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RINGS OF QUOTIENTS OF ENDOMORPHISM RINGS OF PROJECTIVE MODULES

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This paper investigates two related problems. The first is to describe the double centralizer of an arbitrary projective right R-module. This proves to be the ring of left quotients of R with respect to a certain canonical hereditary torsion class of left R-modules determined by the projective module.

The second is to determine the relationship between rings of left quotients of R and S, where S is the endomorphism ring of a finitely generated projective right R-module P_R. It is shown that there exists an inclusion-preserving, one-to-one correspondence between hereditary torsion classes (or localizing subcategories) of left S-modules and hereditary torsion classes of left R-modules which contain the canonical torsion class determined by P_R.

If Q_R and Q_S are rings of left quotients with respect to corresponding classes, then $P \otimes_R Q_R$ is a finitely generated projective right Q_R-module with Q_S as its Q_R-endomorphism ring. Necessary and sufficient conditions are obtained for the maximal rings of left quotients to be related in this manner. In particular, this occurs when P_R is a faithful R-module and R is either a semi-prime ring or a ring with zero left singular ideal. The situation considered includes the case where S is an arbitrary ring, $_SP$ is a left S-generator, and R is the S-endomorphism ring of $_SP$. When $_SP$ is a projective left S-generator, the maximal rings of left quotients of R and S are related in the manner considered above.

We present a brief summary of those aspects of torsion theories and generalized rings of quotients required in the sequel. We include it both for the convenience of the reader and to permit us to establish notation and terminology. This material has been drawn from papers by Dickson [4], Gabriel [8], and the Walkers [18], which may be consulted for a more detailed treatment. Other excellent sources are a paper of Goldman [9] and the recent monograph by Lambek [12], which also includes an extensive bibliography of work in this area.

Throughout this paper all rings will be assumed to be associative and to have identities, and all modules to be unital. In order to eliminate the necessity for opposite rings, module homomorphisms will be written opposite the scalars with which they commute. All other mappings will be written on the right. Also, unless specified otherwise, the notation used is cumulative.

For a ring A, let $_A\mathscr{M}$ denote the category of left A-modules. A *torsion class* in ${}_A\mathscr{M}$ is a nonvoid class $\mathscr{T} \subseteqq {}_A\mathscr{M}$ which is closed under homomorphic images, extensions, and arbitrary direct sums. If $\mathscr{T}$ is also closed under submodules, it is called a *hereditary* torsion class. Corresponding to each torsion class $\mathscr{T}$ in ${}_A\mathscr{M}$, there is a unique *torsion-free class*,

$$\mathscr{F} = \{M \in {}_A\mathscr{M} \mid \operatorname{Hom}_A(N, M) = 0 \quad \text{for all} \quad N \in \mathscr{T}\} .$$

The torsion-free class $\mathscr{F}$ is closed under submodules, extensions, and arbitrary direct products. If $\mathscr{T}$ is hereditary, $\mathscr{F}$ is also closed under injective hulls. For any $M \in {}_A\mathscr{M}$ there is a unique submodule $t(M)$ of M—the $\mathscr{T}$-torsion submodule of M—such that $t(M) \in \mathscr{T}$ and $M/t(M) \in \mathscr{F}$.

Gabriel [8] has exhibited a one-to-one correspondence between hereditary torsion classes in ${}_A\mathscr{M}$ and idempotent filters $\mathscr{I}$ of left ideals of A. The correspondences are

$$\mathscr{T} \longrightarrow \mathscr{I}(\mathscr{T}) = \{I \mid I \text{ is a left ideal of } A \text{ with } A/I \in \mathscr{T}\}$$

and

$$\mathscr{I} \longrightarrow \mathscr{T}(\mathscr{I}) = \{M \in {}_A\mathscr{M} \mid (0\colon m) \in \mathscr{I} \quad \text{for all} \quad m \in M\} ,$$

where $(0\colon m) = \{a \in A \mid am = 0\}$. A filter $\mathscr{I}$ is *faithful* if for any $a \in A$, $(0\colon a) \in \mathscr{I}$ implies $a = 0$. A hereditary torsion class is called faithful when its associated filter of left ideals is faithful. Thus $\mathscr{T}$ is faithful if and only if ${}_AA \in \mathscr{F}$.

Let $\mathscr{T}$ be a hereditary torsion class in ${}_A\mathscr{M}$. A module $M \in {}_A\mathscr{M}$ is *$\mathscr{T}$-injective* if the functor $\operatorname{Hom}_A(\text{—}, M)$ is exact on all short exact sequences $0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0$ with $N'' \in \mathscr{T}$.

Let $\mathscr{A}$ denote the quotient category of ${}_A\mathscr{M}$ with respect to the hereditary torsion class $\mathscr{T}$. (See [8, pp. 365–369].) For any $M \in {}_A\mathscr{M}$, define the *localization of M with respect to $\mathscr{T}$* via

$$L(M) = \operatorname{Hom}_{\mathscr{A}}(A, M) = \varinjlim_{I \in \mathscr{I}} \operatorname{Hom}_A(I, M/t(M)) ,$$

where $\mathscr{I}$ is directed by inverse inclusion. Since $\mathscr{A}$ is an abelian category,

$$Q = \operatorname{Hom}_{\mathscr{A}}(A, A)$$

is a ring, called the *ring of left quotients of A with respect to $\mathscr{T}$*. For each $M \in {}_A\mathscr{M}$, the natural composition

$$\operatorname{Hom}_{\mathscr{A}}(A, A) \times \operatorname{Hom}_{\mathscr{A}}(A, M) \longrightarrow \operatorname{Hom}_{\mathscr{A}}(A, M)$$

makes $L(M)$ a left Q-module. Furthermore, each map of modules in ${}_A\mathscr{M}$ induces a unique Q-homomorphism between their localizations.

Thus L may be viewed as a functor from ${}_A\mathscr{M}$ to ${}_Q\mathscr{M}$. This functor is covariant, additive, and left exact [18, Section 3].

For each $M \in {}_A\mathscr{M}$, there exists a canonical group homomorphism $\sigma(M): M \to L(M)$. Moreover, $\sigma(A): A \to Q$ is a ring homomorphism. Thus each Q-module may also be regarded as an A-module. Hence we may, and often shall, view L as a functor from ${}_A\mathscr{M}$ to ${}_A\mathscr{M}$. When this is done, σ becomes a natural transformation from the identity functor on ${}_A\mathscr{M}$ to the functor L.

For each $M \in {}_A\mathscr{M}$, $L(M)$ is $\mathscr{T}$-injective and $\mathscr{T}$-torsion-free. Furthermore, the kernel and cokernel of $\sigma(M)$ belong to $\mathscr{T}$. These properties characterize $L(M)$, as is shown by the next proposition. We shall deal with $L(M)$ and with Q primarily in terms of this characterization.

PROPOSITION 1.1. *Let M and X belong to ${}_A\mathscr{M}$ and suppose that X is $\mathscr{T}$-injective and $\mathscr{T}$-torsion-free. If there exists an A-homomorphism f from M into X such that $\ker f$ and $\operatorname{coker} f$ are in $\mathscr{T}$, then there is a unique A-isomorphism γ from X to $L(M)$ such that $f \circ \gamma = \sigma(M)$. When $M = A$, $L(M)$ is the ring of left quotients of A with respect to $\mathscr{T}$. In this case, if X is a ring in a manner compatible with its structure as an A-module, γ is a ring isomorphism.*

Proof. Since $\ker f \in \mathscr{T}$ and X is $\mathscr{T}$-torsion-free, $\ker f = t(M)$. Similarly, $\ker \sigma(M) = t(M)$. Thus there exists a unique A-isomorphism γ' of $\operatorname{im} f$ onto $\operatorname{im} \sigma(M)$ such that $f \circ \gamma' = \sigma(M)$. Since $\operatorname{coker} f \in \mathscr{T}$ and $L(M)$ is $\mathscr{T}$-injective, γ' extends to an A-homomorphism γ of X into $L(M)$. Moreover, γ is unique since $\operatorname{Hom}_A(\operatorname{coker} f, L(M)) = 0$. By symmetry, there exists a unique A-homomorphism δ of $L(M)$ into X such that $\sigma(M) \circ \delta = f$. Thus $\gamma \circ \delta$ is an endomorphism of X which is the identity on $\operatorname{im} f$. Hence $\gamma \circ \delta = 1_X$ since $\operatorname{Hom}_A(\operatorname{coker} f, X) = 0$. Similarly, $\delta \circ \gamma = 1_{L(M)}$, and so γ is an isomorphism. The last assertion is immediate from the uniqueness of the ring structure on Q [9, Theorem 4.1].

PROPOSITION 1.2. *For any M and N in ${}_A\mathscr{M}$,* $\operatorname{Hom}_A(L(M), L(N)) = \operatorname{Hom}_Q(L(M), L(N))$. *In particular, since $Q \cong L(A)$ as left Q-modules,* $Q \cong \operatorname{End}_A(L(A))$.

Proof. The first statement is [18, Lemma 3.7]. The remainder is obvious.

Let $M, M', X \in {}_A\mathscr{M}$ and $f: M: \to M'$ be an A-homomorphism. In order to simplify notation, we denote $\operatorname{Hom}_A(f, 1_X): \operatorname{Hom}_A(M', X) \to \operatorname{Hom}_A(M, X)$ by f^*.

