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DIFFERENTIABLE MAPS WITH 0-DIMENSIONAL CRITICAL SET, I

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Let $f: M^n \to N^p$ be C^n with $n - p = 0$ or 1, let $p \geqq 2$, and let $R_{p-1}(f)$ be the critical set of f. If $\dim(R_{p-1}(f)) \leqq 0$ and $\dim(f(R_{p-1}(f)) \leqq p - 2$, then (1.1) at each $x \in M^n$, f is locally topologically equivalent to one of the following maps:

(a) the projection map $\rho: R^n \to R^p$,

(b) $\sigma: C \to C$ defined by $\sigma(z) = z^d$ $(d = 2, 3, \ldots)$, where C is the complex plane, or

(c) $\tau: C \times C \to C \times R$ defined by $\tau(z, w) = (2z \cdot \bar{w}, |w|^2 - |z|^2)$, where $\bar{w}$ is the complex conjugate of w.

In particular, either f is locally topologically equivalent to ρ at each $x \in M^n$, or $(n,p) = (2, 2)$ or $(4, 3)$.

In a sequel the hypothesis on $\dim f(R_{p-1}(f))$ is eliminated.

For a $C^r (r \geqq 1)$ map $f: M^n \to N^p$ let $R_q(f)$ be the set of points $x \in M^n$ at which the rank of (the derivative map of) f is at most q. The *critical* set of f is defined to be $R_{p-1}(f)$ (in case $n < p$, $R_{p-1}(f) = M^n$), and according to the Rank Theorem [2, p. 155] at each $x \in M^n - R_{p-1}(f)$, f is locally C^r equivalent (1.3) to the map $\rho: R^n \to R^p$ defined by $\rho(x_1, x_2, \cdots x_n) = (x_1, x_2, \cdots, x_p)$. Thus (1.1) is a generalization of the Rank Theorem for $n - p = 0$ or 1, and $p \geqq 2$; moreover for $n - p = 1$ it answers a question of Milnor (1.7).

Note that while f is only C^n, the maps ρ, σ, and τ are real analytic. Simple examples (1.4) show that "topologically" cannot be replaced by "C^n" and that no reasonable classification is possible if $p = 1$. Propositions more general than (1.1) are also given ((4.7) and (4.9)).

Theorem (1.1) was announced in the talks [2] and [20]. For $n = p = 2$ (1.1) was essentially proved by Stoilow [2, pp. 147 and 148] and for $n = p \geqq 3$ by Church [2, p. 155]. Both [6, p. 72, (1.5)] and [2, p. 159] deal with maps having a small singular set, and [13, §11] discusses maps with isolated critical points. The map τ is due to N. Kuiper [13, p. 102] and it is topologically equivalent to the cone map $c(\psi)$ of the Hopf fibration $\psi: S^3 \to S^2$ (1.10).

Convention 1.2. A symbol such as M^n denotes a separable n-manifold, without boundary unless otherwise specified (except for obvious cases). A manifold with boundary may have empty boundary.

The boundary of a space X is denoted by X or ∂X (in case

X is a manifold), the interior of X by int X, and the closure of X by $\bar{X}$ or $Cl[X]$. The distance between two points is $d(x, y)$, and $S(x, \varepsilon) = \{y: d(x, y) < \varepsilon\}$. The space of real (resp., complex) numbers is denoted by R (resp., C), euclidean n-space by R^n, its origin by 0, the closed ball $Cl[S(0, 1)]$ in R^n by D^n, and the sphere ∂D by S^{n-1}.

A map is a continuous function, the restriction of a function f to X is denoted by $f \mid X$, and the composition of two functions by gf or $g \circ f$. Homeomorphism of topological spaces and isomorphism of groups is denoted by $\approx$. The map $\pi: X \times Y \to X$ is projection, and ι is used for the identity map on a space.

Given maps $\psi: X \to Y$ and $\phi: U \to V$, define $\psi \times \phi: X \times U \to V \times Y$ by $(\psi \times \phi)(x, u) = (\psi(x), \phi(u))$. Define the *open cone* $c(X)$ as the identification space obtained from $X \times [0, 1)$ by identifying $X \times \{0\}$ to a point x^*, and let the *cone map* $c(\psi): c(X) \to c(Y)$ be the map induced by $\psi \times \iota$.

DEFINITION 1.3. If $f: M^n \to N^p$ and $g: K^n \to L^p$ are C^r maps on C^r manifolds ($r = 0, 1, \cdots$), then f and g are C^r *equivalent* if and only if there are C^r diffeomorphisms $\alpha: M^n \to K^n$ and $\beta: N^p \to L^p$ such that $g \circ \alpha = \beta \circ f$. The map f at x is *locally* C^r *equivalent* to g at u if there are open neighborhoods U of x and V of $f(x)$ such that $f \mid U: U \to V$ is C^r equivalent to g with $\alpha(x) = u$. A C^0 diffeomorphism is a homeomorphism, and *(locally) topologically equivalent* means *(locally)* C^0 *equivalent*.

