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# AN ADJUNCTION THEOREM FOR LOCALLY EQUICONNECTED SPACES

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**The locally equiconnected spaces (LEC spaces) can be characterized as the spaces $X$ with the property that if $f_0, f_1: Z \to X$ are mappings which are "sufficiently close together" and which agree on a subspace $A$ of $Z$, then $f_0$ is homotopic to $f_1$ relative to $A$ ($f_0 \cong f_1 \operatorname{rel} A$); i.e., there is a morphism $F: Z \times I \to X$ with $F \mid Z \times 0 = f_0, F \mid Z \times 1 = f_1$ and $F(a, t) = f_0 a$ for all $a \in A$ and $t \in I$.**

**The notion of "close" is measured by a morphism $\varphi: X \times X \to I$ with $\varphi(x, x') = 0$ if and only if $x = x'$. We then require that $\varphi(f_0\xi, f_1\xi) < 1$ for all $\xi \in Z$ implies that $f_0 \cong f_1 \operatorname{rel} A$.**

**There is a universal test pair $(u, D)$: let $u = \varphi^{-1}[0, 1>$ and $D = \varphi^{-1}0$. Let $f_0$ and $f_1$ be the restrictions to $u$ of the projections $X \times X \to X$ onto the first and second coordinates. Then $f_0$ and $f_1$ agree precisely on $D$. A homotopy $f_0 \cong f_1 \operatorname{rel} D$ exists if and only if $D$ is a strong deformation retract of $u$ in $X \times X$.**

**We note two things. First, the existence of a homotopy $f_0 \cong f_1 \operatorname{rel} D$ implies the existence of the homotopies in the general case described in the first paragraph. Second, the existence of a map $\varphi$ and homotopy $f_0 \cong f_1 \operatorname{rel} D$ is equivalent to the diagonal map**

$$\Delta: X \longrightarrow X \times X$$

**being a *cofibration*.**

We say more on this point below.

The class of LEC spaces has been the subject of considerable investigation (see Dugundji [1] for background) and such spaces have a number of convenient homotopy theoretic properties. To establish contact with a more familiar class of spaces, we recall that every metric absolute neighborhood retract (ANR) is LEC and that every finite dimensional metric LEC space is an ANR. (See [1]).

One of the beautiful results on ANR's is the Whitehead Adjunction Theorem for compact ANR's [7]. This has been the subject of several generalizations [3] and [4]. The object of this paper is to present an adjunction theorem for LEC spaces analogous to Whitehead's but with no restriction on the LEC spaces involved. A corollary is that every cell-complex is locally equiconnected.

**I. Preliminaries.** We do not wish to be too specific about the category of spaces under consideration. Specific categories for which all of the ensuing arguments are valid are the category of compactly generated Hausdorff spaces [6] (which includes metrizable spaces and locally compact Hausdorff spaces) and the category of quasi-topological spaces [5]. A third category in which they are valid is the category of compactly generated weakly Hausdorff spaces; i.e., spaces in which the topology is generated by continuous maps of compact Hausdorff spaces; there appears to be no material in print on this category.

The terms continuous mapping and morphism will be used interchangeably. A morphism $f: A \to X$ is an *injection* provided it is one-to-one into and has the property that a sufficient condition for a function $g: Y \to A$ to be a continuous mapping is that the composition $fg$ be one. If $f$ is an inclusion and an injection, then $A$ is said to have the *subspace topology* or the *induced topology*. Dually, a morphism $p: X \to B$ is a *projection* provided it is onto and has the property that a sufficient condition for a function $q: B \to Y$ to be a continuous mapping is that the composition $qp$ be one. In this case $B$ is said to have the *quotient* or *decomposition topology* determined by $p$.

A subset $A$ of the space $X$ has a *halo* in $X$ if there is a morphism $q: X \to I$ such that $A = q^{-1}0$. Such a morphism is called a haloing function for $A$. Note that only $G_\delta$ closed subsets can have haloes. If, in addition to having a halo in $X$, $A$ is also a retract of $X$, then $A$ is said to be a halo retract of $X$.

A morphism $k: X \to I$ is a *numeration* of an open set $U$ in $X$ provided $U = X - k^{-1}1$. An open set $U$ is numerable if there exists a numeration of it.

In each of the above categories, there are arbitrary function spaces $M(X, Y)$ in which the exponential law is valid and the evaluation map is a morphism.

For a space $X$ the *path space* $PX$ is the subspace of $M(R^+, X) \times R^+$, where $R^+$ is the half-line of nonnegative real numbers, of all pairs $(\alpha, l)$ with $\alpha(t) = \alpha(l)$ for all $t \geqq l$. A *path* is a point $(\alpha, l)$ of $PX$; the number $l$ is said to be the *length* of the path $(\alpha, l)$. We define two morphisms

$$\eta_0: PX \longrightarrow X$$

and

$$\eta_\tau: PX \longrightarrow X$$

by $\eta_0(\alpha, l) = \alpha 0$ and $\eta_\tau(\alpha, l) = \alpha l$.

We next state several lemmas and propositions to be used later. We omit proofs of the more routine of these.

LEMMA I.1. *Let $\varphi: X \to I$ be a morphism and $\psi: X \to I$ be a func-*

*tion such that*

(i) $\psi | u$ *is a morphism*, $u = \varphi^{-1} < 0, 1]$, *and*

(ii) $\psi \leqq \varphi$.