LEMMA 1.3. *Let f be an A-homomorphism from M to M' with* $\ker f$ *and* $\operatorname{coker} f$ *in* $\mathscr{T}$. *If X is a $\mathscr{T}$-injective and $\mathscr{T}$-torsion-free A-module, then f^* is an isomorphism from* $\operatorname{Hom}_A(M', X)$ *to* $\operatorname{Hom}_A(M, X)$.

Proof. Since X is $\mathscr{T}$-injective and $\operatorname{coker} f \in \mathscr{T}$, applying the functor $\operatorname{Hom}_A(\ , X)$ to the exact sequences

$$0 \longrightarrow \ker f \longrightarrow M \xrightarrow{\hat{f}} \operatorname{im} f \longrightarrow 0$$

and

$$0 \longrightarrow \operatorname{im} f \xrightarrow{i} M' \longrightarrow \operatorname{coker} f \longrightarrow 0$$

yields exact sequences

$$0 \longrightarrow \operatorname{Hom}_A(\operatorname{im} f, X) \xrightarrow{\hat{f}^*} \operatorname{Hom}_A(M, X) \longrightarrow \operatorname{Hom}_A(\ker f, X)$$

and

$$0 \longrightarrow \operatorname{Hom}_A(\operatorname{coker} f, X) \longrightarrow \operatorname{Hom}_A(M', X) \xrightarrow{i^*} \operatorname{Hom}_A(\operatorname{im} f, X) \longrightarrow 0 .$$

Since X is $\mathscr{T}$-torsion-free and both $\ker f$ and $\operatorname{coker} f$ are in $\mathscr{T}$, $\operatorname{Hom}_A(\ker f, X) = 0$ and $\operatorname{Hom}_A(\operatorname{coker} f, X) = 0$. Thus $\hat{f}^*$ and i^* are isomorphisms. Composing these maps gives an isomorphism of $\operatorname{Hom}_A(M', X)$ onto $\operatorname{Hom}_A(M, X)$; a direct verification shows that this composition equals f^*.

Among torsion classes in ${}_A\mathscr{M}$ the $E(A)$-torsion class is of special importance. A left ideal I of A is *dense* if $\operatorname{Hom}_A(A/I, E(A)) = 0$. The dense ideals of A form an idempotent filter which contains all faithful idempotent filters of A. Thus the corresponding hereditary torsion class is maximal among all faithful hereditary torsion classes in ${}_A\mathscr{M}$. This class is called the *$E(A)$-torsion class*. The ring of quotients of A with respect to the $E(A)$-torsion class is called the *maximal ring of left quotients* of A and is denoted by $Q(A)$. If Q' is a ring of left quotients of A with respect to a faithful hereditary torsion class in ${}_A\mathscr{M}$, there is a unique ring homomorphism of Q' into Q extending the identity map on A. In fact, this is true if Q' is a rational extension of A in the sense of Lambek [11].

The functor

$$F = P \otimes_R (\ ): {}_R\mathscr{M} \longrightarrow {}_S\mathscr{M}$$

has a right adjoint

$$H = \operatorname{Hom}_S(P, \ ): {}_S\mathscr{M} \longrightarrow {}_R\mathscr{M} .$$

That is, there is an isomorphism

$$\mathrm{Hom}_S(F(M), N) \cong \mathrm{Hom}_R(M, H(N)),$$

natural in $M \in {}_R\mathscr{M}$ and $N \in {}_S\mathscr{M}$ [13]. This is equivalent to the existence of natural transformations

$$\beta: I_{{}_R\mathscr{M}} \longrightarrow HF \quad \text{and} \quad \alpha: FH \longrightarrow I_{{}_S\mathscr{M}}$$

such that

$$F(\beta(M)) \circ \alpha(F(M)) = 1_{F(M)} \quad \text{and} \quad \beta(H(N)) \circ H(\alpha(N)) = 1_{H(N)}$$

for all $M \in {}_R\mathscr{M}$ and $N \in {}_S\mathscr{M}$ [13, Proposition 8.5]. In this case, for $N \in {}_S\mathscr{M}$ one may define

$$\alpha(N): P \otimes_R \mathrm{Hom}_S(P, N) \longrightarrow N$$

via $(p \otimes g)(\alpha(N)) = (p)g$ for all $p \in P$ and $g \in \mathrm{Hom}_S(P, N)$. Similarly, for $M \in {}_R\mathscr{M}$, one may define

$$\beta(M): M \longrightarrow \mathrm{Hom}_S(P, P \otimes M)$$

via $(p)((m)\beta(M)) = p \otimes m$ for $m \in M$ and $p \in P$.

If the module P_R is finitely generated and projective, the functor F defined above also has a left adjoint

$$G = P^* \otimes_S (\quad): {}_S\mathscr{M} \longrightarrow {}_R\mathscr{M},$$

where $P^* = \mathrm{Hom}_R(P, R)$. That is, there is an isomorphism

$$\mathrm{Hom}_R(G(N), M) \cong \mathrm{Hom}_S(N, F(M)),$$

natural in $M \in {}_R\mathscr{M}$ and $N \in {}_S\mathscr{M}$. This is equivalent to the existence of natural transformations

$$\beta': GF \longrightarrow I_{{}_R\mathscr{M}} \quad \text{and} \quad \alpha': I_{{}_S\mathscr{M}} \longrightarrow FG$$

such that

$$\alpha'(F(M)) \circ F(\beta'(M)) = 1_{F(M)} \quad \text{and} \quad G(\alpha'(N)) \circ \beta'(G(N)) = 1_{G(N)}$$

for all $M \in {}_R\mathscr{M}$ and $N \in {}_S\mathscr{M}$. In this case, for $M \in {}_R\mathscr{M}$ one may define

$$(g \otimes p \otimes m)\beta'(M) = g(p)m$$

for $g \in P^*$, $p \in P$, and $m \in M$. Similarly, for $N \in {}_S\mathscr{M}$ one may define

$$(n)\alpha'(N) = \sum_i x_i \otimes f_i \otimes n$$

for $n \in N$, where $\{x_i\}$ and $\{f_i\}$ are a "dual basis" for P_R. (See [3, Chapter II, Proposition 4.5].) Since $S = \mathrm{End}_R(P_R)$, both α and α' are natural equivalences of functors when P_R is finitely generated and projective.

If P_R is projective, the *trace ideal* T of P is $\sum_f \operatorname{im}(f)$, where $f \in P^* = \operatorname{Hom}_R(P, R)$. Thus T is an ideal of R, and it is immediate from the "dual basis lemma" that $P \cdot T = P$ and $T^2 = T$. Furthermore, when P_R is finitely generated, T is also the trace ideal of ${}_RP^*$.

For the functor $F = P \otimes_R (\ \ )$, let

$$\operatorname{Ker} F = \{M \in {}_R\mathscr{M} \mid F(M) = 0\} .$$

If P is a projective module with trace ideal T, then it is easily verified that

$$\operatorname{Ker} F = \{M \in {}_R\mathscr{M} \mid T \cdot M = 0\} .$$

PROPOSITION 1.4. *Let P_R be a projective module with trace ideal T. Then* $\operatorname{Ker} F$ *is a hereditary torsion class in ${}_R\mathscr{M}$ whose associated filter of left ideals is $\{I \mid I$ is a left ideal of R and $I \supseteq T\}$. Thus* $\operatorname{Ker} F$ *is faithful if and only if T is a dense left ideal of R. This occurs if and only if P_R is a faithful module.*

Proof. Since F is additive, exact, and commutes with direct sums, it is easy to see that $\operatorname{Ker} F$ is a hereditary torsion class. A left ideal I is in its associated filter iff $R/I \in \operatorname{Ker} F$ iff $T \cdot R = T \subseteq I$. The next statement follows since the filter of dense left ideals is a faithful filter which contains all faithful idempotent filters. Finally, since the torsion submodule of R with respect to $\operatorname{Ker} F$ is $\{r \in R \mid P_R = 0\}$, it is clear that $\operatorname{Ker} F$ is faithful iff P_R is faithful.

When P_R is projective, we shall denote the torsion class $\operatorname{Ker} F$ by $\mathscr{T}_T$, the associated torsion submodule by t_T, and the corresponding torsion-free class and filter by $\mathscr{F}_T$ and $\mathscr{/}_T$ respectively. The localization functor for this torsion class will be denoted by L_T and the ring of left quotients of R with respect to $\mathscr{T}_T$ by Q_T.

Unless otherwise indicated, throughout the rest of this paper P_R is a projective right R-module, $S = \operatorname{End}_R(P_R)$, and T is the trace ideal of P in R. For the rest of this section and all of §§ 3 and 4, it will be assumed in addition that P_R is finitely generated. We note that if S is an arbitrary ring, ${}_SP$ is a generator for ${}_S\mathscr{M}$, and $R = \operatorname{End}_S({}_SP)$, then all of the above hypotheses are satisfied [3, Chapter II, Propositions 4.1, 4.4, and Theorem 3.4]. The notation introduced in this section will be employed freely throughout the rest of the paper.

