PACIFIC JOURNAL OF MATHEMATICS
Vol. 41, No. 3, 1972

ON COMMUTATIVE *P.P.* RINGS

M. W. EVANS

The purpose of this paper is to study further the ideal and module structure of a commutative ring with identity, in which every principal ideal is projective. Results concerning particular modules being projective are also obtained, e.g. if R is a commutative ring with identity, then $Z_R(R_R) = 0$ and every finitely generated nonsingular R-module is projective if and only if R is semihereditary and K, the classical ring of quotients of R, is selfinjective.

A ring R is said to be a right *P.P.* ring if every right principal ideal of R is projective. These rings have been considered by Hattori [6] and by Endo [4], [5].

If R is a commutative ring with identity it can be shown that R is a *P.P.* ring if and only if for each $x \in R$, $r(x) = \{t \in R \mid xt = 0\} = eR$ for some idempotent $e \in R$. This latter property was used by Kist [9] to define 'commutative Baer rings'. In this paper, however by a *Baer ring* we will mean a ring R, with identity, such that for each subset $S \subseteqq R$, $r(S) = \{t \in R \mid St = 0\} = eR$, where e is an idempotent of R. This is the definition used by Kaplansky [8, p. 2].

1. Notations and terminologies. Throughout this paper, unless otherwise indicated, a ring R is an associative ring with identity; all modules are unitary.

Given a subset S of a module M we set, as usual, $r.\operatorname{ann}_R(S) = \{x \in R \mid Sx = 0\}$ and we abbreviate this to $r(S)$ if no ambiguity arises. The notion $l.\operatorname{ann}_R(S) = l(S)$ is similarily defined; over a commutative ring no distinctinction is made between $l(S)$ and $r(S)$. If N is a submodule of M we set $(N: M) = r.\operatorname{ann}_R(M/N)$.

For all homological notions used in this paper, the reader is referred to [10].

Throughout this paper, K will denote the classical ring of quotients of a commutative ring R. Spec R will denote the space of prime ideals of commutative ring R, while Minp R will denote the space of minimal prime ideals of R. Details of Minp R may be found in [7]. If R is a commutative *P.P.* ring, let e_x be the unique idempotent such that $r(x) = e_xR$.

By a *regular ring* we mean a von Neumman regular ring, that is a ring with the property that every finitely generated right (left) ideal is generated by an idempotent. Regular rings, thus are in particular *P.P.* rings.

2. Quasi-regular rings. Quasi-regular rings were first discussed by Endo in [5].

DEFINITION 2.1. *A commutative ring R is said to be quasi-regular if the classical quotient ring K of R is a regular ring.*

THEOREM 2.2. *For a commutative ring R, the following are equivalent*:

(1) *For all $x \in R$ there exists $x' \in R$ such that $rr(x) = r(x')$ and R is a semiprime ring.*

(2) *For all $x \in R$ there exists a nonzero divisor $d \in R$ such that $xd = x^2$.*

(3) *R is a quasi-regular ring.*

Proof. (1) implies (2). It will first be shown that $x + x'$ is a nonzero divisor of R. If $(x + x')s = 0$, then $xs = -x's$ and hence $xs \in rr(x) \cap r(x)$. It is a consequence of R being a semiprime ring that $rr(x) \cap r(x) = 0$ and hence $s \in r(x)$. Similarily $s \in r(x') = rr(x)$ and so $s = 0$. The result follows by observing $x(x + x') = x^2$.

(2) implies (3). Let $xd^{-1} \in K$ where $x \in R$ and d is a nonzero divisor of R. By (2), there exists a nonzero divisor $u \in R$ such that $xu = x^2$. Hence $x^2(d^{-1})^2du^{-1} = xd^{-1}$, which implies K is a regular ring.

(3) implies (1). Let $x \in R \subseteqq K$. Then, as K is a $P.P.$ ring $r.\mathrm{ann}_K(x) = (sd^{-1})K$ where sd^{-1} is an idempotent of K, $s \in R$ and d a nonzero divisor of R. Hence $r.\mathrm{ann}_R(r.\mathrm{ann}_R(x)) = r.\mathrm{ann}_K(r.\mathrm{ann}_K(x)) \cap R = r.\mathrm{ann}_K(sd^{-1}) \cap R = r.\mathrm{ann}_R(s)$. Finally R is semiprime as K is.