*Then* $\psi$ *is a morphism.*

LEMMA I.2. *Let* $\varphi: X \to I$, $\psi: Y \to I$, *and* $h: X \times I \to Z$ *be morphisms such that*

$$h(x, t) \text{ is independent of } t \text{ if } \varphi(x) = 0 .$$

*Let* $W$ *be the subspace of* $X \times Y \times I$

$$W = \{x, y, t \in X \times Y \times I \,|\, t\psi y \leqq \varphi x\}$$

*and* $l: W \to Z$ *be the function*

$$l(x, y, t) = \begin{cases} h(x, t\psi(y)/\varphi x) & \text{if } \varphi x \neq 0 \\ h(x, 0) & \text{if } \varphi x = 0 . \end{cases}$$

*Then* $l$ *is a morphism.*

*Proof.* Let $\gamma: C \to W$ be a morphism, where $C$ is compact Hausdorff. The map $\gamma$ is given by coordinates

$$\gamma_1: C \longrightarrow X, \gamma_2: C \longrightarrow Y \text{ and } \gamma_3: C \longrightarrow I .$$

The definition of $W$ requires that

$$\gamma_3 \cdot (\psi\gamma_2) \leqq \varphi\gamma_1 .$$

Let $\alpha: C \times I \to I \times I$ be defined by $\alpha(c, t) = (t\varphi\gamma_1 c, , \gamma_3 c \cdot \psi\gamma_2 c)$ and $D = \alpha^{-1}\{\Delta \cup 0 \times I\}$. Since $\alpha$ is continuous, $D$ is compact; and so, $\pi_1: D \to C$ is a projection since it is onto. The composition

$$D \xrightarrow{\pi_1} C \xrightarrow{\gamma} W \xrightarrow{l} Z$$

is

$$\begin{aligned} l\gamma\pi_1(c, t) = l\gamma c &= l(\gamma_1 c, \gamma_2 c, \gamma_3 c) \\ &= \begin{cases} h\left(\gamma_1 c, \gamma_3 c \dfrac{\psi\gamma_2 c}{\varphi\gamma_1 c}\right) & \text{if } \varphi\gamma_1 c \neq 0 \\ h(\gamma_1 c, 0) & \text{if } \varphi\gamma_1 c = 0 \end{cases} \\ &= \begin{cases} h(\gamma_1 c, t) & \text{if } \varphi\gamma_1 c \neq 0 \\ h(\gamma_1 c, 0) & \text{if } \varphi\gamma_1 c = 0 \end{cases} \\ &= h(\gamma_1 c, t) , \end{aligned}$$

which is a morphism. Since $\pi_1: D \to C$ is a projection, this implies $l\gamma$ is a morphism. But since this is true for every morphism $\gamma: C \to W$, it follows that $l$ is a morphism.

LEMMA I.3. *For $C$ compact Hausdorff, the function*

$$\sup\colon M(C, R) \longrightarrow R$$

*defined by* $\sup f = \sup \{fc \mid c \in C\}$ *is a morphism.*

PROPOSITION I.4. *The function*

$$j\colon PY \longrightarrow M(I, Y) \times R^{+}$$

*defined by* $j(\alpha, l) = (\hat{\alpha}, l)$, *where* $\hat{\alpha}t = \alpha(t\cdot l)$, *is an injection.*

*Proof.* It is clear that $j$ is a morphism and is a set-theoretic injection. We have only to show that if $f\colon C \to PY$ is a function, where $C$ is compact Hausdorff, such that $jf$ is a morphism, then $f$ is a morphism.

We assume $jf$ is a morphism. This implies both $\pi_1 jf$ and $\pi_2 jf$ are morphisms.

If $\sup \pi_2 jf = 0$, then $\operatorname{Im} f$ is contained in the subspace of $PY$ of paths of length 0. This subspace is homeomorphic to $Y$ and is mapped by $\pi_1 j$ homeomorphically onto the subspace of constant paths in $M(I, Y)$. Thus, since $\pi_1 jf$ is a morphism, so is $f$.

Otherwise, let

$$\varphi\colon C \longrightarrow I \quad \text{be} \quad \varphi c = \pi_2 jfc/\sup \pi_2 jf\ .$$

Let $D \subset C \times I$ be $\{c, t \mid t \leqq \varphi c\}$. The composition

$$D \longrightarrow C \times I \xrightarrow{\tilde{f}} Y$$

is a morphism, where $\tilde{f}(c, t) = fc(t\pi_2 jfc)$ is adjoint to $\pi_1 jf$ and the first morphism is defined by $(c, t) \to (c, t/\varphi c)$.

Let $E$ be the subspace of $C \times R^{+}$ of all pairs $(c, s)$ with $s \leqq \pi_2 jfc$ and map $E$ to $D$ by sending $(c, s)$ to $(c, s/\sup \pi_2 jf)$. Finally, define $\gamma\colon C \times R^{+} \to E$ by

$$\gamma(c, u) = \begin{cases} c,\ u, & \text{if} \quad u \leqq \pi_2 jfc \\ c,\ \pi_2 jfc\ , & \text{if} \quad \pi_2 jfc \leqq u\ . \end{cases}$$

The composition

$$C \times R^{+} \xrightarrow{\gamma} E \longrightarrow D \longrightarrow Y$$

is a morphism taking $(c, u)$ to $(fc)(u)$. Its adjoint is $\pi_1 f\colon C \to M(R^{+}, Y)$, which is thus a morphism. Since $\pi_2 f$ is the morphism $\pi_2 jf$, it follows that $f$ is a morphism.

LEMMA I.5. *Let* $g: X \to PY$ *and* $f: X \to R^+$ *be morphisms such that* $f^{-1}0 \subset (\pi_2 g)^{-1}0$. *Then the function*

$$\hat{g}: X \longrightarrow PY$$

*defined by* $\hat{g}x(t \cdot fx) = gx(t \cdot \pi_2 gx)$, $t \in I$, *with* $\pi_2 \hat{g} = f$, *is a morphism.*

*Proof.* $\hat{g}$ is well-defined since $f^{-1}0 \subset (\pi_2 g)^{-1}0$. $\pi_1 j\hat{g} = \pi_1 jg$ and $\pi_2 j\hat{g} = f$ are morphisms; thus, $j\hat{g}$ is a morphism. Since $j$ is an injection, it follows that $\hat{g}$ is a morphism.

The following companion theorems are proved by variants of a method of G. S. Young in [8]. We recall that a morphism $f: A \to X$ is a *cofibration* provided the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ \downarrow{\scriptstyle \varepsilon_0} & & \downarrow{\scriptstyle \varepsilon_0} \\ A \times I & \xrightarrow{f \times I} & X \times I \end{array}$$

is a weak pushout; i.e., given $F: A \times I \to Y$ and $g: X \to Y$ with $F\varepsilon_0 = gf$, there exists a $G: X \times I \to Y$ (not necessary unique) such that $G\varepsilon_0 = g$ and $Gf \times I = F$.