LEMMA 1.5. *For any $M \in {}_R\mathscr{M}$, the exact sequences*

$$0 \longrightarrow \ker \beta'(M) \longrightarrow GF(M) \xrightarrow{\beta'(M)} M \longrightarrow \operatorname{coker} \beta'(M) \longrightarrow 0$$

$$0 \longrightarrow \ker \beta(M) \longrightarrow M \xrightarrow{\beta(M)} HF(M) \longrightarrow \operatorname{coker} \beta(M) \longrightarrow 0$$

have $\ker \beta'(M)$, $\ker \beta(M)$, $\operatorname{coker} \beta'(M)$, *and* $\operatorname{coker} \beta(M)$ *all in* $\mathscr{T}_T$.

Proof. Since both α and α' are natural equivalences, $\alpha(F(M))$ and $\alpha'(F(M))$ are both isomorphisms. Thus from the adjointness relations $F(\beta(M))\circ\alpha(F(M)) = 1_{F(M)}$ and $\alpha'(F(M)\circ F(\beta'(M)) = 1_{F(M)}$, we conclude that $F(\beta(M))$ and $F(\beta'(M))$ are isomorphisms. The result is immediate from this observation and the exactness of F.

REMARK. When P_R is projective, but not necessarily finitely generated, it follows from the adjointness relation $F(\beta(M))\circ\alpha(FM)) = 1_{F(M)}$ and the exactness of F that $\ker \beta(M)$ belongs to $\mathscr{T}_T$.

PROPOSITION 1.6. *For any left S-module N, H(N) is in* $\mathscr{F}_T$ *and is* $\mathscr{T}_T$*-injective. Thus for any left R-module M, HF(M) is isomorphic to* $L_T(M)$ *via a map* γ *such that* $\beta(M)\circ\gamma = \sigma(M)$. *Hence if M is in* $\mathscr{F}_T$ *and is* $\mathscr{T}_T$*-injective,* $\beta(M)$ *is an isomorphism.*

Proof. Let $M' \in \mathscr{T}_T$. Then $\operatorname{Hom}_R(M', H(N)) \cong \operatorname{Hom}_S(F(M'), N) = \operatorname{Hom}_S(0, N) = 0$. Thus $H(N) \in \mathscr{F}_T$. To show $H(N)$ is $\mathscr{T}_T$-injective, it suffices to prove that $\operatorname{Ext}^1_R(R/I, H(N)) = 0$ for each $I \in \mathscr{I}_T$. The usual exact sequence

$$0 \longrightarrow I \xrightarrow{i} R \longrightarrow R/I \longrightarrow 0$$

yields an exact sequence

$$\begin{aligned} 0 \longrightarrow \operatorname{Hom}_R(R/I, H(N)) &\longrightarrow \operatorname{Hom}_R(R, H(N)) \xrightarrow{i^*} \operatorname{Hom}_R(I, H(N)) \\ &\longrightarrow \operatorname{Ext}^1_R(R/I, H(N)) \longrightarrow 0\,. \end{aligned}$$

Since F is exact and $F(R/I) = 0$, $F(i)$ is an isomorphism. Thus $\operatorname{Hom}_R(F(i), 1_N)$ is an isomorphism. By adjointness $\operatorname{Hom}_R(i, H(1_N)) = \operatorname{Hom}_R(i, 1_{H(N)}) = i^*$ is an isomorphism. Hence $\operatorname{Ext}^1_R(R/I, H(N)) = 0$.

Lemma 1.5 implies that $\ker \beta(M)$ and $\operatorname{coker} \beta(M)$ are in $\mathscr{T}_T$. Combining these facts with those established in the preceding paragraph and applying 1.1 yields the desired isomorphism γ. The last statement is now immediate.

REMARK. The first assertion of Proposition 1.6 remains valid when P_R is projective but not necessarily finitely generated.

2. Double centralizers of projective modules.

Each right R-module M_R is in a natural way a left module over its endomorphism ring $C = \operatorname{Hom}_R(M, M)$. The endomorphism ring $D = \operatorname{Hom}_C(M, M)$ of M as a left C-module is called the *double centralizer* of the module M. There is a canonical ring homomorphism $\rho(M)$ of R into D, given by sending each element of R onto the right multiplication which it defines on M. The module M is said to have the *double centralizer property* if $\rho(M)$ is onto.

In this section we describe the double centralizer of a projective module. In particular, we determine those faithful projective modules that have the double centralizer property. These results yield generalizations of theorems of Fuller [6], Tachikawa [16], and Mochizuki [14].

Throughout this section P_R denotes a projective right R-module which is *not* assumed to be finitely generated, $S = \mathrm{End}_R(P_R)$, and $\hat{R} = \mathrm{End}_S(P)$ is the double centralizer of P. We recall that $\mathscr{T}_T$ denotes the hereditary torsion class in ${}_R\mathscr{M}$ consisting of all modules whose annihilators contain the trace ideal T of P_R. We use freely the notation and terminology introduced in section one.

THEOREM 2.1. *Let P_R be a projective right R-module and Q_T be the ring of left quotients of R with respect to $\mathscr{T}_T$. Then there exists a ring isomorphism γ of the double centralizer $\hat{R}$ of P_R onto Q_T such that $\rho(P)\circ\gamma = \sigma(R)$. Thus $\hat{R}$ may be described by $\hat{R} \cong \mathrm{End}_R(T/t_T(T))$.*

Proof. Since $P \otimes_R R \cong P$, $HF(R) = \mathrm{Hom}_S(P, P\otimes_R R) \cong \mathrm{Hom}_S(P, P) = \hat{R}$. A direct verification shows that the composition of $\beta(R)\colon R \to HF(R)$ with this isomorphism is $\rho(P)$. It, therefore, follows from the remark following 1.6 that ${}_R\hat{R}$ is $\mathscr{T}_T$-injective and is in $\mathscr{F}_T$. Further, the remark following 1.5 implies that $\ker \rho(P) \in \mathscr{T}_T$. Thus 1.1 will imply the existence of γ if it can be shown that $\mathrm{coker}\, \rho(P) \in \mathscr{T}_T$. It, therefore, suffices to see that $T\hat{R} \subseteq \mathrm{im}\, \rho(P)$. This follows from the fact that $f(x)\hat{r} = (f((x)\hat{r}))\rho(P)$ for all $x \in P_R$, $f \in \mathrm{Hom}_R(P, R)$, and $\hat{r} \in \hat{R}$. To verify this, it must be shown that these functions have the same value at each $y \in P$. In order to do this, we define a mapping s_y of P into itself by $s_y(w) = yf(w)$ for all $w \in P$. A direct verification shows that $s_y \in S$. Hence $(y)(f((x)\hat{r}))\rho(P) = yf((x)\hat{r}) = s_y((x)\hat{r}) = (s_y(x))\hat{r} = (yf(x))\hat{r} = (y)(f(x)\hat{r})$.

Since the filter $\mathscr{J}_T$ of left ideals corresponding to $\mathscr{T}_T$ has T as minimal element, it follows directly from the definition of the quotient category that $Q_T = \mathrm{Hom}_R(T, R/t_T(R))$. However, $T^2 = T$ implies that for any $g \in \mathrm{Hom}_R(T, R/t_T(R))$, $\mathrm{im}\, g \subseteq T/t_T(R) \cap T = T/t_T(T)$. Further, since $T/t_T(T) \in \mathscr{F}_T$, any such g must have $t_T(T) \subseteq \ker g$. Thus $Q_T \cong \mathrm{End}_R(T/t_T(T))$.

A ring R is said to be *semi-prime* if R has no nonzero nilpotent ideals. Equivalently, R is semi-prime if for any $0 \neq r \in R$, there is an $r' \in R$ such that $rr'r \neq 0$. R is *prime* if any nonzero ideal of R has zero annihilator.

COROLLARY 2.2. *Let P_R be a projective right R-module with double centralizer $\hat{R}$. If R is semi-prime, then $\hat{R} \cong \mathrm{End}_R({}_RT)$. Thus R semi-prime* (prime) *implies $\hat{R}$ is semi-prime* (prime).

Proof. Assume R is semi-prime. Since $t_T(T) = \{t \in T \mid Tt = 0\}$, $(t_T(T))^2 = 0$ and hence $t_T(T) = 0$. Thus the first assertion follows from 2.1. The second assertion is now immediate from [19, Proposition 1.2].

COROLLARY 2.3. *If P_R is a faithful projective right R-module, its double centralizer is $\hat{R} \cong \{q \in Q(R) \mid Tq \subseteqq T\}$, where $Q(R)$ is the maximal ring of left quotients of R.*

Proof. Since P_R is faithful, the torsion class $\mathscr{T}_T$ is faithful by 1.4. Thus ${}_RR$, and hence ${}_RT$, is in $\mathscr{F}_T$. Since T is a dense left ideal of R by 1.4, $\hat{R} \cong \mathrm{End}_R(T) \cong \{q \in Q(R) \mid Tq \subseteqq T\}$, where the first isomorphism follows from 2.1 and the second from [11, Proposition 5, p. 97].

COROLLARY 2.4. *If P_R is a faithful projective right R-module, then P_R has the double centralizer property if and only if* $\mathrm{Ext}^1_R(R/T, R) = 0$.

Proof. Since P_R is faithful, 1.4 implies that ${}_RR \in \mathscr{F}_T$. Hence $R = Q_T$ iff R is $\mathscr{T}_T$-injective. However, since T is the minimal element of the filter $\mathscr{f}_T$ and ${}_RR \in \mathscr{F}_T$, this occurs iff $\mathrm{Ext}^1_R(R/T, R) = 0$. The conclusion is now immediate from 2.1.