REMARK 1. Since every quasi-regular ring is semiprime, condition (2) expresses the fact that for each $x \in R$ there is a nonzero divisor $d \in R$ such that $x \leqq d$, where $\leqq$ is the partial ordering defined on any semiprime ring by $x \leqq y$ if and only if $xy = x^2$, [1].

REMARK 2. Condition (1) was introduced in Theorem 3.4 of [7]. If R is a semiprime ring this condition implies Minp R is compact. It has been stated in the paper of Henriksen and Jerison [7] and later in the paper of Mewborn [12] that an example of a semiprime ring R with Minp R compact, but which does not satisfy condition (1) of the Theorem, has not been found.

COROLLARY. *Every commutative $P.P.$ ring is a quasi-regular ring.*

Proof. For each $x \in R$, $rr(x) = r(e_x)$ where $r(x) = e_xR$ and the other half of condition (1) of the Theorem 2.2 is proved in the following lemma.

LEMMA 2.3. *Let R be a commutative ring in which every principal ideal is flat. Then R is a semiprime ring.*

Proof. Let $x^2 = 0$. Consider the exact sequence $0 \to r(x) \to R \to xR \to 0$. As xR is flat $0 \to r(x) \otimes_R xR \to R \otimes_R xR \to xR \otimes_R xR$ is an exact sequence. Hence by Proposition 1 of §5.4 of [10] it follows that $0 \to r(x).xR \to xR \to x^2R \to 0$ is exact: i.e. $0 \to 0 \to xR \to x^2R \to 0$ is exact and as $x^2 = 0$, $x = 0$.

REMARK 3. The above Corollary is contained in Endo's Proposition 1 [4, p. 168], which we record here as we shall have occasion to remark on it again.

PROPOSITION 2.4. *If R is a commutative ring, then R is a P.P. ring if and only if K is regular and R_V is an integral domain for each maximal ideal V of R.*

Quasi-regular rings are analogous to distributive $*$-lattices [16]. The following proposition has an analogue in distributive lattices. This has been given in [16] and so the proof will not be given here.

PROPOSITION 2.5. *If R is a semiprime ring, then the following are equivalent:*

(1) *For all $x \in R$, there exists $x' \in R$ such that $rr(x) = r(x')$.*

(2) *For all $x \in R$, there exists an $x' \in R$ such that $xx' = 0$ and $x + x'$ is a non zero divisor.*

(3) *If P is a prime ideal of R, which contains only zero divisors then P is a minimal prime ideal.*

REMARK 4. The Baer extension [9, p. 46] of a quasi-regular ring is simply the ring generated by R and the idempotents of K.

3. Modules in which every cyclic submodule is projective. A right R module is said to be a *C.P.* module if every right cyclic submodule is projective.

PROPOSITION 3.1. *If R is a ring and A_R is a right R-module, then the following are equivalent:*

(1) *A_R is a C.P. module.*

(2) *For each $x \in A_R$, $r(x) = eR$ for some idempotent $e \in R$.*

Proof. (1) implies (2). Consider the exact sequence $0 \to r(x) \xrightarrow{i} R \xrightarrow{j} xR \to 0$, where i is the imbedding map and j: $a \to xa$. Then, as

xR is projective, the exact sequence splits and $r(x)$ is a direct summand of R.

(2) implies (1). Suppose xR is a cyclic submodule. Then $xR \cong R/r(x) = R/eR$. As $R/eR \cong (1-e)R$ by the correspondence $r/eR \leftrightarrow (1-e)r$, $xR \cong (1-e)R$. Hence xR is projective.

THEOREM 3.2. *For a ring R the following are equivalent:*
(1) *R is right P.P.*
(2) *Every free right R-module F_R is a C.P. module.*
(3) *Every projective right R-module P_R is a C.P. module.*

Proof. (1) implies (2). It suffices to prove (2) in the case $F_R = R^{(n)} = \{(x_1, x_2 \cdots x_n) \nmid x_i \in R\}$ for some positive integer n. Thus suppose $n > 1$ and let $xR \subseteqq R^{(n)}$ where $x = (x_1, \cdots, x_n) \in R^{(n)}$. Let $\pi\colon R^{(n)} \to R$ be the map given by $\pi(r_1, \cdots, r_n) = r_1$ (i.e. the projection on to the first component of $R^{(n)}$) and let $\bar{\pi} = \pi | xR$. Then the exact sequence $0 \to \ker \bar{\pi} \xrightarrow{i} xR \xrightarrow{\bar{\pi}} \operatorname{Im} \bar{\pi} \to 0$ splits as $\operatorname{Im} \bar{\pi}$ is a principal right ideal of R. It follows that $xR \cong \ker \bar{\pi} \oplus \operatorname{Im} \bar{\pi}$, where $\ker \bar{\pi}$ is a cyclic submodule of R^{n-1} and so projective by the induction assumption.

xR is a projective R-module as it is a direct sum of two projective right R-modules.