THEOREM I.6. *Every cofibration* $f: A \to X$ *is an injection of* $A$ *on a closed subset of* $X$ *which has a halo.*

THEOREM I.7. *Let* $A$ *be a closed subset of* $X$ *and* $i: A \to X$ *be the injection. Then the following properties are equivalent*:

( i ) $i$ *is a cofibration.*

(ii) $A \times I \cup X \times 0$ *is a retract of* $X \times I$,

(iii) *there exist a halo* $U$ *of* $A$ *and a morphism*

$$h: X \times I \longrightarrow X$$

*such that*

$$h(x, 0) = x,\ h(a, t) = a,\ h(u, 1) \in A$$

*for* $x \in X$, $a \in A$, $t \in I$, *and* $u \in U$,

(iv) *there exist a halo* $V$ *of* $A$ *and a morphism*

$$h: V \times I \longrightarrow X$$

*such that*

$$h(v, 0) = v,\ h(a, t) = a,\ h(v, 1) \in A$$

*for* $v \in V$, $a \in A$, *and* $t \in I$, *and*

(v) *there exist a haloing function $\varphi$ for $A$ in $X$ and a morphism*

$$h\colon V \longrightarrow PX\,, \quad \textit{where} \quad V = \varphi^{-1}[0, 1>\,,$$

*such that*

$$\eta_0 h = 1_v,\ \eta_\tau h(v) \subset A,\ lh = \varphi \,|\, V\,.$$

**II. Properties of LEC spaces.** We shall define a space $X$ to be *locally equiconnected* (abbreviated LEC) provided the diagonal map

$$\Delta\colon X \longrightarrow X \times X$$

is a cofibration. This is in agreement with earlier usage [1].

THEOREM II.1. *The space $X$ is* LEC *if and only if there exist morphisms*

$$k\colon X \times X \longrightarrow I$$
$$g\colon V \longrightarrow PX\,, \quad \text{for} \quad V = k^{-1}[0, 1 >\,,$$

*such that*
(i) $k^{-1}0 = D$, *the diagonal in $X \times X$ and*
(ii) $\eta_0 g = \pi_1 | V$, $\eta_\tau g = \pi_2 | V$ *and* $lg = k$, *where $\pi_1$ and $\pi_2$ are the projections of $X \times X$ onto its first and second factors and for a path $\alpha$, $l\alpha$ is the length of the path.*

We shall refer to a pair of morphisms $(k, g)$ having the properties of this theorem as LEC-data for $X$.

*Proof of Theorem.* If $\Delta\colon X \rightarrow X \times X$ is a cofibration, then by condition (v) of Theorem I.6 there exist morphisms

$$\varphi\colon X \times X \longrightarrow I\,, \quad \text{with} \quad D = \varphi^{-1}0\,, \quad \text{and}$$
$$h\colon V \longrightarrow P(X \times X)\,, \quad \text{where} \quad V = \varphi^{-1}[0, 1>\,,$$

such that $hv$ is a path from $v$ to $D$ of length $\varphi v$. Let $k = 2\varphi$ and define

$$g\colon V \longrightarrow PX$$

to be $g = \pi_1 h - \pi_2 h$. Since $hv0 = v$, $gv$ is a path in $X$ from $\pi_1 v$ to $\pi_2 v$. Its length is $2\varphi v = kv$.

Suppose $(k, g)$ are LEC-data for $X$. Let $\varphi = k$ and define

$$h\colon V \longrightarrow P(X \times X),\ V = \varphi^{-1}[0, 1>\,,$$

to be $(g, \pi_2)$. Since $gv$ is a path from $\pi_1 v$ to $\pi_2 v$, $hv$ is a path from

$v$ to $(\pi_2 v, \pi_2 v) \in D$. Its length is $kv = \varphi v$.

THEOREM II.2. *If* $X_1, X_2, \cdots, X_n$ *are* LEC, *then* $X = \prod_{i=1}^{n} X_i$ *is* LEC.

*Proof.* Since $\Delta_i: X_i \to X_i \times X_i$ is a cofibration for each $i$,

$$\prod_i \Delta_i: X \longrightarrow \prod_i (X_i \times X_i)$$

is a cofibration. This morphism composed with the twisting homeomorphism

$$T: \prod_i (X_i \times X_i) \longrightarrow X \times X$$

is the diagonal map for $X$. Thus, $\Delta_X$ is a cofibration and $X$ is LEC.

THEOREM II.3. *A* LEC *space* $X$ *can be covered by numerated open sets contractible in* $X$. *Also,* $X$ *is Hausdorff.*

*Proof.* Let $k, g$ be LEC-data for $X$ and $V = k^{-1}[0, 1>$.

For $x \in X$, let $V_x = \{y \in X \mid (x, y) \in V\}$. Define $k_x: X \to I$ by $k_x(y) = k(x, y)$. Then the open set $V_x$ is numerated by $k_x$. Define

$$C_x: V_x \times I \longrightarrow X$$

by $C_x(y, t) = g(x, y)t$. This is a deformation in $X$ of $V_x$ into $x$.

As in §4, number 4 of page 253 of [2], it follows that if $X$ is also compact, then it is metrizable.

For $x \neq y$, let $S_x = \{\xi \in X \mid k_x(\xi) < k_y(\xi)\}$ and $S_y = \{\xi \in X \mid k_y(\xi) < k_x(\xi)\}$. The sets $S_x$ and $S_y$ are disjoint open sets in $X$ containing $x$ and $y$, respectively.

THEOREM II.4. *In a* LEC *space the path components of a numerated open set are open.*

*Proof.* Let $k, g$ be LEC-data for $X$ and $V_x$ be as in the previous proof.

For $\psi: X \to I$, $U =$ Support $\psi$ and $x \in U$, define

$$\bar{\psi}: V_x \longrightarrow I$$

by $\bar{\psi} y = \inf \{\psi g(x, y)t \mid t \in I\}$. Then $0 < \psi(x) = \bar{\psi}(x)$. If $0 < \bar{\psi} y$, then $g(x, y)I \subset U$. Thus, the path component of $x$ in $U$ contains the support of $\bar{\psi}$, which is open.

COROLLARY II.5. *If* $X$ *is* LEC, *the decomposition space* $\Pi X$ *of* *path components of X is discrete.*

*Proof.* The function $1: X \to I$ has support $X$. By the previous theorem the path components of $X$ are open.