Let M be an R-module which has a direct sum decomposition $M = \bigoplus \sum_{\alpha \in A} M_\alpha$ with the endomorphism ring of each M_α a local ring. If $\{M_\beta\}_{\beta \in \Gamma}$ is a set of representatives for distinct isomorphism classes of M_α's, the *basic submodule* of M is defined to be $\bar{M} = \bigoplus \sum_{\beta \in \Gamma} M_\beta$. It follows from Azumaya's generalization of the Krull-Schmidt theorem [1, Theorem 1] that the basic submodule of M is uniquely determined to within isomorphism.

The next several results will be concerned with right perfect rings. The definition and basic properties of these rings, as well as any terminology not defined here, may be found in [2].

COROLLARY 2.5. *If R is a right perfect ring, there exists a faithful, finitely generated projective right R-module P_R whose double centralizer is isomorphic to the maximal ring of left quotients of R.*

Proof. Let ${}_RM$ be the projective cover of the basic submodule of the left socle of R, and let $P_R = \mathrm{Hom}_R(M, R)$. Since ${}_RM$ is finitely generated and projective, so is P_R by [3, Chapter II, Proposition 4.1 and Theorem 3.4]. Moreover, they have the same trace ideal T. This corollary will follow from 2.1 if we show that the filter $\mathscr{f}_T$ is equal to the filter of dense left ideals of R. Since T is the minimal element of the filter $\mathscr{I}_T$, it will suffice to show that T is a dense left ideal which is contained in all dense left ideals of R.

If X is a minimal left ideal of R, $T\cdot X = X$ since X is a homomorphic image of M and $T\cdot M = M$. Thus the right annihilator of T in R intersects the left socle of R in zero. Since the left socle of a right perfect ring is an essential left ideal, the right annihilator of T in R is zero. Thus P_R is faithful, and hence T is a dense left ideal of R by 1.4.

Let D be a dense left ideal of R, i.e., assume $\mathrm{Hom}_R(R/D, E(R)) = 0$. Suppose $\mathrm{Hom}_R(M, R/D) \neq 0$. Since M is finitely generated, this implies that R/D contains a submodule ${}_RK$ which has a simple epimorphic image isomorphic to a simple epimorphic image of M. But each of these is in the left socle of R, so $\mathrm{Hom}_R(R/D, E(R)) \neq 0$. This contradiction implies $\mathrm{Hom}_R(M, R/D) = 0$. Thus for all $f \in \mathrm{Hom}_R(M, R)$, $\mathrm{im}\, f \subseteqq D$ and hence $T \subseteqq D$.

If R is right perfect and P_R is a projective right R-module, $P_R \cong \bigoplus \sum_{\lambda \in \Lambda} e_\lambda R$ where e_λ is a primitive idempotent in R for each λ in the index set Λ. Since the endomorphism ring $e_\lambda R e_\lambda$ of $e_\lambda R$ is a local ring, the basic submodule $\bar{P}_R$ of P_R is defined. $\bar{P}_R$ is a finitely generated projective module having the same trace ideal T as P_R and is a direct summand of any projective right R-module having T as trace ideal. We note that a simple left R-module belongs to $\mathscr{T}_T$ if and only if it is not a homomorphic image of $\bar{P}^* = \mathrm{Hom}_R(\bar{P}, R)$. Thus the following theorem generalizes half of [6, Theorem 4].

THEOREM 2.6. *If R is a right perfect ring and P_R is a faithful projective right R-module with trace ideal T, then P_R has the double centralizer property if and only if* $\mathrm{Ext}^1_R(X, R) = 0$ *for every simple left R-module X in $\mathscr{T}_T$.*

Proof. We first show that for any $M \in {}_R\mathscr{M}$, $\mathrm{Hom}_R(R/T, M) = 0$ iff $\mathrm{Hom}_R(X, M) = 0$ for all simple modules X in $\mathscr{T}_T$. Since T is the minimal element of the filter $\mathscr{I}_T$, $\mathrm{Hom}_R(R/T, M) = 0$ iff $M \in \mathscr{F}_T$. As R is right perfect, M has an essential socle and hence M belongs to $\mathscr{F}_T$ iff its socle does. Thus we conclude that $\mathrm{Hom}_R(R/T, M) = 0$ iff $\mathrm{Hom}_R(X, M) = 0$ for all simple modules X in $\mathscr{T}_T$.

Since P_R is faithful, T is a dense left ideal of R by 1.4 and hence $\mathrm{Hom}_R(R/T, E(R)) = 0$. Thus $\mathrm{Hom}_R(X, E(R)) = 0$ for all simple modules X in $\mathscr{T}_T$. The exact sequence

$$0 \longrightarrow R \longrightarrow E(R) \longrightarrow E(R)/R \longrightarrow 0$$

gives exact sequences

$$(1)\quad \begin{aligned} 0 &\longrightarrow \operatorname{Hom}_R(R/T, R) \longrightarrow \operatorname{Hom}_R(R/T, E(R)) \longrightarrow \operatorname{Hom}_R(R/T, E(R)/R) \\ &\longrightarrow \operatorname{Ext}^1_R(R/T, R) \longrightarrow 0 \end{aligned}$$

and

$$(2)\quad \begin{aligned} 0 &\longrightarrow \operatorname{Hom}_R(X, R) \longrightarrow \operatorname{Hom}_R(X, E(R)) \longrightarrow \operatorname{Hom}_R(X, E(R)/R) \\ &\longrightarrow \operatorname{Ext}^1_R(X, R) \longrightarrow 0 \end{aligned}$$

for any simple module X in $\mathscr{T}_T$. Since $\operatorname{Hom}_R(R/T, E(R)) = 0$, sequence (1) gives $\operatorname{Hom}_R(R/T, E(R)/R) \cong \operatorname{Ext}^1_R(R/T, R)$. Thus by 2.4, P_R has the double centralizer property iff $\operatorname{Hom}_R(R/T, E(R)/R) = 0$. By the result of the first paragraph, this is equivalent to $\operatorname{Hom}_R(X, E(R)/R) = 0$ for all simple modules X in $\mathscr{T}_T$. Since $\operatorname{Hom}_R(X, E(R)) = 0$ for all such X, sequence (2) gives $\operatorname{Hom}_R(X, E(R)/R) \cong \operatorname{Ext}^1_R(X, R)$ for all simple modules X in $\mathscr{T}_T$. Thus P_R has the double centralizer property iff $\operatorname{Ext}^1_R(X, R) = 0$ for all simple modules X in $\mathscr{T}_T$.

The next result generalizes one half of [6, Theorem 5].

COROLLARY 2.7. *If R is right perfect, a necessary and sufficient condition for every faithful projective right R-module to have the double centralizer property is that* $\operatorname{Ext}^1_R(X, R) = 0$ *for every simple left R-module X which is not isomorphic to a left ideal of R.*

Proof. Let P_R be the module defined in the proof of 2.5. Then a simple module X is not isomorphic to a left ideal of R iff $X \in \mathscr{T}_T$. It therefore follows from 2.6 that the condition is necessary.

Conversely, suppose P_R is an arbitrary faithful projective module. If M is a minimal left ideal of R, $P \otimes_R M \cong P \cdot M \neq 0$. Thus $M \notin \mathscr{T}_T$. Hence if X is a simple module in $\mathscr{T}_T$, X is not isomorphic to a minimal left ideal of R. Thus the condition is sufficient by 2.6.

Finally, we obtain a generalization of theorems of Tachikawa [16, Theorem 1.4] and Mochizuki [14, Theorem 3.1]. For a given module W, a module M is said to have *W-dominant dimension* $\geqq n$ if there is an exact sequence $0 \rightarrow M \rightarrow X_1 \rightarrow \cdots \rightarrow X_n$, where each X_i is a direct product of copies of W.

COROLLARY 2.8. *If the $E(R)$-dominant dimension of ${}_RR$ is $\geqq 2$, every faithful projective right R-module has the double centralizer property. If R is right perfect, the converse is true.*

Proof. Assume the $E(R)$-dominant dimension of R is $\geqq 2$. Then since $E(R)$ is injective, there is an exact sequence $0 \rightarrow E(R)/R \rightarrow \Pi E(R)$. This gives an exact sequence

$$0 \to \mathrm{Hom}_R(R/T, E(R)/R) \to \mathrm{Hom}_R(R/T, \Pi E(R)) \cong \Pi\, \mathrm{Hom}_R(R/T, E(R)) .$$

Since P_R is faithful, ${}_RT$ is a dense left ideal of R by 1.4. Thus $\mathrm{Hom}_R(R/T, E(R)) = 0$, and hence $\mathrm{Hom}_R(R/T, E(R)/R) = 0$. This implies $\mathrm{Ext}^1_R(R/T, R) = 0$, as in the proof of 2.6. Thus the conclusion follows from 2.4.