(2) implies (3). Trivial.

(3) implies (1). Trivial.

Next in this section we obtain a characterization of commutative P.P. rings. The following lemma will be required for the proof of this characterization. For modules $N \subseteqq M$ we write $N \subseteqq' M$ when M is an essential extension of N. For the definition of essential the reader is referred to the book of Lambek [10, p. 90].

LEMMA 3.3. *Let $R \subseteqq S$ be rings (with the same identity) such that $R_R \subseteqq' S_R$. Then for each $s \in S$, $r.ann_R(s)$ is generated by an idempotent of R if and only if $r.ann_S(s)$ is generated by an idempotent of R.*

Proof. Let $s \in S$ and suppose $r.\text{ann}_R(s) = eR$, where $e^2 = e \in R$. Since $r.\text{ann}_R(s) \subseteqq' r.\text{ann}_S(s)$ (as R-modules) it follows that $eS \subseteqq' r.\text{ann}_S(s)$ (as S-modules) and so $r.\text{ann}_S(s) = eS$.

Next suppose $r.\text{ann}_S(s) = eS$ for some $e^2 = e \in R$. We then have $eR \subseteqq' eS \cap R = r.\text{ann}_R(s)$ and so $eR = r.\text{ann}_R(s)$.

THEOREM 3.4. *For a commutative ring R with total quotient ring K the following are equivalent:*
(1) *R is a P.P. ring.*
(2) *K is a P.P. ring and every idempotent of K is an element*

of R.

(3) *Any ring* Λ *such that* $R \subseteqq \Lambda \subseteqq K$, *is a P.P. ring.*

[Note: All rings in (3) share the identity of R.]

Proof. (1) implies (2). By Proposition 3.1 it suffices to that for each $k \in K$, $r.\text{ann}_K(k) = eK$ where e is an idempotent element of R. Thus let $k \in K$ and so $k = ad^{-1}$ where $a \in R$ and d is a nonzero divisor of R. The map $kR \rightarrow R$ given by $kr \rightarrow dkr$ is a module imbedding and so kR is projective. It follows that the epimorphism $R \rightarrow kR$ given by $r \rightarrow kr$, splits and so $r.\text{ann}_R(k) = \ker(R \rightarrow kR) = eR$ for some $e^2 = e \in R$. Lemma 3.3 now gives $r.\text{ann}_K(k) = eK$.

(2) implies (3). This is a clear consequence of Lemma 3.3.

(3) implies (1). Trivial.

We now have a corollary to Proposition 2.4. [Endo].

COROLLARY. *Suppose* R *is a quasi-regular ring. For any ring* Λ *such that* $R \subseteqq \Lambda \subseteqq K$ *the following are equivalent*:

(1) R_V *is an integral domain for every maximal ideal* V *of* R.

(2) Λ_V *is an integral domain for every maximal ideal* V *of* Λ.

4. Baer ideals and torsion free R**-modules.** Throughout this section R is assumed to have a right ring of quotients as defined by Levy [11, p. 133]. Any commutative ring has such a ring of quotients.

If M is a right R-module, let $T(M) = \{m \in M \nmid md = 0;$ for some nonzero divisor d of $R\}$. $T(M)$ is a submodule of R [11, Theorem 1.4]. M is said to be *torsion-free* if $T(M) = 0$.

If J is a right ideal of R, let J_B be the right ideal of R such that $T(R/J) = J_B/J$. Ideals such that $J_B = J$, have previously been used by Cateforis and Sandomierski [2, p. 162].

PROPOSITION 4.2. *Let* R *be a ring,* S *the right ring of quotients of* R *and* J *a right ideal of* R. *Then the following are equivalent*:

(1) $T(R/J) = 0$.

(2) *There exists a right ideal* J' *of* S *such that* $J' \cap R = J$.