For a compact space $C$, a closed subset $A$ of $C$ and a map $\varphi: A \to X$, denote by $(C, X, \varphi)$ the subspace of the space $M(C, X)$ of maps of $C$ into $X$ of those functions $f: C \to X$ such that $f \mid A = \varphi$.

THEOREM II.6. *If $X$ is* LEC, *then so is $(C, X, \varphi)$.*

*Proof.* Let $k, g$ be LEC-data for $X$. For $f, f' \in M(C, X)$, let

$$\bar{k}(f, f') = \sup \{k(fc, f'c) \mid c \in C\} .$$

The function $\bar{k}$ is a morphism since it is the composition of the morphism

$$\sup: M(C, I) \longrightarrow I$$

with the adjoint of the composition

$$\begin{aligned} C \times M(C, X) \times M(C, X) &\xrightarrow{\Delta_C \times 1} C \times C \times M(C, X) \times M(C, X) \\ &\xrightarrow{1 \times T \times 1} C \times M(C, X) \times C \times M(C, X) \\ &\xrightarrow{e \times e} X \times X \xrightarrow{k} I , \end{aligned}$$

where $e: C \times M(C, X) \to X$ is the evaluation map.

For $f, f' \in M(C, X)$, $\bar{k}(f, f') = 0$ is equivalent to $fc = f'c$ for all $c \in C$; i.e., $\bar{k}(f, f') = 0$ if and only if $f = f'$.

For $\bar{k}(f, f') < 1$, define $\bar{g}((f, f') \in PM(C, X)$ by

$$\bar{g}(f, f')(t)(c) = g(fc, f'c)(t) .$$

Letting $U = \bar{k}^{-1}[0, 1>$, we conclude that $\bar{g}: U \to PM(C, X)$ is a morphism from the diagram

$$\begin{array}{ccccc} C \times U \times R & \longrightarrow & V \times R & \longrightarrow PX \times R \longrightarrow X \\ \downarrow & & \downarrow \subset & \\ C \times M(C, X) \times M(C, X) \times R & \longrightarrow & X \times X \times R . & \end{array}$$

We define $\hat{k}: (C, X, \varphi) \times (C, X, \varphi) \to I$ to be the restriction of $\bar{k}$ and $\hat{g}: \hat{k}^{-1}[0, 1> \to P(C, X, \varphi)$ to be the restriction of $\bar{g}$. We note that for $f, f' \in (C, X, \varphi)$ such that $\bar{k}(f, f') < 1$, $\bar{g}(f, f')(t)(a) = g(\varphi a, \varphi a)(t) = \varphi a$. Thus, the restriction of $\bar{g}$ to $\hat{k}^{-1}[0, 1>$ factors as asserted in defining $\hat{g}$.

THEOREM II.7. *If $X$ is* LEC *and $A$ is a halo retract of $X$, then*

*$A$ is* LEC *and the map $A \to X$ is a cofibration.*

*Proof.* Let $(k, g)$ be LEC-data for $X$, $\eta: X \to I$ be a halo function for $A$, and $r: U \to A$ be a retraction, where $U = \eta^{-1}[0, 1>$.

Define $\psi: X \times X \to I$ by

$$\psi(x, y) = \begin{cases} \sup_{s \in I} \{\eta(g(x, y)s), k(x, y)\} & \text{for} \quad k(x, y) < 1 \\ 1 & \text{for} \quad k(x, y) = 1 . \end{cases}$$

Then $1 - \psi$ is a morphism on Support $(1 - k)$ and $1 - \psi \leqq 1 - k$. Thus, $1 - \psi$ and $\psi$ are morphisms. If $\psi(x, y) < 1$, the path $g(x, y)$ is defined and lies in $U$. Also, $\psi(x, y) = 0$ if and only if $x = y \in A$.

Let $\hat{k} = \psi | A \times A$. Then $\hat{k}^{-1}0$ is the diagonal in $A \times A$. Let $W = \hat{k}^{-1}[0, 1>$ and define $\hat{g}: W \to PA$ by

$$\hat{g} = (rg, \hat{k}) .$$

This new parametrization is a morphism by Lemma 1.5 since the zeroes of $\hat{k}$ are the same as those of $k | A \times A$. Thus, $(\hat{k}, \hat{g})$ is LEC-data for $A$.

To verify that $A \to X$ is a cofibration define $\chi: X \to I$ by

$$\chi(x) = \begin{cases} \sup \{\eta x, k(x, rx)\} & \text{for} \quad \eta x < 1 \\ 1 & \text{for} \quad \eta x = 1 . \end{cases}$$

Exactly as before, $\chi$ is a morphism. $A = \chi^{-1}0$ and if $\chi x < 1$, then $\eta x < 1$, $rx$ is defined and $k(x, rx) < 1$. Let $T = \chi^{-1}[0, 1>$ and define $h: T \to PX$ to be the composition

$$T \xrightarrow{\Delta} T \times T \xrightarrow{1_T \times r} V \xrightarrow{g} PX ,$$

where $V = k^{-1}[0, 1>$. Then $\chi$ is haloing for $A$, $T = \chi^{-1}[0, 1>$ and $h: T \to PX$ is a morphism such that $\eta_0 h = 1_T$, $\mathrm{Im}(\eta_{\tau} h) \subset A$, and $h | A = i_A$, where $i_A: A \to PA$ injects each point to the path of zero length at that point. Since for $t \in T$, $\chi(t) = 0$ if and only if $kht = 0$, we can reparametrize the paths $ht$ to have length $\chi(t)$. Then $\chi$, $T$, $h$ satisfy condition (v) of the cofibration Theorem I.7.

COROLLARY II.8. *If $X$ is* LEC *and $x$ is a point of $X$, then the injection $x \to X$ is a cofibration.*

*Proof.* By the previous theorem, it suffices to show $x$ is a halo retract of $X$. Since a point is a retract of any set containing it, it suffices to show $x$ has a halo in $X$.

Let $k$, $g$ be LEC-data for $X$. The function $X \times x \to X \times X$ defined by $(x', x) \to (x', x)$ is a morphism which composed with $k$ defines a halo for $x$.

**III. The adjunction theorem.** J. H. C. Whitehead gave the first proof of this type result for compact, metric ANRs in [7]. Several generalizations have since been made (see [3] and [4]).

