda' u &= 0 \quad \text{in } G'_{(k)} \\ u &= 0 \quad \text{on } \partial G'_{(k)} \end{aligned}$$

satisfies $\lambda'_1 = 0$. This completes the proof.

An application of the Sturmian comparison theorem for elliptic equations [7] yields the following result.

COROLLARY 2.3. *If* (2.3) *is oscillatory at* $r = \infty$ *and* $u(x)$ *is a solution of* (2.1), *then* $u(x)$ *has a zero in* E_R *for every* $R > 0$.

3. The case $n = 2$. In case $n = 2$ equation (2.3) becomes

$$(3.1)\qquad \frac{d}{dr}\left(r\frac{dv}{dr}\right) + r\gamma(r)v = 0 ,$$

and the Leighton oscillation criterion [8] asserts that (3.1) is oscillatory at $r = \infty$ if

$$(3.2)\qquad \int_0^\infty r\gamma(r)dr = +\infty .$$

Recalling the definition of $\gamma(r)$ we have

$$(3.3)\qquad \int_0^\infty r\gamma(r)dr = \frac{1}{2\pi}\int_0^\infty\left[\int_0^{2\pi} c(r, \theta)d\theta\right]r\, dr = \frac{1}{2\pi}\iint_{E^2} c(x, y)dx\, dy .$$

These observations together with Theorem 2.2 yield

THEOREM 3.1. *If*

$$\iint_{E^2} c(x, y)dxdy = +\infty$$

then $\Delta u + cu = 0$ *is nodally oscillatory in* E^2.

The transformation $t = \ln r$ transforms equation (3.1) into

$$\frac{d^2v}{dt^2} + e^{2t}\gamma(e^t)v(t) = 0 \tag{3.4}$$

and leaves oscillation invariant. A result of Hille [5] asserts that (3.4) is oscillatory if

$$\frac{1}{4} < \liminf_{t\to\infty} t\int_t^\infty e^{2s}\gamma(e^s)ds \ . \tag{3.5}$$

Using the change of variables $s = \ln r$ and equation (3.3) we obtain

THEOREM 3.2. *If*

$$\frac{\pi}{2} < \liminf_{r\to\infty}(\ln r)\int_{E_r}\int c(x, y)dxdy$$

then $\Delta u + cu = 0$ *is nodally oscillatory in* E^2.

Nehari [9] shows that if $\gamma(r) \geqq 0$ and

$$\limsup_{t\to\infty} t\int_t^\infty e^{2s}\gamma(e^s)ds = \infty \tag{3.6}$$

then

$$\frac{d^2v}{dt^2} + \lambda e^{2t}\gamma(e^t)v(t) = 0$$

is oscillatory for all positive λ. Using Nehari's result, the change of variables $s = \ln r$, and Theorem 2.2, we obtain

THEOREM 3.3. *If* $c(x, y) \geqq 0$ *and*

$$\limsup_{r\to\infty}(\ln r)\int_{E_r}\int c(x, y)dxdy = \infty$$

then $\Delta u + \lambda c(x, y)u = 0$ *is nodally oscillatory for all positive* λ.

4. The case $n > 2$. In the case $n > 2$ equation (2.3) does not satisfy the conditions of the Leighton oscillation criterion [8]. However this difficulty can be overcome by making the oscillation-preserving transformation $\varphi(r) = r^{(n-1)/2}v(r)$. Equation (2.3) then becomes

$$\varphi'' + [\gamma(r) - (n-1)(n-3)/4r^2]\varphi = 0 \ . \tag{4.1}$$

The Leighton oscillation criterion applied to (4.1) together with Theorem 2.2 yields

THEOREM 4.1. *If*

$$\int^{\infty} \gamma(r)dr = \infty$$

then $\Delta u + cu = 0$ *is nodally oscillatory in* E^n.

THEOREM 4.2. *If*

$$\frac{(n-2)^2}{4} < \liminf_{r\to\infty} r^2\gamma(r) \tag{4.2}$$

then $\Delta u + cu = 0$ *is nodally oscillatory in* E^n.

Proof. Condition (4.2) implies that there exist constants r_0 and k such that

$$\frac{(n-2)^2}{4} < k < r^2\gamma(r) \text{ for } r_0 < r .$$

Thus the Euler equation $(r^{n-1}v')' + kr^{n-3}v = 0$ is oscillatory and (2.3) is oscillatory by Sturm's comparison theorem.

Theorem 4.2 improves similar results by Glazman [2] and Headley [4]. We shall now extend Theorem 3.2 to the case $n > 2$. A simple application of Hille's classical result (3.4) to equation (4.1), together with Theorem 2.2, yields

THEOREM 4.3. *If*

$$\frac{(n-2)^2}{4} < \liminf_{r\to\infty} r\int_r^{\infty} \gamma(r)dr$$

then $\Delta u + cu = 0$ *is nodally oscillatory in* E^n.

Theorem 4.3 improves a result of Swanson [10].