Conversely, assume R is right perfect and each faithful projective right R-module has the double centralizer property. Let P_R be the module defined in the proof of 2.5. Then it follows from 2.6 that $\mathrm{Ext}^1_R(X, R) = 0$ for every simple module X in $\mathscr{T}_T$. As in the proof of 2.6, this implies $\mathrm{Hom}_R(X, E(R)/R) = 0$ for every such X. But P_R was chosen so that the simple modules in $\mathscr{T}_T$ are precisely those simple modules not isomorphic to minimal left ideals of R. Since R is right perfect, $E(R)/R$ has an essential socle and, as we have just shown, each simple submodule of $E(R)/R$ is isomorphic to a minimal left ideal of R. Hence there exists a monomorphism from $E(R)/R$ into a direct product of copies of $E(R)$. Thus ${}_RR$ has $E(R)$-dominant dimension $\geqq 2$.

3. Correspondence of torsion classes.

If $\mathscr{T}_R$ is a hereditary torsion class in ${}_R\mathscr{M}$, define

$$F(\mathscr{T}_R) = \{N \in {}_S\mathscr{M} \mid N \cong F(M) \quad \text{for some} \quad M \in \mathscr{T}_R\} .$$

Similarly, for a hereditary torsion class $\mathscr{T}_S$ in ${}_S\mathscr{M}$, define

$$F^{\leftarrow}(\mathscr{T}_S) = \{M \in {}_R\mathscr{M} \mid F(M) \in \mathscr{T}_S\} .$$

These definitions will be used to establish a one-to-one correspondence between the hereditary torsion classes in ${}_S\mathscr{M}$ and those in ${}_R\mathscr{M}$ containing $\mathscr{T}_T$.

LEMMA 3.1. *If $\mathscr{T}_R \supseteqq \mathscr{T}_T$ and $M \in \mathscr{T}_R$, then $GF(M) \in \mathscr{T}_R$ and $HF(M) \in \mathscr{T}_R$. Thus if $N \in F(\mathscr{T}_R)$, $H(N) \in \mathscr{T}_R$ and $G(N) \in \mathscr{T}_R$.*

Proof. The sequence $0 \to \ker \beta'(M) \to GF(M) \to M$ is exact, $M \in \mathscr{T}_R$, and by 1.5, $\ker \beta'(M) \in \mathscr{T}_T \subseteqq \mathscr{T}_R$. Since $\mathscr{T}_R$ is closed under submodules and extensions, this implies that $GF(M) \in \mathscr{T}_R$. The proof that $HF(M) \in \mathscr{T}_R$ is similar. The last statement is now immediate from the definition of $F(\mathscr{T}_R)$.

PROPOSITION 3.2. *$F^{\leftarrow}(\mathscr{T}_S)$ is a hereditary torsion class in ${}_R\mathscr{M}$ containing $\mathscr{T}_T$. If $\mathscr{T}_R \supseteqq \mathscr{T}_T$, then $F(\mathscr{T}_R)$ is a hereditary torsion class in ${}_S\mathscr{M}$.*

Proof. Since $\{0\} \subseteqq \mathscr{T}_S$ and $F^{\leftarrow}(\{0\}) = \mathscr{T}_T$, it is clear that $F^{\leftarrow}(\mathscr{T}_S) \supseteqq \mathscr{T}_T$. That $F^{\leftarrow}(\mathscr{T}_S)$ is closed under direct sums is immediate from the fact that $\mathscr{T}_S$ has this property, since F commutes with direct sums. Finally, if $0 \to M' \to M \to M'' \to 0$ is exact in ${}_R\mathscr{M}$, $0 \to F(M') \to F(M) \to F(M'') \to 0$ is exact in ${}_S\mathscr{M}$. Thus $F(M) \in \mathscr{T}_S$ iff $F(M')$ and $F(M'') \in \mathscr{T}_S$. Hence $M \in F^{\leftarrow}(\mathscr{T}_S)$ iff M' and $M'' \in F^{\leftarrow}(\mathscr{T}_S)$. It follows that $F^{\leftarrow}(\mathscr{T}_S)$ is a hereditary torsion class.

Now let $\mathscr{T}_R \supseteq \mathscr{T}_T$. Clearly $F(\mathscr{T}_R)$ is closed under direct sums. Let $0 \to N' \to N \to N'' \to 0$ be an exact sequence in ${}_S\mathscr{M}$. Then $0 \to H(N') \to H(N)$ and $G(N) \to G(N'') \to 0$ are exact sequences in ${}_R\mathscr{M}$. If $N \in F(\mathscr{T}_R)$, 3.1 implies that $H(N)$ and $G(N) \in \mathscr{T}_R$. Thus $H(N')$ and $G(N'') \in \mathscr{T}_R$. Since $N' \cong FH(N') \in F(\mathscr{T}_R)$ and $N'' \cong FG(N'') \in F(\mathscr{T}_R)$, the class $F(\mathscr{T}_R)$ is closed under both submodules and homomorphic images. Suppose that both N' and $N'' \in F(\mathscr{T}_R)$. Then 3.1 implies that $H(N')$ and $H(N'') \in \mathscr{T}_R$. Since the sequence $0 \to H(N') \to H(N) \to H(N'')$ is exact, $H(N) \in \mathscr{T}_R$. Thus $N \cong FH(N) \in F(\mathscr{T}_R)$, and so $F(\mathscr{T}_R)$ is closed under extensions. Hence $F(\mathscr{T}_R)$ is a hereditary torsion class in ${}_S\mathscr{M}$.

THEOREM 3.3. *The mappings $\mathscr{T}_R \to F(\mathscr{T}_R)$ and $\mathscr{T}_S \to F^{\leftarrow}(\mathscr{T}_S)$ are inclusion-preserving, inverse one-to-one correspondences between the hereditary torsion classes in ${}_S\mathscr{M}$ and those hereditary torsion classes in ${}_R\mathscr{M}$ containing $\mathscr{T}_T$.*

Proof. The mappings clearly preserve inclusion. Thus by 3.2, it suffices to show that $F(F^{\leftarrow}(\mathscr{T}_S)) = \mathscr{T}_S$ and $F^{\leftarrow}(F(\mathscr{T}_R)) = \mathscr{T}_R$ if $\mathscr{T}_R \supseteq \mathscr{T}_T$. Clearly $F(F^{\leftarrow}(\mathscr{T}_S)) \subseteq \mathscr{T}_S$. Suppose $N \in \mathscr{T}_S$. Since $N \cong FH(N)$, $H(N) \in F^{\leftarrow}(\mathscr{T}_S)$ and hence $N \in F(F^{\leftarrow}(\mathscr{T}_S))$. Thus $\mathscr{T}_S = F(F^{\leftarrow}(\mathscr{T}_S))$. It is apparent that $\mathscr{T}_R \subseteq F^{\leftarrow}(F(\mathscr{T}_R))$. Let $M \in F^{\leftarrow}(F(\mathscr{T}_R))$. Then $F(M) \in F(\mathscr{T}_R)$, and so 3.1 implies $HF(M) \in \mathscr{T}_R$. Since the sequence $0 \to \ker \beta(M) \to M \to HF(M)$ is exact, $HF(M) \in \mathscr{T}_R$, and $\ker \beta(M) \in \mathscr{T}_T \subseteq \mathscr{T}_R$ by 1.5, we have $M \in \mathscr{T}_R$. Thus $\mathscr{T}_R = F^{\leftarrow}(F(\mathscr{T}_R))$.

The most useful rings of left quotients are those constructed with respect to faithful hereditary torsion classes. We thus ask under what circumstances the above correspondence gives a one-to-one correspondence of the faithful hereditary torsion classes in ${}_S\mathscr{M}$ with those in ${}_R\mathscr{M}$ containing $\mathscr{T}_T$.

PROPOSITION 3.4. *Let $\mathscr{T}_R$ and $\mathscr{T}_S$ correspond as in Theorem 3.3. Then $M \in \mathscr{F}_R$ if and only if $M \in \mathscr{F}_T$ and $F(M) \in \mathscr{F}_S$.*

Proof. Let $M \in \mathscr{F}_R$. Since $\mathscr{T}_R \supseteq \mathscr{T}_T$, we have $\mathscr{F}_R \subseteq \mathscr{F}_T$, and so $M \in \mathscr{F}_T$. For any $N \in \mathscr{T}_S = F(\mathscr{T}_R)$, $\mathrm{Hom}_S(N, F(M)) \cong \mathrm{Hom}_R(G(N), M) = 0$ since $M \in \mathscr{F}_R$ and $G(N) \in \mathscr{T}_R$ by 3.1. Thus $F(M) \in \mathscr{F}_S$.

Conversely, let $M \in \mathscr{F}_T$ and $F(M) \in \mathscr{F}_S$, but suppose $M \notin \mathscr{F}_R$. Then there is a nonzero submodule K of M with $K \in \mathscr{T}_R$, but $K \notin \mathscr{T}_T$.

Hence $F(K)$ is a nonzero submodule of $F(M)$ with $F(K) \in F(\mathscr{T}_R) = \mathscr{T}_S$. This contradicts the fact that $F(M) \in \mathscr{F}_S$. Thus $M \in \mathscr{F}_R$.

REMARK. Note that if $\mathscr{T}_R$ and $\mathscr{T}_S$ correspond as in 3.3 and $\mathscr{T}_R$ is faithful, then $\mathscr{T}_S$ is faithful. For since ${}_RR \in \mathscr{F}_R$, we must have ${}_SP = F({}_RR) \in \mathscr{F}_S$ by 3.4. Since ${}_SP$ is a generator, ${}_SS \in \mathscr{F}_S$. The converse is false. The following examples illustrate how a faithful $\mathscr{T}_S$ can correspond to a non-faithful $\mathscr{T}_R$.