ADJUNCTION THEOREM. *If $X$ and $Y$ are* LEC, *$A$ is a halo retract in $X$ and $f: A \to Y$ is a morphism, then any* LEC-*data for $Y$ can be extended to* LEC-*data for $X \bigcup_f Y$.*

We note this result states not only that $X \bigcup_f Y$ is LEC but also relates a LEC structure on it to one on $Y$.

*Proof.* The proof of this theorem follows from five constructions, stated here. We let $k$, $g$ be LEC-data for $X$ and $\eta: X \to I$ be a haloing function for $A$ in $X$.

*Step* 1. There exists a morphism $\bar{k}: X \times X \to I$ such that
(i) $\eta x \leqq \bar{k}(x, y)$ and $\eta y \leqq \bar{k}(x, y)$,
(ii) $\varDelta A = \bar{k}^{-1}0$
(iii) for $T = \bar{k}^{-1}[0, 1\rangle$, the function $G: T \times I^2 \to X$ given by

$$G(x, y)(s, t) = \hat{g}(\hat{g}(x, y)s, r\hat{g}(x, y)s)t$$

is defined and a morphism, where $\hat{g}$ is the composition $\pi_1 jg$, $j$ being the injection $j: PX \to M(I, X) \times R^+$.

*Step* 2. There exists a morphism $\tilde{k}: X \times X \to I$ such that
(i) $k \leqq \tilde{k}$
(ii) $\tilde{k}^{-1}0 = \varDelta X$
(iii) $\bar{k} = 1$ and $\tau = 0$ implies $\tilde{k} = 1$, where $\tau = \inf(\eta\pi_1, \eta\pi_2)$.

We note that (ii) implies paths $g$ can be linearly reparametrized by $\tilde{k}$ to give LEC-data $g$, $\tilde{k}$ for $X$. Condition (i) implies $g$ is defined on $S = \tilde{k}^{-1}[0, 1\rangle$.

*Step* 3. In $S$, let

$$S_0 = (\tau \geqq \tilde{k}) \cup (\tau \geqq 1 - \bar{k}) \quad \text{and}$$
$$S_1 = (\tau \leqq \tilde{k}) \cap (\tau \leqq 1 - \bar{k}) \,.$$

There exists a *function* $f: S \to [0, 1]$ such that

$$f \,|\, S_0 = 0 \,,$$
$$f \,|\, (\tau^{-1}0 - \varDelta A) \cap S = 1 \,, \quad \text{and}$$
$$f \,|\, S - \varDelta A \quad \text{is a morphism} \,.$$

*Step* 4. There exist morphisms $k_1, k_2: S \to [0, 1]$ such that

(i) $k_1 + k_2 = \tilde{k}$ and $k_1 = k_2$ on $S_0$,

(ii) $k_1 + k_2 \leqq \tilde{k}$ on $S$, and

(iii) $k_1 = 0$ if and only if $\eta\pi_1 = 0$ or $\tilde{k} = 0$ and $k_2 = 0$ if and only if $\eta\pi_2 = 0$ or $\tilde{k} = 0$.

*Step* 5. There exist morphisms $g_1, g_2: S \to PX$ and closed sets $M$ and $N$ in $S_1$ such that

(i) $lg_1 = k_1$, $lg_2 = k_2$,

(ii) $g_1 + g_2 = g$ on $S_0$,

(iii) $\eta_0 g_1 = \pi_1$ and $\eta_\tau g_2 = \pi_2$,

(iv) if $\eta\pi_1 = 0$, then $\eta_0 g_2 = r\pi_2$ and if $\eta\pi_2 = 0$, then $\eta_\tau g_1 = r\pi_1$, and

(v) $S_0 \cap S_1 \subset M$, $\eta_\tau g_1 = \eta_0 g_2$ on $M$, $\eta_\tau g_1 \in A$ and $\eta_0 g_2 \in A$ on $N$, and $\tau^{-1}0 \subset N$.

Before proving these five assertions, let us prove that the Adjunction Theorem is implied by them.

Extensions of LEC-data $k_Y$, $g_Y$ for $Y$ to $X \cup_f Y$ will be defined on $(X \amalg Y) \times (X \amalg Y)$ so as to agree under the identifications imposed by the pushout diagram

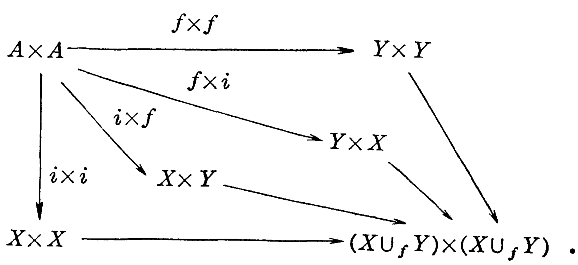


FIGURE 1

Define $k'$ by the following:
on $X \times X$,

$$k' = \begin{cases} \tilde{k} & \text{on } \tilde{k}^{-1}1 \cup S_0 \\ k_1 + k_2 & \text{on } M \\ k_1 + k_Y(f\eta_\tau g_1, f\eta_0 g_2) + k_2 & \text{on } N\,, \end{cases}$$

on $X \times Y$, $k' = k_1(\pi_1, r\pi_1) + k_Y(fr\pi_1, \pi_2)$ ,

on $Y \times X$, $k' = k_Y(\pi_1, fr\pi_2) + k_2(r\pi_2, \pi_2)$ , and

on $Y \times Y$, $k' = k_Y$ .

On the set $k'^{-1}[0, 1>$, define $g'$ by:

on $X \times X$,

$$g' = \begin{cases} g & \text{on } S_0 \\ g_1 + g_2 & \text{on } M \\ g_1 + g_Y(f\eta_\tau g_1, f\eta_0 g_2) + g_2 & \text{on } N, \end{cases}$$

on $X \times Y$, $\quad g' = g_1(\pi_1, r\pi_1) + g_Y(fr\pi_1, \pi_2)$ ,

on $Y \times X$, $\quad g' = g_Y(\pi_1, fr\pi_2) + g_2(r\pi_2, \pi_2)$ , and

on $Y \times X$, $\quad g' = g_Y$ .

Checking the conditions as stated in Steps 1 through 5 when relevant shows the functions as defined are morphisms, agree as required on $A \times A$, and define LEC-data on $X \bigcup_f Y$.

We next establish the constructions asserted in Steps 1 through 5.