Using (3.5) and equation (4.1) we obtain

THEOREM 4.4. *If*

$$\limsup_{r\to\infty} r\int_r^{\infty} \gamma(r)dr =$$

then $\Delta u + \lambda cu = 0$ *is nodally oscillatory in* E^n *for all positive* λ.

5. A more general equation. In this section we shall sketch the proof of oscillation criteria for the more general elliptic equation

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) + c(x)u = 0 \tag{1.1}$$

in E^n. We denote by $N(r, \theta)$ the largest eigenvalue of the matrix $A(x) = (a_{ij}(x))$ at $x = (r, \theta)$ and define

$$\nu(r) = \frac{1}{\sum_n}\int_{\Omega} N(r, \theta)d\theta .$$

By classical variational principles the first eigenvalue of

$$\begin{aligned} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial}{\partial x_i}\right) + c(x)u + \lambda u &= 0 \quad \text{in } G \\ u &= 0 \quad \text{on } \partial G \end{aligned} \tag{5.1}$$

is no larger than the first eigenvalue of

$$\begin{aligned} \nabla\cdot(N(r, \theta)\nabla u) + c(r, \theta)u + \lambda u &= 0 \quad \text{in } G \\ u &= 0 \quad \text{on } \partial G . \end{aligned} \tag{5.2}$$

Thus to show that (5.1) is nodally oscillatory in E^n it is sufficient to show that (5.2) is nodally oscillatory. To that end, consider the eigenvalue problem

$$\begin{aligned} \nabla\cdot(\nu(r)\nabla v) + \gamma(r)v + \mu v &= 0 \quad \text{in } G \\ v &= 0 \quad \text{on } \partial G \end{aligned} \tag{5.3}$$

and the related elliptic equation

$$\nabla\cdot(\nu(r)\nabla v) + \gamma(r)v = 0 \tag{5.4}$$

where $\gamma(r)$ is defined as in §2. Equation (5.4) has solutions $v(r)$ which are independent of θ and can be found by solving the ordinary differential equation

$$\frac{d}{dr}\left(r^{n-1}\nu(r)\frac{dv}{dr}\right) + r^{n-1}\gamma(r)v = 0 . \tag{5.5}$$

THEOREM 5.1. *If the ordinary differential equation* (5.5) *is oscillatory at* $r = \infty$, *then* (1.1) *is nodally oscillatory in* E^n.

Proof. As in the proof of Theorem 2.2 let $r_1 < r_2 < \cdots$ be the zeros of a solution $v_1(r)$ and consider the domain

$$G_{(k)} = \{(r, \theta) : r_k < r < r_{k+1}\} .$$

Then $v_1(r)$ is an eigenfunction of (5.3) corresponding to $\mu_1 = 0$. Since the level curves (surfaces) of v_1 and of $\nabla v_1 = \partial v/\partial r$ are circles (spheres) it follows as before that

$$\int_{G(k)} \gamma v_1^2 dx = \int_{G(k)} c v_1^2 dx$$

and

$$\int_{G(k)} N|\nabla v_1|^2\, dx = \int_{G(k)} \nu|\nabla v_1|^2\, dx$$

for $k = 1, 2, \cdots$. The first eigenvalue of (5.2) is given by

$$\lambda_1 = \min_{u\in\Phi} \frac{\int_{G(k)} (N|\nabla u|^2 - cu^2)dx}{\int_{G(k)} u^2 dx} \leqq \frac{\int_{G(k)} (N|\nabla v_1|^2 - cv_1^2)dx}{\int_{G(k)} v_1^2 dx}$$

$$= \frac{\int_{G(k)} (\nu|\nabla v_1|^2 - \gamma v_1^2)dx}{\int_{G(k)} v_1^2 dx} = \mu_1 = 0\ .$$

Therefore $\lambda_1 \leqq 0$, and the remainder of the proof follows as in Theorem 2.2.

6. Unbounded domains. In this section we shall study the oscillatory behavior of solutions of $\nabla u + cu = 0$ in unbounded domains G. For the sake of simplicity we shall restrict our attention to the case $n = 2$, even though some of the results clearly generalize to E^n.

Consider first a conical domain

$$G = \{(r, \theta) \in E^2 \colon \alpha < \theta < \beta\}$$

and suppose that $c(x)$ is continuous in G. Defining

$$\gamma(r) = \frac{1}{\beta - \alpha}\int_\alpha^\beta c(r, \theta)d\theta\ ,$$

we shall obtain oscillation criteria for certain solutions of $\Delta u + cu = 0$ in G. If the ordinary differential equation

$$\frac{d}{dr}\left(r\frac{dv}{dr}\right) + r\gamma(r)v = 0 \tag{6.1}$$

is oscillatory at $r = \infty$, then $v(r)$ is also a solution of

$$\Delta v + \gamma(r)v = 0 \text{ in } G \tag{6.2}$$

satisfying $\partial v/\partial\nu = 0$ on ∂G, where $\partial v/\partial\nu$ denotes the exterior normal derivative. Furthermore if $v_1(r)$ is a solution of (6.1) with zeros at $r_1 < r_2 < \cdots$, $\lim_{k\to\infty} r_k = \infty$, then $v_1(r)$ defines a sequence of domains