(3) $J_B = J$.

Proof. (1) implies (2). Clearly $J \subseteqq JS \cap R$. Conversely if $x \in JS \cap R$, then $x = asd^{-1}$, where $a \in J, s \in R$ and d is a nonzero divisor of R. Hence $xd = as \in J$. Thus as $T(R/J) = 0, x \in J$.

(2) implies (3). If $xd \in J$, then $xdd^{-1} \in JS \subseteqq J'$ which implies $x \in J$; i.e. $J_B = J$.

(3) implies (1). If M is any right R-module $T(M/T(M)) = 0$. Hence $T(R/J_B) = 0$.

PROPOSITION 4.3. *Let R be a ring, J a right ideal of R. Then*

(1) *$J_B = R$ if and only if J contains a nonzero divisor.*

(2) *$T(R/J_B) = 0$, and if $J \subseteq I$ a right ideal such that $T(R\backslash I) = 0$, then $J_B \subseteq I$.*

Proof (1). Since $T(R/J) = R/J$ if and only if J contains a nonzero divisor, the result follows.

(2) From Proposition 4.2. $T(R/J_B) = 0$. Now suppose $x \in J_B$. Then $xd \in J \subseteq I$ for some nonzero divisor d of R. Hence as $T(R/J) = 0$, $x \in I$.

We now look at ideals J of a ring R such that $T(R/J) = 0$ when R is a quasi-regular or commutative *P.P.* ring. Properties of these ideals in quasi-regular rings have been looked at by Endo [5, p. 111–112]. From Proposition 3.5 we have the following.

COROLLARY 4.4. *If R is a commutative semiprime ring then the following are equivalent:*

(1) *R is a quasi-regular ring.*

(2) *If P is a prime ideal of R, $T(R/P) = 0$ if and only if it is a minimal prime ideal.*

COROLLARY 4.5. *If R is a commutative quasi-regular ring then the following are equivalent:*

(1) *If J is an ideal of R, then R, then $T(R/J) = 0$.*

(2) *R is a regular ring.*

Proof. (1) implies (2). From Corollary 4.4 this means that every prime ideal of R is a minimal prime ideal, and hence maximal ideal of R. Hence R_V is a field for each maximal ideal V of R. The result follows from Theorem 1 of Endo [5].

(2) implies (1). Every nonzero divisor of a regular ring is a unit.

PROPOSITION 4.6. *Let R be a commutative P.P. ring, J an ideal of R then the following are equivalent:*

(1) *$T(R/J) = 0$.*

(2) *If $x - y \in J$ then $e_x - e_y \in J$.*

(3) *R/J is a flat R-module.*

(4) *If $x \in J$ then $rr(x) \subseteq J$.*

(5) *$J = \cap \{M \in Minp\, R \nmid J \subseteq M\}$.*

Proof. (1) implies (2). It will first be shown that if $s, x \in R$, then $sx \in J$ if and only if $se_x - s \in J$. If $se_x - s \in J$ then $-xs = x(se_x - s) \in J$. Hence $xs \in J$.

Conversely if $xs \in J$, then $(se_x - s)(x + e_x) = -sx + se_x - se_x =$

$sx \in J$.

Now if $x - y \in J$, then $x + J = y + J$. From the above it can be seen that this implies $(e_x + J)R/J = r.\text{ann}_{R/J}(x/J) = r.\text{ann}_{R/J}(y/J) = (e_y + J)R/J$ i.e. $e_x - {}_y \in J$.

(2) implies (3). Let $x \in J$. Then $1 - e_x \in J$ and $x(1 - e_x) = x$. Thus R/J is flat [11, Ex 3, p. 135].

(3) implies (1). If $xd \in J$, then by [11, Ex 3. p. 135] there exists a $c \in J$ such that $xdc = xd$, which implies $x = xc \in J$.

The equivalence of (2), (4) and (5) has been shown by T. P. Speed [17].

REMARK 1. Conditions (1), (4) and (5), of the above theorem, are equivalent for a quasi-regular ring. Furthermore it can be shown as a corollary of Theorem 3.1 of Mewborn [12], that conditions (4) and (5) are equivalent for a semiprime ring R if and only if Minp R is compact.

In [18] ideals satisfying property (2) were called *Baer ideals*. We will continue to use this name.