*Step* 1. Let $k$, $g$ be LEC-data for $X$, $\eta$ a halo for $A$ in $X$ and $r: U \to A$ a retraction, $U = \eta^{-1}[0, 1>$.

Define $\hat{k}: X \times X \to I$ by

$$\hat{k}(x, y) = \begin{cases} \sup_{s \in I} \{k(x, y), \eta g(x, y)s\} & \text{for } k(x, y) < 1 \\ 1 & \text{for } k(x, y) = 1 . \end{cases}$$

The $1 - \hat{k} \leqq 1 - k$ and $1 - \hat{k}$ is a morphism on the support of $1 - k$. Thus, $\hat{k}$ is a morphism.

Define $\check{k}: X \times X \to I$ by

$$\check{k}(x, y) = \begin{cases} \sup_{s \in I} \{(\hat{k}(x, y), k(g(x, y)s, rg(x, y)s)\} & \text{for } \hat{k}(x, y) < 1 \\ 1 & \text{for } \hat{k}(x, y) = 1 . \end{cases}$$

As above, $\check{k}$ is a morphism.

Finally, define $\bar{k}: X \times X \to I$ by

$$\bar{k}(x, y) = \sup_{s, s', t, t' \in I^4} \{\check{k}(x, y), k(g(g(x, y)s, rg(x, y)s)t, \\ g(g(x, y)s', rg(x, y)s')t')\}$$

for $\check{k} < 1$ and $\bar{k}(x, y) = 1$ for $\hat{k}(x, y) = 1$. As before $\bar{k}$ is a morphism.

Observe that $\eta x \leqq \bar{k}(x, y)$ and $\eta y \leqq \bar{k}(x, y)$; also, $\bar{k}^{-1}0 = \Delta A$. The conditions imposed by $\bar{k}$ being less than 1 are sufficient to establish $G$ has the asserted properties.

*Step* 2. Let $\tilde{k} = \sup \{k, k/1 + \tau + k - \bar{k}\}$. The denominator $1 + \tau + k - \bar{k} \neq 0$ since $\bar{k} = 1$ and $\tau = 0$ imply $k > 0$. Thus, $\tilde{k}$ is a morphism. By definition, $k \leqq \tilde{k}$. Also, $\tilde{k} = 0$ if and only if $k = 0$; and so, $\tilde{k}^{-1}0 = \Delta X$. Clearly, for $\bar{k} = 1$ and $\tau = 0$, $\tilde{k} = 1$.

*Step* 3. For $S = \tilde{k}^{-1}[0, 1\rangle$, let

$$S_0 = \{(\tau \geqq \tilde{k}) \cup (\tau \geqq 1 - \bar{k})\} \cap S \quad \text{and}$$
$$S_1 = \{(\tau \leqq \tilde{k}) \cap (\tau \leqq 1 - \bar{k})\} \cap S .$$

Define the function $d: S \to I$ by $d|S_0 = 0$ (note that $\Delta A \subset S_0$)

$$d = (\tilde{k} - \tau)(1 - \bar{k} - \tau)/\tilde{k}\cdot(1 - \bar{k}) \quad \text{on} \quad S_1 - \Delta A .$$

The two definitions agree on $(S_0 - \Delta A) \cap (S_1 - \Delta A)$. These two sets are closed in $S - \Delta A$. To check that $d|S - \Delta A$ is a morphism it suffices to check that $d|S_1 - \Delta A$ is one.

Since $\tilde{k} < 1$, either $\bar{k} < 1$ or $\tau > 0$. In $S_1$, $\bar{k} = 1$ implies $\tau = 0$. Thus, $\bar{k} < 1$. If $\tilde{k} = 0$, then $\tau = 0$, and the point is in $\Delta A$. Hence, in $S_1 - \Delta A$, $\tilde{k}\cdot(1 - \bar{k}) > 0$; and so, $d|S_1 - \Delta A$ is a morphism.

Since $(\tau^{-1}0 - \Delta A) \cap S \subset S_1 - \Delta$, $d|(\tau^{-1}0 - \Delta A) \cap S = 1$.

*Step* 4. Define $\bar{\bar{k}}: S \to I$ by

$$\bar{\bar{k}} = \begin{cases} \inf\{\tilde{k}, (1 - d)\cdot\tilde{k} + d\cdot(\eta\pi_1 + \eta\pi_2)\} & \text{on Support } \tilde{k} \\ 0 & \text{on } \tilde{k}^{-1}0 . \end{cases}$$

Then $\bar{\bar{k}} \leqq \tilde{k}$ and $\bar{\bar{k}}$ is a morphism on Support $\tilde{k}$. Thus $\bar{\bar{k}}$ is a morphim on $S$.

Let $\hat{\varphi} = \inf(1, \eta\pi_1 + \eta\pi_2)$ and define $k_1': S \to I$ by

$$k_1' = \begin{cases} \bar{\bar{k}}\cdot\eta\pi_1/\hat{\varphi} & \text{for} \quad \hat{\varphi} > 0 \\ 0 & \text{for} \quad \hat{\varphi} = 0 . \end{cases}$$

That $k_1'$ is a morphism follows from Lemma I.2 by the following argument:

$$\begin{aligned} &\hat{\varphi}: X \times X \longrightarrow I && \text{is a morphism ,} \\ \eta = \hat{\psi}: &X \longrightarrow I && \text{is a morphism, and} \\ &\hat{h}: X \times X \times I \longrightarrow I && \text{by} \quad \hat{h} = (\bar{\bar{k}}\circ(\rho_1 \times \rho_2))\cdot\rho_3 , \end{aligned}$$

the $\rho_i$'s being projections, is a morphism. If $\hat{\varphi} = 0$, then $\tilde{k} = 0$ or $d = 1$. In either case $\bar{\bar{k}} = 0$; and so, $\hat{h}$ is independent of $t \in I$ if $\hat{\varphi} = 0$.