EXAMPLES 3.5. (a). Let S be any ring with nontrivial $E(S)$-torsion theory and $0 \neq {}_SX$ an S-module which is $E(S)$-torsion. Then ${}_SP = {}_SS \oplus {}_SX$ is a generator in ${}_S\mathscr{M}$. Let $R = \mathrm{End}_S({}_SP)$. Then P_R is faithful, finitely generated, and projective. Hence $\mathscr{T}_T$ is faithful by 1.4. By 3.4, each faithful hereditary torsion class in ${}_R\mathscr{M}$ containing $\mathscr{T}_T$ corresponds to a faithful hereditary torsion class with respect to which ${}_SP$ is torsion-free. Since ${}_SP$ is *not* torsion-free with respect to the $E(S)$-torsion class, this torsion class cannot correspond to a faithful torsion class in ${}_R\mathscr{M}$.

(b). If P_R is not faithful, then $\mathscr{T}_T$ is not faithful and hence neither is any torsion class containing $\mathscr{T}_T$. Thus no torsion class in ${}_S\mathscr{M}$ corresponds to a faithful torsion class in ${}_R\mathscr{M}$.

We use the notation $E(S)$-torsion-free to mean torsion-free with respect to the $E(S)$-torsion class in ${}_S\mathscr{M}$.

THEOREM 3.6. *The correspondence of Theorem* 3.3 *induces a one-to-one correspondence of the faithful hereditary torsion classes in* ${}_S\mathscr{M}$ *and the faithful hereditary torsion classes in* ${}_R\mathscr{M}$ *containing* $\mathscr{T}_T$ *if and only if* P_R *is faithful and* ${}_SP$ *is* $E(S)$*-torsion-free.*

Proof. Assume the correspondence is as desired. Since $\{0\}$ is a faithful hereditary torsion class in ${}_S\mathscr{M}$, $F^{\leftarrow}(\{0\}) = \mathscr{T}_T$ is faithful. Hence P_R must be faithful by 1.4. Furthermore, the $E(S)$-torsion class corresponds to a torsion class $\mathscr{T}_R$ with ${}_RR \in \mathscr{F}_R$. Thus 3.4 implies $F({}_RR) = {}_SP$ is $E(S)$-torsion-free.

Conversely, let P_R be faithful and ${}_SP$ be $E(S)$-torsion-free. Then $\mathscr{T}_T$ is faithful by 1.4. By the remark just preceding 3.5, it suffices to show that $\mathscr{T}_S$ faithful implies $\mathscr{T}_R = F^{\leftarrow}(\mathscr{T}_S)$ faithful. If $\mathscr{T}_S$ is faithful, $\mathscr{F}_S$ contains all $E(S)$-torsion-free modules, and so ${}_SP \in \mathscr{F}_S$. Thus ${}_RR \in \mathscr{F}_R$ by 3.4, and hence $\mathscr{T}_R$ is faithful.

COROLLARY 3.7. *If* P_R *is faithful and* ${}_SP$ *is* $E(S)$*-torsion-free, the* $E(S)$*- and* $E(R)$*-torsion classes correspond under the correspondence of Theorem* 3.3.

Proof. Both are maximal faithful hereditary torsion classes.

COROLLARY 3.8. *If $_SP$ is an $E(S)$-torsion-free generator and $R = \mathrm{End}_S(_SP)$, the correspondence of Theorem 3.3 induces a one-to-one correspondence of the faithful hereditary torsion classes in $_S\mathscr{M}$ and those in $_R\mathscr{M}$ containing $\mathscr{T}_T$. In particular, the $E(S)$-and $E(R)$-torsion classes correspond.*

REMARK. A left S-module N is *torsionless* if there is a monomorphism of N into a direct product of copies of S. Hence all torsionless and, in particular, projective left S-modules are $E(S)$-torsion-free. Thus Corollary 3.8 is valid for torsionless or projective generators.

Two modules are *similar* if each is isomorphic to a direct summand of a finite direct sum of copies of the other.

LEMMA 3.9. *If $_SP$ is a generator and $\mathrm{End}_S(_SP)$ is a semi-prime ring, then $_SP$ is torsionless.*

Proof. Since $_SP$ is a generator and $_SS$ is finitely generated, there is an epimorphism of a finite direct sum of copies of $_SP$ onto S. Since $_SS$ is projective, this epimorphism splits and so this finite direct sum of copies of $_SP$ has the form $S \oplus X$. Thus $_SP$ is similar to a module of this type. But similar modules have Morita-equivalent endomorphism rings [7, Theorem 1.5], and a ring Morita equivalent to a semi-prime ring is semi-prime [19, Proposition 1.2]. Hence it suffices to prove this lemma for modules of the type $S \oplus X$. In particular, it is enough to show $_SX$ torsionless.

If $\mathrm{End}_S(S \oplus X)$ is semi-prime, a standard matrix argument shows that for any nonzero $\hat{x} \in \mathrm{Hom}_S(S, X) \cong X$, there exists $g \in \mathrm{Hom}_S(X, S)$ such that $\hat{x}g\hat{x} \neq 0$. If $x = (1)\hat{x}$, this yields $(x)gx \neq 0$. In particular, $(x)g \neq 0$. Since $x \neq 0$ is arbitrary, X is torsionless.

We note in passing that the condition, for any nonzero $x \in X$ there is a homomorphism $g: X \to S$ such that $(x)gx \neq 0$, is a generalization to modules of the concept of semi-prime rings of some independent interest. Such modules might reasonably be termed semi-prime modules. A similar condition, for any nonzero x and y in X there is a homomorphism $g: X \to S$ such that $(x)gy \neq 0$, is a generalization to modules of the concept of prime rings. This arises in the proof of 3.9 if we assume that $\mathrm{End}_S(S \oplus X)$ is prime.

Finally, we obtain conditions on the ring R alone giving a one-to-one correspondence between the faithful hereditary torsion classes in $_S\mathscr{M}$ and those in $_R\mathscr{M}$ containing $\mathscr{T}_T$.

PROPOSITION 3.10. *If R is a semi-prime ring, P_R is a faithful* *finitely generated projective right R-module, and* $S = \mathrm{End}_R(P_R)$, *then the correspondence of Theorem 3.3 induces a one-to-one correspondence between the faithful hereditary torsion classes in* ${}_S\mathscr{M}$ *and those in* ${}_R\mathscr{M}$ *containing* $\mathscr{T}_T$. *In particular, the* $E(S)$*-and* $E(R)$*-torsion clasess correspond.*

Proof. We showed in 2.2 that $\mathrm{End}_S({}_SP) \cong \mathrm{End}_R({}_RT)$. Since ${}_RT$ is torsionless and R is semi-prime, $\mathrm{End}_S({}_SP) \cong \mathrm{End}_R({}_RT)$ is semi-prime by [19, Proposition 1.2]. By 3.9, ${}_SP$ is torsionless, whence ${}_SP$ is $E(S)$-torsion-free. The result then follows by 3.6.

REMARK. If R is prime and $P_R \neq 0$, the condition that P_R be faithful is redundant.

4. Endomorphism rings. We recall that unless we indicate otherwise P_R will denote a finitely generated projective module and that $S = \mathrm{End}_R(P_R)$. We also make the standing assumption that $\mathscr{T}_S$ and $\mathscr{T}_R \supseteqq \mathscr{T}_T$ are torsion classes in ${}_S\mathscr{M}$ and ${}_R\mathscr{M}$, with torsion-free classes $\mathscr{F}_S$ and $\mathscr{F}_R$, respectively, which correspond as in Theorem 3.3. We denote that associated localization functors by L_S and L_R and the corresponding rings of left quotients by Q_S and Q_R, respectively.

LEMMA 4.1. *If* M *is in* $\mathscr{F}_T$ *and* M *is* $\mathscr{T}_R$*-injective,* $F(M)$ *is* $\mathscr{T}_S$*-injective.*

Proof. Let $f: {}_SN' \to {}_SN$ be a monomorphism with $\mathrm{coker}\, f \in \mathscr{T}_S$, and let $g: N' \to F(M)$. Then $H(f): H(N') \to H(N)$ is a monomorphism with $\mathrm{coker}\, H(f)$ isomorphic to a submodule of $H(N/N')$. But $H(N/N') \in \mathscr{T}_R$ by 3.1, which implies $\mathrm{coker}\, H(f) \in \mathscr{T}_R$. Now $M \in \mathscr{F}_T$ by hypothesis, and since ${}_RM$ is $\mathscr{T}_R$-injective and $\mathscr{T}_R \supseteqq \mathscr{T}_T$, M is $\mathscr{T}_T$-injective. Therefore, 1.6 implies that $M \cong HF(M)$. Thus $HF(M)$ is $\mathscr{T}_R$-injective, and so there exists $l: H(N) \to HF(M)$ such that $H(f) \circ l = H(g)$. Applying the functor F and recalling that FH is naturally equivalent to the identity functor on ${}_S\mathscr{M}$, we obtain $\bar{l}: N \to F(M)$ such that $f \circ \bar{l} = g$. Thus $F(M)$ is $\mathscr{T}_S$-injective.