REMARK 2. In [18] a Baer homomorphism between two commutative *P.P.* rings, R and R' was defined to be a ring homomorphism f satisfying the additional property $f(e_x) = e_{f(x)}$, where $r.\text{ann}_{R'}(f(x)) = e_{f(x)}R'$, for all $x \in R$. It was shown that an ideal J of R is the kernel of a Baer homomorphism if and only if J satisfied condition (2) of the above theorem.

The Baer ideals of R, a commutative *P.P.* ring, form a pseudo complemented lattice [17], which we will denote by $I^B(R)$. If S is a commutative ring, denote the lattice of ideals of S by $I(S)$ and let $B(S)$ denote the Boolean algebra of idempotents of S.

PROPOSITION 4.7. *Let R be a commutative P.P. ring. Then $I(K) \cong I(B(K)) \cong I(B(R)) \cong I^B(R)$, where the isomorphisms are lattice isomorphisms.*

Proof. It can be seen from Proposition 4.2 (2) that $I(K) \cong I^B(R)$. It is well known that $I(K) \cong I(B(K))$ and $I(B(K)) \cong I(B(R))$ as $B(K) = B(R)$ (Theorem 3.4).

Finally in this section we look at the decomposition of commutative *P.P.* rings.

A submodule M of a right R-module A_R is said to be *large* if $M \cap N \neq 0$ for each nonzero submodule N of A_R. A large ideal is a large submodule of R. If R is a commutative semiprime ring, J is a large ideal if and only if $r(J) = 0$. A right R-module A_R is said to be *nonsingular* if $Z_R(A_R) = 0$ where $Z_R(A_R) = \{x \in R \mid r(x)$ is a large ideal of $R\}$.

THEOREM 4.8. *Over a commutative P.P. ring the following are equivalent*:

(1) *R has the a.c.c. on Baer ideals.*
(2) *R is a finite direct sum of integral domains.*
(3) *Every torsion-free R-module is a C.P. module.*
(4) *Every torsion-free R-module is a nonsingular module.*
(5) $Z(M) = T(M)$ *for every module M.*
(6) *Every large ideal contains a nonzero divisor of R.*

Proof. (1) implies (2). This follows from the lemma of Hattori [6, p. 156].

(2) implies (3). If A_R is a torsion-free R-module then each cyclic submodule of R is a torsion-free R-module. Hence if $x \in R$, $r(x)$ is a Baer ideal, (Proposition 4.6). It is a consequence of R being a finite direct sum of integral domains that all Baer ideals of R are idempotently generated.

(3) implies (4). Free modules over nonsingular rings are nonsingular modules.

(4) implies (5). If an ideal J contains a nonzero divisor d then $J \subseteq' R$ and so $T(M) \subseteq Z(M)$. Now $Z(M)/T(M) \subseteq Z(M/T(M))$ and since $T(M/T(M)) = 0$ we have $Z(M/T(M)) = 0$ or $Z(M) \subseteq T(M)$.

(5) implies (6). If J is a large ideal of R, then $Z(R/J) = R/J$ and so $T(R/J) = R/J$. Hence there is a nonzero divisor $d \in R$ such that $d(1 + J) = 0$ or $d \in J$.

(6) implies (1). If B is a large ideal of K then $B \cap R$ is a large ideal of R and so B contains a nonzero divisor of R. It follows that $B = K$, as d^{-1} exists in K, and so has no large ideals $\neq K$. K is thus artinian semisimple. Hence the ideals of K satisfy the a.c.c. It now follows from Proposition 4.7 that the Baer ideals of R satisfy the a.c.c.

COROLLARY 4.9. *If R is a commutative hereditary* (*semihereditary*) *ring R with identity, then R is a finite direct sum of Dedekind* (*Prüfer*) *domains if and only if R has the a.c.c. on Baer ideals.*

5. **Finitely generated nonsingular R-modules of a commutative semiprime ring.** In this section we introduce a third torsion theory. If A_R is an R-module of a commutative ring R, let $U(A) = \{x \in A \nmid lr(x) = 0\}$. This is the definition of 'torsion submodule' used by Pierce [13, p. 80–83]. Note that if R is an integral domain $U(A) = 0$ if and only if $T(A) = 0$. Also, if R is a semiprime ring $U(A) = 0$ if and only if $Z(A) = 0$ [11, Ex. 1. p. 108].