Let $W \subset X \times X \times X \times I$ be

$$W = \{(x_1, x_2), x, t \mid t\cdot\hat{\psi}x \leqq \hat{\varphi}(x_1, x_2)\} .$$

By Lemma I.2 $c: W \to I$ defined by

$$c((x_1, x_2), (x, t) = \begin{cases} \bar{\bar{k}}(x_1, x_2)\cdot t\cdot\eta x/\hat{\varphi}(x_1, x_2) & \text{for} \quad \hat{\varphi} > 0 \\ 0 & \text{for} \quad \hat{\varphi} = 0 \end{cases}$$

is a morphism. Define $j: S \to W$ by $j(x_1, x_2) = ((x_1, x_2), x_1, 1)$. Then

$$c \cdot j(x_1, x_2) = \begin{cases} \bar{k}(x_1, x_2) \cdot \eta x_1 / \hat{\varphi}(x_1, x_2) & \text{for } \hat{\varphi}(x_1, x_2) \neq 0 \\ 0 & \text{for } \hat{\varphi}(x_1, x_2) = 0 \end{cases}$$
$$= k_1'$$

is a morphism on $S$.

Define $\gamma: X \times X \to I$ by

$$\gamma = \begin{cases} 1/\eta\pi_1 + \eta\pi_2 & \text{for } 1 \leqq \eta\pi_1 + \eta\pi_2 \\ 1 & \text{for } \eta\pi_1 + \eta\pi_2 \leqq 1 . \end{cases}$$

Since $\gamma$ is a morphism, so is $\tilde{k}_1 = k_1' \cdot \gamma$. Thus,

$$\tilde{k}_1 = \begin{cases} \bar{k} \cdot \eta\pi_1 / \eta\pi_1 + \eta\pi_2 & \text{for } 0 < \eta\pi_1 + \eta\pi_2 \\ 0 & \text{for } 0 = \eta\pi_1 + \eta\pi_2 \end{cases}$$

and

$$\tilde{k}_2 = \begin{cases} \bar{k} \cdot \eta\pi_2 / \eta\pi_1 + \eta\pi_2 & \text{for } 0 < \eta\pi_1 + \eta\pi_2 \\ 0 & \text{for } 0 = \eta\pi_1 + \eta\pi_2 \end{cases}$$

are morphisms on $S$. Clearly, $\tilde{k}_1 + \tilde{k}_2 = \bar{k}$ where $0 < \eta\pi_1 + \eta\pi_2$. If $0 = \eta\pi_1 + \eta\pi_2$, then either $\tilde{k} = 0$ or $\tau = 0$ and $d = 1$. In either case $\bar{k} = 0$. Thus, $\tilde{k}_1 + \tilde{k}_2 = \bar{k}$ on $S$; and so, $\tilde{k}_1 + \tilde{k}_2 = \bar{k} \leqq \tilde{k}$ for all $(x, y) \in S$. For $(x, y) \in S_0$, $d = 0$ and $\bar{k} = \tilde{k}$; thus, on $S_0$ $\tilde{k}_1 + \tilde{k}_2 = \tilde{k}$. $\tilde{k}_1 = 0$ implies $\bar{k} = 0$ or $\eta\pi_1 = 0$, which implies $\tilde{k} = 0$ or $\eta\pi_1 = 0$, which implies $\bar{k} = 0$ or $\eta\pi_1 = 0$, which implies $\tilde{k}_1 = 0$. Thus, $\tilde{k}_1 = 0$ if and only if $\tilde{k} = 0$ or $\eta\pi_1 = 0$. Similarly, $\tilde{k}_2 = 0$ if and only if $\tilde{k} = 0$ or $\eta\pi_2 = 0$.

Let $k_1 = (\tilde{k}_1 - (1/2)\tilde{k}) \cdot d + (1/2)\tilde{k}$ and $k_2 = (\tilde{k}_2 - (1/2)\tilde{k}) \cdot d + (1/2)\tilde{k}$. Since $k_i$, $i = 1, 2$, is a morphism on Support $2\tilde{k}$ and $k_i \leqq 2\tilde{k}$, $k_i$ is a morphism on $S$. Also, $0 \leqq k_i \leqq 1$. Furthermore,

$$k_1 + k_2 = (\tilde{k}_1 + \tilde{k}_2 - \tilde{k}) \cdot d + \tilde{k} \leqq \tilde{k} .$$

On $S_0$, $d = 0$ and so, $k_1 = k_2 = (1/2)\tilde{k}$. Finally, $k_1 = 0$ if and only if $\tilde{k}_1 d = ((1/2)d - 1/2)\tilde{k}$. But the latter is true if and only if $\tilde{k}_1 = 0$ and $\tilde{k} = 0$ or $\tilde{k}_1 = 0$ and $d = 1$ or $d = 0$ and $\tilde{k} = 0$. This is true if and only if $\tilde{k}_1 = 0$. Thus, $k_1 = 0$ if and only if $\eta\pi_1 = 0$ or $\tilde{k} = 0$ and $k_2 = 0$ if and only if $\eta\pi_2 = 0$ or $\tilde{k} = 0$.

*Step* 5. Let $M = d^{-1}[0, 2/\pi \operatorname{arc\,tan} 2] \cap S_1$ and

$$N = \{y^{-1}[2/\pi \operatorname{arc\,tan} 2, 1] \cap S_1\} \cup \Delta A .$$

Then $S_0 \cap S_1 \subset M$ and $S_0 \cap N = \Delta A$. Define $a$ and $b$ mapping $I$ in $PI^2$ by

$$a(t)s = \begin{cases} \left(\frac{1}{2}s, \frac{1}{2}s \tan \pi t/2\right) & \text{for} \quad 0 \leqq \pi t/2 \leqq \text{arc} \tan 2 \\ (s \cot \pi t/2, s) & \text{for} \quad \text{arc} \tan 2 \leqq \pi t/2 \leqq \pi/2 , \end{cases}$$

and

$$b(t)s = \begin{cases} \left(\frac{1}{2} + \frac{1}{2}s, \left(\frac{1}{2} - \frac{1}{2}s\right)\tan \pi t/2\right) & 0 \leqq \pi t/2 \leqq \text{arc} \tan 2 \\ (1 + (s-1) \cot \pi t/2, 1 - s) & \text{arc} \tan 2 \leqq \pi t/2 \leqq \pi/2 . \end{cases}$$

The following assertions are easily verified:

(1) for $0 \leqq \pi t/2 \leqq \text{arc} \tan 2$, $a(t)1 = b(t)0$,

(2) for $\text{arc} \tan 2 \leqq \pi t/2 \leqq 1$, $\pi_2 a(t)1 = 1 = \pi_2 b(t)0$, and

(3) $a(t)(0) = (0, 0)$ and $b(t)(1) = (1, 0)$ for all $t$.