PROPOSITION 4.2. *For any* $M \in {}_R\mathscr{M}$, *there exists a unique isomorphism* γ *from* $F(L_R(M))$ *to* $L_S(F(M))$ *such that* $F(\sigma(M)) \circ \gamma = \sigma(F(M))$.

Proof. Since $L_R(M)$ is in $\mathscr{F}_R$ and is $\mathscr{T}_R$-injective, $F(L_R(M))$ is in $\mathscr{F}_S$ by 3.4 and is $\mathscr{T}_S$-injective by 4.1. Furthermore, since $\ker \sigma(M)$ and $\mathrm{coker}\, \sigma(M)$ are in $\mathscr{T}_R$ and F is exact, $\ker F(\sigma(M))$ and $\mathrm{coker}\, F(\sigma M))$ are in $\mathscr{T}_S$. The desired conclusion is now immediate from 1.1.

THEOREM 4.3. *Let P_R be a finitely generated projective right R-module, with $S = \mathrm{End}_R(P_R)$. If Q_R and Q_S are rings of left quotients of R and S with respect to hereditary torsion classes $\mathscr{T}_R$ and $\mathscr{T}_S$ which correspond as in Theorem 3.3, then $P \otimes_R Q_R$ is a finitely generated projective right Q_R-module with $Q_S \cong \mathrm{End}_{Q_R}(P \otimes_R Q_R)$ and $Q_R \cong \mathrm{End}_{Q_S}(P \otimes_R Q_R)$.*

Proof. It is clear that $P \otimes_R Q_R$ is a finitely generated projective right Q_R-module. By 4.2, $P \otimes_R Q_R \cong L_S({}_SP)$. Thus $P \otimes_R Q_R$ is $\mathscr{T}_S$-injective and is in $\mathscr{F}_S$. It follows [9, Corollary 4.2] that $P \otimes_R Q_R$ has a unique structure as a left Q_S-module which extends its natural structure as a left S-module. Since ${}_SP$ is a generator and ${}_SS$ is finitely generated, ${}_SS$ is a direct summand of a finite direct sum of copies of ${}_SP$. Hence since L_S is an additive functor, $L_S({}_SS) = Q_S$ is a direct summand of a finite direct sum of copies of $L_S({}_SP)$. Thus $L_S({}_SP)$, and hence also $P \otimes_R Q_R$, is a left Q_S-generator.

It is immediate from 4.2 that $\ker F(\sigma(R))$ and $\mathrm{coker}\, F(\sigma(R))$ are in $\mathscr{T}_S$. Composing $F(\sigma(R))$ with the canonical isomorphism of P onto $P \otimes_R R$ yields an S-homomorphism $f\colon {}_SP \to {}_SP \otimes_R Q_R$ given by $(p)f = p \otimes 1_{Q_R}$. Furthermore, $\ker f$ and $\mathrm{coker}\, f$ are in $\mathscr{T}_S$. Thus 1.3 implies that $\mathrm{Hom}_S(P, P \otimes_R Q_R) \cong \mathrm{End}_S(P \otimes_R Q_R)$. By 1.6, we also have $Q_R \cong \mathrm{Hom}_S(P, P \otimes_R Q_R)$ via $\beta(Q_R)$. Composing these maps gives an abelian group isomorphism $Q_R \cong \mathrm{End}_S(P \otimes_R Q_R) = \mathrm{End}_{Q_S}(P \otimes_R Q_R)$, with the equality coming via 1.2. However, a direct verification shows that this composition is just the canonical mapping taking each element of Q_R into the right multiplication it defines on $P \otimes_R Q_R$. Thus it is a ring isomorphism.

Since $P \otimes_R Q_R$ is a left Q_S-generator with $Q_R \cong \mathrm{End}_{Q_S}(P \otimes_R Q_R)$, and since generators have the double centralizer property [3, Chapter II, Proposition 4.4 and Theorem 3.4], $Q_S \cong \mathrm{End}_{Q_R}(P \otimes_R Q_R)$.

COROLLARY 4.4. *Let R be a semi-prime ring, P_R be a faithful finitely generated projective right R-module, and $S = \mathrm{End}_R(P_R)$. If $Q(R)$ and $Q(S)$ are the maximal rings of left quotients of R and S, respectively, then $P \otimes_R Q(R)$ is a faithful finitely generated projective right $Q(R)$-module and $Q(S) \cong \mathrm{End}_{Q(R)}(P \otimes_R Q(R))$.*

Proof. Immediate from 3.10 and 4.3.

If R is a prime ring, the assumption that P_R is faithful is redundant.

COROLLARY 4.5. *Let S be an arbitrary ring, ${}_SP$ an $E(S)$-torsion-free left S-generator, and $R = \mathrm{End}_S({}_SP)$. If $Q(R)$ and $Q(S)$ are the maximal rings of left quotients of R and S, respectively, then $P \otimes_R Q(R)$*

is a left $Q(S)$-generator and $Q(R) \cong \mathrm{End}_{Q(S)}(P \otimes_R Q(R))$.

Proof. Immediate from 3.7 and 4.3.

In particular, Corollary 4.5 is valid when ${}_SP$ is a projective left S-generator. This raises the question of whether, in this case, $P \otimes_R Q(R)$ is a projective left $Q(S)$-generator. The following example shows that, in general, the answer is no.

EXAMPLE 4.6. Let S be the full ring of linear transformations of an infinite-dimensional vector space, $P = \bigoplus \sum_{\alpha \in A} {}_SS$, and $R = \mathrm{End}_S({}_SP)$. Since S is a left self-injective regular ring, the localization of each $E(S)$-torsion-free module with respect to the $E(S)$-torsion class is its injective hull. In particular $Q(S) = S$. However, if A is chosen so that $|A| > |S|$, ${}_SP \otimes_R Q(R) \cong E({}_SP)$ is not a projective left S-module.

Our proof of this fact depends on Kaplansky's characterization of projective modules over regular rings [10] and the fact that the (up to isomorphism) unique minimal left ideal of S is not Σ-injective.

However, if the filter $\mathscr{F}_S$ corresponding to $\mathscr{T}_S$ has a cofinal set of finitely generated left ideals, ${}_SP$ projective does imply that $P \otimes_R Q_R$ is left Q_S-projective. For in this case, L_S commutes with direct sums [18, Lemma 3.1], and hence when ${}_SP$ is projective, $P \otimes_R Q_R \cong L_S({}_SP) \cong Q_S \otimes_S P$.

COROLLARY 4.7. *Let S be an arbitary ring, ${}_SP$ be a projective left S-generator, and $R = \mathrm{End}_S({}_SP)$. Let $\mathscr{T}_S$ be a hereditary torsion class in ${}_S\mathscr{M}$ whose filter $\mathscr{F}_S$ has a cofinal set of finitely generated left ideals, and let $\mathscr{T}_R$ correspond to $\mathscr{T}_S$ as in Theorem* 3.3. *If Q_S and Q_R are the rings of left quotients of S and R with respect to $\mathscr{T}_S$ and $\mathscr{T}_R$, respectively, then $Q_S \otimes_S P$ is a projective left Q_S-generator and* $Q_R \cong \mathrm{End}_{Q_S}(Q_S \otimes_S P)$.

REMARK. In particular, suppose ${}_SP$ is a projective left S-generator and the filter of dense left ideals of S contains a cofinal set of finitely generated left ideals. If $Q(S)$ and $Q(R)$ are the maximal rings of left quotients of S and R, respectively, then $Q(S) \otimes_S P$ is a projective left $Q(S)$-generator with $Q(R) \cong \mathrm{End}_{Q(S)}(Q(S) \otimes_S P)$. This occurs, for example, when S is left Noetherian or $Q(S)$ is semi-simple Artinian. (See [15, Theorem 1.6].)

A left S-module ${}_SP$ is a *progenerator* if it is a finitely generated projective generator. If ${}_SP$ is a progenerator and $R = \mathrm{End}_S({}_SP)$, then P_R is also a progenerator [3, Chapter II, Theorem 3.4]. Two rings R and S are said to be *Morita equivalent* if there exists a category equivalence between ${}_R\mathscr{M}$ and ${}_S\mathscr{M}$. This can occur if and only if there is a left S-progenerator ${}_SP$ with $R \cong \mathrm{End}_S({}_SP)$, in which case the equivalence is given by the functor $F = P \otimes_R (\ )$. (See [3, Chapter II].)

Our results give the following generalization of a theorem of Turnidge [17, Theorem 2.4].

COROLLARY 4.8. *Let R and S be Morita equivalent via the bimodule ${}_SP_R$. Then the correspondence of Theorem 3.3 is a one-to-one correspondence of the hereditary torsion classes in ${}_R\mathscr{M}$ and ${}_S\mathscr{M}$ which induces a one-to-one correspondence of the faithful classes. If Q_R and Q_S are rings of quotients with respect to corresponding classes, then Q_R and Q_S are Morita equivalent via $Q_S \otimes_S P \cong P \otimes_R Q_R$. In particular, the maximal rings of left quotients $Q(R)$ and $Q(S)$ are Morita equivalent via $Q(S) \otimes_S P \cong P \otimes_R Q(R)$.*

Proof. Since P_R is a generator, $T = R$ whence $\mathscr{T}_T = \{0\}$. Thus the correspondence of 3.3 is a one-to-one correspondence between the hereditary torsion classes in ${}_R\mathscr{M}$ and ${}_S\mathscr{M}$. If $\mathscr{T}_R$ and $\mathscr{T}_S$ are corresponding classes, it is immediate from 3.8 that $\mathscr{T}_R$ is faithful iff $\mathscr{T}_S$ is faithful. Finally, ${}_SP$ finitely generated and projective implies $L_S({}_SP) \cong Q_S \otimes_S P$ since L_S is an additive functor. The remaining assertions follow from 4.2 and 4.3.