It was mentioned in the introduction of this paper, that by a Baer ring, we will mean a ring R with identity, such that for each subset $S \subseteqq R$, $r(S) = \{t \in R \nmid St = 0\} = eR$, where e is an idempotent of R. A commutative *P.P.* ring is a Baer ring if and only if its Boolean algebra of idempotents is complete. We will need the following lemma of Sandomierski [15, p. 226]. For the definition of closed submodule the reader is referred to [15].

LEMMA 5.1. *If B_R is a submodule of an R-module A_R, such tvht $Z(A/B) = 0$, then B is closed in A.*

THEOREM 5.2. *Let R be a commutative ring. Then the following are equivalent*:

(1) *R is a Baer ring.*

(2) *$Z_R(R) = 0$, and if A_R is a nonsingular R-module, then A_R is a C.P. module.*

(3) *If A_R is an R module such that $U(A) = 0$, then A is a C.P. module.*

(4) *If $S \subseteqq R$, then $R/r(S)$ is a projective R-module.*

Proof. (1) implies (2). Since R is a semiprime ring, the closed ideals of R are exactly the annihilator ideals of R. If A_R is a nonsingular R-module, then $x \in A_R$ implies $r(x)$ is an annihilator ideal of R (Lemma 5.1). The result now follows by Proposition 3.1 as R is a Baer ring.

(2) implies (3). Since $Z(R) = 0$, $U(A) = 0$ if and only if $Z(A) = 0$.

(3) implies (4). Let $S \subseteqq R$, where S is an arbitrary subset of R. It will first be shown that $U(R/r(S)) = 0$. Suppose that $r(S) \neq R$. Then there is an $x \in R$, such that $x \notin r(S)$. Now $r(x + r(S)) = \{t \in R \mid xt \in r(S)\} = r(xS)$. Hence if $rr(x + r(S)) = 0$, then $r(xS) = R$, which implies $xS = 0$, i.e. $x \in r(S)$. Thus $U(R/r(S)) = 0$ for $r(S) \neq R$. Hence as $R/r(S)$ is a cyclic R-module it is projective.

(4) implies (1). Since $R/r(S)$ is projective the exact sequence $0 \rightarrow r(S) \rightarrow R \rightarrow R/r(S) \rightarrow 0$ splits. Hence $r(S)$ is a direct summand of R, i.e. $r(S) = eR$ for some idempotent $e \in R$.

K. M. Rangaswamy and N. Vanaja have also considered the condition that every cyclic nonsingular R-module is projective. The above result generalised Proposition 1 of [14] for the commutative case.

The following proposition will be used in the proof of Theorem 5.4.

PROPOSITION 5.3. *Let J be a finitely generated ideal of a commutative quasi-regular ring R. Then if J does not contain a nonzero* *divisor*, $r(J) \neq 0$.

Proof. Let $J = \sum_{i=1}^{n} a_i R$ and suppose $Jx = 0$ implies $x = 0$. Then J is a large ideal of R and so JK is a large ideal of K. Since K is regular and JK is a finitely generated K-ideal (generated by the a_i) it follows that $JK = K$. Thus $1 \in JK$ and $1 = \sum_{i=1}^{n} x_i k_i = \sum_{i=1}^{n} x_i(a_i d_i^{-1})$ where $x_i \in J$, d_i is a nonzero divisor of R. Now by [12, Lemma 1.3] there exist $b_i \in R$ and a nonzero divisor $d \in R$ such that $a_i d_i^{-1} = b_i d^{-1}$ and so $1 = \sum x_i b_i d^{-1}$. It follows that $d \in J$, a contradiction.

Let Q denote the complete ring of quotients of R[10, p. 40]. Cateforis and Sandomierski have introduced the condition $Z_R(Q \otimes_R Q) = 0$. [2, p. 151]. The next proposition gives an equivalent condition to this when R is a quasi-regular ring.

THEOREM 5.4. *Let R be a quasi-regular ring, Q the complete ring of quotients of R, then the following are equivalent*:

(1) $Z_R(Q \otimes_R Q) = 0$.