$$G_{(k)} = \{(r, \theta): r_k < r < r_{k+1}; \alpha < \theta < \beta\}$$

in which the first eigenvalue of

$$\begin{aligned} \Delta v + \gamma(r)v + \mu v &= 0 \quad \text{in } G_{(k)} \\ v &= 0 \quad \text{on } \partial G_{(k)} \cap \{r: r = r_k \text{ or } r = r_{n+1}\} \\ \frac{\partial v}{\partial \nu} &= 0 \quad \text{on } \partial G_{(k)} \cap \{\theta: \theta = \alpha \text{ or } \theta = \beta\} \end{aligned} \tag{6.3}$$

satisfies $\mu_1 = 0$ for $k = 1, 2, \cdots$.

By an argument analogous to that used in Theorem 2.2 one concludes that the first eigenvalue of

$$\begin{aligned} \Delta u + c(x)u + \lambda u &= 0 \text{ in } G_{(k)} \\ u &= 0 \text{ on } \partial G_{(k)} \cap \{r: r = r_k \text{ or } r = r_{k+1}\} \\ \frac{\partial u}{\partial \nu} &= 0 \text{ on } \partial G_{(k)} \cap \{\theta: \theta = \alpha \text{ or } \theta = \beta\} \end{aligned} \tag{6.4}$$

satisfies $\lambda_1 \leqq 0$ for $k = 1, 2, \cdots$. An application of the Sturmian comparison theorem of [7] yields the following result.

THEOREM 6.1. *If $u(x)$ is a solution of $\Delta u + cu$ in the cone G satisfying $\partial u/\partial \nu = 0$ on ∂G, and if (6.1) is oscillatory at $r = \infty$, then $u(x)$ has a zero in G_R for every $R > 0$.*

THEOREM 6.2. *If $u(x)$ is a solution of $\Delta u + cu = 0$ in the cone G satisfying $\partial u/\partial \nu = 0$ on ∂G, and if*

$$\iint_G c(x, y)dxdy = +\infty ,$$

then $u(x)$ has a zero in G_R for every $R > 0$.

Proof. From the definition of $\gamma(r)$,

$$\int_0^\infty r\gamma(r)dr = \frac{1}{\beta - \alpha}\int_0^\infty\left[\int_\alpha^\beta c(r, \theta)d\theta\right]r\, dr = \frac{1}{\beta - \alpha}\iint_G c(x, y)dx\, dy .$$

By the Leighton oscillation criterion [8], the fact that $\int_0^\infty r\gamma(r)dr = +\infty$ implies that (6.1) is oscillatory at $r = \infty$. An application of Theorem 6.1 now completes the proof.

In E^2 we can establish oscillation criteria in more general domains by means of conformal mapping. As an example of such results, suppose Γ is an unbounded domain in E^2 and that

$$\xi = \xi(x, y);\ \eta = \eta(x, y)$$

defines a conformal map of Γ onto the cone G. Assuming that the Jacobian $J(\xi, \eta/x, y)$ is bounded assures that the singularity at ∞ is preserved under such a map. Furthermore, solutions of

$$\begin{aligned} u_{xx} + u_{yy} + c(x, y)u &= 0 \quad \text{in } \Gamma \\ \frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial\Gamma \end{aligned} \tag{6.5}$$

are transformed into solutions of

$$\begin{aligned} u_{\xi\xi} + u_{\eta\eta} + C(\xi, \eta)J\left(\frac{x, y}{\xi, \eta}\right) &= 0 \quad \text{in } G \\ \frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial G \end{aligned}$$

where $C(\xi, \eta) = c(x(\xi, \eta), y(\xi, \eta))$. Since

$$\iint_\Gamma c(x, y)dx\, dy = \iint_G C(\xi, \eta)J\left(\frac{x, y}{\xi, \eta}\right)d\xi\, d\eta\ ,$$

the condition

$$\iint_\Gamma c(x, y)dx\, dy = +\infty$$

assures that solutions of

$$\begin{aligned} \Delta u + cu &= 0 \quad \text{in } \Gamma \\ \frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial\Gamma \end{aligned}$$

will have zeros in Γ_R for all $R > 0$.

Added in revision. The authors have learned of some recent work of E. Noussair, to appear in the Journal of Differential Equations, which establishes related results for elliptic equations of even order. In particular, our Theorem 3.1 is a special case of Noussair's results. We note, however, that Noussair's techniques are different, requiring substantial machinery of a variational nature, and that our techniques can also be extended to deal with the case of even order elliptic equations.

Added in Proof. Noussair's paper has since appeared in the *J. Differential Equations*, 10 (1971), 100–111. C. A. Swanson has also shown us a shorter proof of Theorem 2.2 based on a paper by Swanson and Headley in the *Pacific Journal of Mathematics*, 27 (1968), 501–506. This proof will appear in the *Canadian Mathematical Bulletin*.

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UNIVERSITY OF CALIFORNIA, DAVIS
AND
VANDERBILT UNIVERSITY