For a ring A, the *essential* (or large) left ideals of A form a filter contaning the filter of dense ideals. (See [8, pp. 416–420].) In general, the filter of essential left ideals is neither idempotent nor faithful. The essential left ideals of A form an idempotent faithful filter if and only if the *left singular ideal* of A,

$$Z({}_AA) = \{a \in A \mid (0\colon a) \quad \text{is essential in } A\}\ ,$$

is zero. Since the assumption that $Z(A) = 0$ plays a key role in many results concerning the maximal ring of left quotients of A, we examine it briefly.

For a left A-module X, the *singular submodule* of X is defined to be

$$Z({}_AX) = \{x \in X \mid (0\colon x) \quad \text{is essential in } A\}\ .$$

THEOREM 4.9. *Let P_R be a faithful finitely generated projective right R-module with $S = \mathrm{End}_R(P_R)$. Then $Z({}_RR) = 0$ if and only if $Z({}_SP) = 0$. In particular, $Z({}_RR) = 0$ implies $Z({}_SS) = 0$ and $Q(S) \cong \mathrm{End}_{Q(R)}(P \otimes_R Q(R))$, where $Q(R)$ and $Q(S)$ are the maximal rings of left quotients of R and S, respectively.*

Proof. Assume $Z({}_RR) = 0$. Since $\hat{R} = \mathrm{End}_S({}_SP)$ may be identified with a subring of $Q(R)$ containing R by 2.3, it follows from [5, Proposition 3, p. 70] that $Z({}_{\hat{R}}\hat{R}) = 0$. Recall that in proving 3.9 it was shown that since ${}_SP$ is a generator, it is similar to a module of the form $S \oplus X$. Further, similar modules have Morita-equivalent endomorphism rings [7, Theorem 1.5]. But 4.7 implies that the property of having zero singular ideal is preserved under Morita equivalence. This follows from the Morita invariance of regularity since a ring has zero singular ideal if and only if its maximal ring of left quotients is regular [5, Theorem 1, p. 69 and Proposition 3, p. 70]. Thus it will suffice to show that if ${}_SP = S \oplus X$ and $R = \mathrm{End}_S(S \oplus X)$ with $Z({}_RR) = 0$, then $Z({}_SS) = 0$ and $Z({}_SX) = 0$.

In this case, R has the form

$$R = \begin{pmatrix} S & X \\ \mathrm{Hom}_S(X, S) & \mathrm{End}_S(X) \end{pmatrix},$$

where we have made the usual identifications of $\mathrm{Hom}_S(S, S)$ with S and $\mathrm{Hom}_S(S, X)$ with X. If $s \in Z({}_SS)$ and $x \in Z({}_SX)$, a direct verification shows that the left annihilators of the elements

$$\begin{pmatrix} s & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$$

are essential in R, and so these elements belong to $Z({}_RR)$. Thus $s = 0$ and $x = 0$, and hence $Z({}_SS) = 0$ and $Z({}_SX) = 0$.

Conversely, since $Z({}_SP) = 0$ and ${}_SP$ is a left S-generator, $Z({}_SS) = 0$. Thus the filter of dense left ideals and the filter of essential left ideals of S coincide. Hence $Z({}_SP) = 0$ implies ${}_SP$ is $E(S)$-torsion-free. By 3.7, 4.3, and 4.2, $Q(R) \cong \mathrm{End}_S(L_S({}_SP))$. Since $Z({}_SS)$ and $Z({}_SP)$ are both zero, $L_S({}_SP) \cong E({}_SP)$, the injective hull of ${}_SP$. Thus [5, Lemma G, p. 69] implies that $Q(R)$ is regular, and hence $Z({}_RR) = 0$.

The last assertion is immediate from 3.7 and 4.3.

A module M is called *finite dimensional* if there do not exist infinitely many nonzero submodules of M whose sum is direct.

COROLLARY 4.10. *Let P_R be a faithful, finitely generated projective right R-module with $S = \mathrm{End}_R(P_R)$. Then the maximal ring of left quotients of R is semi-simple Artinian if and only if*

(i) *$Z({}_SP) = 0$, and*

(ii) *${}_SP$ is finite dimensional.*

If these conditions are satisfied, the maximal rings of left quotients $Q(R)$ and $Q(S)$ of R and S, respectively, are Morita equivalent via the module $P \otimes_R Q(R)$.

Proof. Assume $Q(R)$ is semi-simple Artinian. Then $Q(R)$ is regular, and hence $Z({}_RR) = 0$ [5, Proposition 3, p. 70]. Thus 4.8 implies $Z({}_SP) = 0$. It follows from 3.7 and 4.2 that $P \otimes_R Q(R) \cong L_S({}_SP)$, where the localization is with respect to the $E(S)$-torsion theory. However, since $Z({}_SS)$ and $Z({}_SP)$ are both zero, $L_S({}_SP) \cong E({}_SP)$, the injective hull of ${}_SP$. Hence by 1.2 and 4.3, $Q(R) \cong \mathrm{End}_S(E({}_SP))$. Thus ${}_SP$ is finite dimensional. For otherwise, $\mathrm{End}_S(E({}_SP))$ contains arbitrarily large finite sets of orthogonal idempotents; but this is impossible since $Q(R)$ is semi-simple Artinian.

Conversely, assume $Z({}_SP) = 0$ and ${}_SP$ is finite dimensional. Then $Z({}_RR) = 0$ by 4.8 and hence $Q(R)$ is regular [5, Theorem 1, p. 69]. It follows exactly as in the preceding paragraph that $Q(R) \cong \mathrm{End}_S(E({}_SP))$. Since ${}_SP$ is finite dimensional, this implies $Q(R)$ is semi-perfect by [11, Proposition 2, p. 103]. Hence $Q(R)$ is semi-simple Artinian.

If $Q(R)$ is semi-simple Artinian, 4.8 implies $Q(S) \cong \mathrm{End}_{Q(R)}(P \otimes_R Q(R))$. By 4.3, $P \otimes_R Q(R)$ is a finitely generated projective faithful right $Q(R)$-module. Since $Q(R)$ is semi-simple Artinian, this implies $P \otimes_R Q(R)$ is a $Q(R)$-progenerator. Thus $Q(R)$ and $Q(S)$ are Morita equivalent via $P \otimes_R Q(R)$.

REFERENCES

1. G. Azumaya, *Corrections and supplementaries to my paper concerning Krull-Remak-Schmidt's theorem*, Nagoya Math. J., **1** (1950), 117-124.

2. H. Bass, *Finitistic dimension and a homological generalization of semi-primary rings*, Trans. Amer. Math. Soc., **95** (1960), 466-488.

3. ————, *Algebraic K-theory*, Benjamin, New York. 1968.

4. S. E. Dickson, *A torsion theory for abelian categories*, Trans. Amer. Math. Soc., **121** (1966), 223-235.

5. C. Faith, *Lectures on Injective Modules and Quotient Rings*, Springer, Berlin-Heidelberg-New York, 1967.

6. K. R. Fuller, *Double centralizers of injectives and projectives over Artinian rings*, Ill. J. Math., **14** (1970), 658-664.

7. K. Hirata, *Some types of separable extensions of rings*, Nagoya Math. J., **33** (1968), 107-115.

8. P. Gabriel, *Des catégories abeliennes*, Bull. Soc. Math. France **90** (1962), 323-448.

9. O. Goldman, *Rings and modules of quotients*, J. Algebra, **13** (1969), 10-47.

10. I. Kaplansky, *Projective modules*, Ann. Math., **68** (1958), 372-377.

11. J. Lambek, *Lectures on Rings and Modules*, Blaisdell, Waltham, Mass., 1966.

12. ————, *Torsion Theories, Additive Semantics, and Rings of Quotients*, Springer, Berlin-Heidelberg-New York, 1971.

13. S. Mac Lane, *Categorical algebra*, Bull. Amer. Math. Soc., **71** (1965), 40-106.

14. H. Y. Mochizuki, *On the double commutator algebra of QF-3 algebras*, Nagoya Math. J., **25** (1965), 221-230.

15. F. L. Sandomierski, *Semi-simple maximal quotient rings*, Trans. Amer. Math. Soc., **128** (1967), 112-120.

16. H. Tachikawa, *On dominant dimension of QF-3 algebras*, Trans. Amer. Math. Soc., **112** (1964), 249-266.

17. D. R. Turnidge, *Torsion theories and rings of quotients of Morita equivalent rings*, Pacific J. Math., **37** (1971), 225-234.
18. C. L. Walker and E. A. Walker, *Quotient categories and rings of quotients*, Rocky Mtn. J. Math., (to appear).
19. J. M. Zelmanowitz, *Endomorphism rings of torsionless modules*, J. Algebra, **5** (1967), 325-341,

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