(2) $Q = K$, *the classical ring of quotients of R.*

Proof. (1) implies (2). Q is a flat R-module as Minp R is compact [12, Thm 3.1]. As R is also semiprime it is possible to use Theorem 1.6 of [3]. Hence $Z(Q \otimes_R Q)$ is equivalent to the condition that for each $q \in Q$, $(R:_R q) = \{r \in R \mid rq \in R\}$ contains a finitely generated large submodule, J. Q is an essential extension of R and therefore J is a large ideal of R. Now if J is a finitely generated large ideal of R, $r(J) = 0$ and hence by Proposition 5.3 J must contain a nonzero divisor d. Finally, if $q \in Q$ there exists a nonzero divisor d of R such that $dq \in R \subseteqq K$ which implies $q \in K$.

(2) implies (1). As $K \otimes_R K \cong K$ it follows that $Z_R(Q \otimes_R Q) = 0$.

If R is a semiprime ring with Minp R compact and $Z_R(Q \otimes_R Q) = 0$, where Q is the complete ring of quotients of R, then $Q = \bar{K}$, the classical ring of quotients of the Baer extension of R. The Baer extension has been introduced in [9].

The following Proposition is due to Cateforis [3].

PROPOSITION 5.5. *For a ring R, the following are equivalent*:

(1) $Z(R_R) = 0$ *and every finitely generated nonsingular right R-module is projective.*

(2) *R is semihereditary, Q_R is flat and* $Z(Q \otimes_R Q) = 0$.

COROLLARY 5.6. *Let R be a commutative ring. Then the following are equivalent*:

(1) *If* A_R *is a finitely generated R-module and* $U(A) = 0$, *then* A_R *is projective.*

(2) $Z_R(R_R) = 0$ *and every finitely generated nonsingular R-module is projective.*

(3) *R is semihereditary and K the classical quotient ring of R is self injective.*

Proof. The proof is derived from Theorem 5.2, Theorem 5.4 and Proposition 5.5.

Theorem 24.5 of Pierce [13] can be obtained directly from this corollary. For if R is a regular ring, R is semihereditary and $R = K$.

REFERENCES

1. A. Abian, *Direct sum decomposition of atomic and orthogonally complete rings*, J. Australian Math. Soc., **11** (1970), 357-361.
2. V. C. Cateforis and F. L. Sandomierski, *The singular submodule splits off*, J. Algebra, **10** (1968), 140-165.
3. V. C. Cateforis, *On regular self-injective rings*, Pacific J. Math., **30** (1969), 39-44.
4. S. Endo, *Note on P.P. rings*, Nagoya Math. J., **17** (1960), 167-170.
5. ———, *On semihereditary rings*, J. Math. Soc. Japan, **13** (1961), 109-119.
6. A. Hattori, *A foundation of the torsion theory over general rings*, Nagoya Math. J., **17** (1960), 147-158.
7. M. Henriksen and M. Jerison, *The space of minimal prime ideals of a commutative ring*, Trans. Amer. Math. Soc., **115** (1965), 110-130.
8. I. Kaplansky, *Rings of Operators*, W. A. Benjamin, Inc; 1968.
9. J. Kist, *Minimal prime ideals in commutative semigroups*, Proc. London Math. Soc., (3) **13** (1963), 31-50.
10. J. Lambek, *Lectures on Rings and Modules*, Blaisdell publishing Co., Waltham Mass. 1966.
11. L. Levy, *Torsion free and divisible modules over non integral domains*, Canad. J. Math., **15** (1963), 132-151.
12. A. C. Mewborn, *Some conditions on commutative semiprime rings*, J. Algebra, **13** (1969), 422-431.
13. R. S. Pierce, *Modules over commutative regular rings*, Memoirs Amer. Math. Soc., No. 70 Providence R. I. (1967).
14. K. M. Rangaswamy and N. Vanaja, *Modules over regular rings*, Bull. Australian Math. Soc., **4** (1971), 57-61.
15. F. L. Sandomierski, *Nonsingular rings*, Proc. Amer. Math. Soc., **19** (1968), 225-230.
16. T. P. Speed, *Some remarks on a class of distributive latticess*, J. Australian Math. Soc., **9** (1969), 289-296.
17. ———, *A note on commutative Bear rings II*, (to appear).
18. T. P. Speed and M. W. Evans, *A note on commutative Baer rings*, J. Australian Math. Soc. (to appear).

Received March 19, 1971 and in revised form August 19, 1971. The author thanks the referee for many constructive comments and in particular for a proof of Theorem 3.2. in the noncommutative case. The author also acknowledges the work of W. H. Cornish, who has independently obtained some of the results of this paper.

MONASH UNIVERSITY