Define $g_1: S \to PX$ by

$$g_1(x, x') = \begin{cases} g(x, x')\,|_0^{k_1(x,x')} & \text{with length } k_1 \text{ for } (x, x') \in S_0 \\ G(x, x')\,|\,a \circ d(x, x') & \text{with length } k_1 \text{ for } (x, x') \in S_1 . \end{cases}$$

On $S_0 \cap S_1$, $\tau = \tilde{k} = 1 - \bar{k}$, $d = 0$ and $k_1 = (1/2)\tilde{k}$. $a \circ d$ is the interval $[(0, 0), (1/2, 0)]$. The composition

$$\begin{aligned} G(x, x')\,|\,a \circ d &= \hat{g}(\hat{g}(x, x')s, r\hat{g}(x, x')s)0 , \quad 0 \leqq s \leqq \frac{1}{2} \\ &= \hat{g}(x, x')s \qquad\qquad 0 \leqq s \leqq \frac{1}{2} \\ &= \hat{g}(x, x')\,|_0^{1/2} \\ &= \pi_1 jg(x, x')\,|_0^{(1/2)\tilde{k}(x,x')} \\ &= \pi_1 jg(x, x')\,|_0^{k_1(x,x')} . \end{aligned}$$

Thus, the definitions of $g_1$ on $S_0$ and $S_1$ agree on $S_0 \cap S_1$. Clearly, $g_1 | S_0$ is a morphism. To check that $g_1 | S_1$ is also, it suffices to check $\pi_1 jg | S_1$. On $S_1$, $\bar{k} < 1$. Define a morphism $h: T \times M(I, I^2) \times I \to X$ by the following function space adjointness applications:

$$\begin{aligned} G&: T \times I^2 \longrightarrow X , \\ \hat{G}&: I^2 \longrightarrow M(T, X) , \\ M(I, \hat{G})&: M(I, I^2) \longrightarrow M(I, M(T, X)) \cong M(T, M(I, X)) , \\ M(I, \hat{G})^{\sim}&: T \times M(I, I^2) \longrightarrow M(I, X) , \quad \text{and finally} \\ h = M(I, \hat{G})^{\sim\sim}&: T \times M(I, I^2) \times I \longrightarrow X . \end{aligned}$$

Define $\rho: T \to M(I, I^2)$ by

$$\rho(x, x') = \begin{cases} ad(x, x') & \text{if} \quad \bar{k}(x, x') > 0 \\ [0, 0), (0, 1)] & \text{if} \quad \bar{k}(x', x) = 0 . \end{cases}$$

Then $\bar{k}$, $h$, $\rho$ satisfy the hypotheses of Lemma I.2. Thus,

$$k(x, x', t) = h(x, x', \rho(x, x'), t)$$

is a morphism. But this is $\pi_1 j g_1 | S_1$.

Similarly,

$$g_2(x, x') = \begin{cases} g(x, x') |_{k_1(x,x')}^{\bar{k}(x,x')} & \text{with length } k_2(x, x') \text{ on } S_0 \\ G(x, x') | bd(x, x') & \text{with length } k_2(x, x') \text{ on } S_1 \end{cases}$$

is a morphism.

Verification of (i) and (ii) of Step 5 is immediate. Condition (v) follows from (1) and (2) above and condition (iii) follows from (3). To check condition (iv), observe that $\eta\pi_1 = 0$ implies $d = 1$ or $\bar{k} = 0$. In the latter case $g_2$ is the 0-path at $x = x' = rx = rx'$. In the former,

$$\pi_1 j g_2 s = (\hat{g}(\hat{g}(x, x')1, r\hat{g}(x, x')1)(1 - s) \quad 0 \leqq s \leqq 1 .$$

and so,

$$\eta_0 g_2 = \pi_1 j g_2 0 = rx' = r\pi_2 .$$

COROLLARY III.2. *Every cell-complex is* LEC.

*Proof.* A cell-complex $X$ is the colimit of a sequence of morphisms $X^{(n)} \xrightarrow{f_n} X^{(n+1)}$, where $X^0$ is a discrete space and for each $n$, $f_n$ is defined by the push-out diagram

$$\begin{array}{ccc} \coprod S_\alpha^n & \xrightarrow{\coprod g_\alpha} & X^{(n)} \\ \downarrow{\scriptstyle \coprod i_\alpha} & & \downarrow{\scriptstyle f_n} \\ \coprod D_\alpha^{n+1} & \longrightarrow & X^{(n+1)} \end{array} .$$

$\coprod D_\alpha^{n+1}$ is LEC and as $\coprod i_\alpha$ is a cofibration, $\coprod S_\alpha^n$ is a halo retract in $\coprod D_\alpha^{n+1}$. Thus, by the Adjunction Theorem LEC-data for $X^{(n)}$ extends to LEC-data for $X^{(n+1)}$. $X^{(0)}$, being discrete, is LEC. Inductively, a sequence $\{g^{(n)}, k^{(n)}\}$ of LEC-data is formed for the $\{X^{(n)}\}$ such that each extends its predecessor. The functions $g, k$ defined on the colimit $X$ are thus morphisms and are LEC-data for $X$.

## REFERENCES


1. J. Dugundji, *Locally equiconnected spaces and absolute neighborhood retracts*, Fund. Math., **55** (1965), 187-193.
2. ———, *Topology*, Allyn and Bacon, Boston, 1966.
3. D. M. Hyman, *A category slightly larger than the metric and CW-categories*, Michigan Math. J., **15** (1968), 193-214.
4. W. D. McIntosh, *Adjunction of locally equiconnected spaces*, Fund. Math., **63** (1968), 175-192.
5. E. H. Spanier, *Quasi-Topologies*, Duke Math. J., **30** (1963). 1-14.

6. N. E. Steenrod, *A convenient category of topological spaces*, Michigan Math. J., **14** (1967), 133-152.
7. J. H. C. Whitehead, *Note on a theorem due to Borsuk*, Bull. Amer. Math. Soc., **54** (1948), 1125-1132.
8. G. S. Young, *A condition for the absolute homotopy extension property*, Amer. Math. Monthly, **71** (1964), 896-897.

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