REMARK 1.4. Suppose that f is any one of ρ, σ, or τ, and let $\eta: R^n \to R^n$ be a C^∞ homeomorphism such that η fixes the origin, $\eta \mid (R^n - A)$ is a C^∞ diffeomorphism, where A is the closure of a sequence of points converging to the origin, and the rank of the Jacobian matrix of η at points in A is zero. Since the rank of f at any point in A away from the origin is not maximal, $f \circ \eta$ is not locally C^1 diffeomorphically equivalent at the origin to any of the maps in (1.1). Thus in the statement of (1.1) "topologically" cannot be replaced by "C^n". For examples to show that no reasonable classification is possible if $p = 1$ take height functions on compact manifolds which have an infinite collection of local maxima.

DEFINITION 1.5. Given M^n and N^p manifolds with (possibly empty) boundary, $n \geqq p$, and a map $f: M^n \to N^p$, we now define the *branch set* $B_f \subset M^n$. Let $R_+^m = \{x \in R^m: x_m \geqq 0\}$, let $F = R^{n-p}$ or R_+^{n-p}, and let $G = R^p$ or R_+^p (not respectively). Then $x \notin B_f$ if and only if f at x is locally topologically equivalent to $\pi: F \times G \to G$ at $(0, 0)$. Occasionally the notation $B(f)$ is used.

In (1.1) $B_\rho = \phi$ while B_σ and B_τ (and thus B_f) are discrete.

With the definition of B_f the Rank Theorem [2, p. 155] for $n \geqq p$ becomes:

RANK THEOREM 1.6. *If $f\colon M^n \to N^p$ is $C^r (r \geqq 1)$, $n \geqq p$, and $\partial M^n = \partial N^p = \phi$, then $B_f \subset R_{p-1}(f)$.*

Question 1.7. (Milnor [13, p. 100, first problem]). *Let $f\colon R^n \to R^p$ be a (real) polynomial map with an isolated critical point at* 0. *For what dimensions $n \geqq p \geqq 2$ do nontrivial examples exist?*

The topic of [13], except for § 11, is certain complex polynomial maps $f\colon C^{n+1} \to C$ with 0 as the only critical point. These maps have a deep and very interesting structure related to exotic spheres, and their properties led Milnor to ask about real polynomial maps.

After posing this question Milnor says "It is not quite clear what 'non-trivial' should mean here. Certainly the projection $\cdots$ is a trivial example". One natural definition is: *f is nontrivial at x if and only if $x \in B_f$*, i.e. f is trivial at x if and only if f at x is locally topologically equivalent to the projection map ρ.

In the complex polynomial case the study of a singularity employs a certain fibration, and analogous fibration exists in the real polynomial case. Milnor formulates his ("tentative") definition of non-trivial [13, p. 97 and p. 100] in terms of this fibration; we omit it here because it is technical. While Milnor's definition appears to be quite different from the definition we have given above, Church and Lamotke have shown [4] that they agree (at least for $n \neq 4$). With our formulation we can ask Milnor's question in other contexts.

Let $f\colon M^n \to N^p$ $(n \geqq p \geqq 2)$ be continuous, C^r $(r = 1, 2, \cdots;$ or $\infty)$, or real analytic. For what dimensions (n, p) can f have a nonempty discrete (or 0-dimensional) branch set B_f, and, up to local topological equivalence, what are the examples? For $n - p = 0$ or 1 Theorem (1.1) answers a C^n version of this question, and a fortiori answers Milnor's question for these dimensions. A continuous version is discussed in (4.9).

In sequels [5] the hypothesis on $\dim f(R_{p-1}(f))$ is removed, and analogous results for $n - p = 2$ are proved in both continuous and C^n contexts. At first glance it seems very special to consider only the cases $0 \leqq n - p \leqq 2$ in these theorems, but for every (n, p) with $n - p \geqq 4$ and $p \geqq 2$, Church and Lamotke constructed [4] a continuous counterexample with isolated branch point. Moreover, for $n - p = 3$ and $p \geqq 6$, the nonexistence of examples depends on the Poincaré Conjecture. Thus our restriction $0 \leqq n - p \leqq 2$ in these papers is reasonable.

DEFINITION 1.8. A map $f: X \to Y$ is *proper* if, for each compact set $K \subset Y$, $f^{-1}(K)$ is compact; f is *light* if, for each $y \in Y$, $\dim(f^{-1}(y)) \leqq 0$; and f is *monotone* if each $f^{-1}(y)$ is connected. It is *quasi-monotone* if, for each connected open set $U \subset Y$ and component V of $f^{-1}(U)$, $f(V) = U$ [23, p. 151].

THEOREM 1.9. (Cheeger and Kister [1, p. 151]; see also [2, p. 170]). *If $f: M^n \to N^n$ is a proper map, $n \geqq p$, and $B_f = \phi$, then f is the projection map of a fiber bundle.*

(While they assume that f is monotone, this hypothesis is not used in their proof. The manifolds may have nonempty boundary.)

REMARK 1.10. *If $\psi: S^3 \to S^2$ is the Hopf fibration, then τ in* (1.1) *is topologically equivalent to the cone map $c(\psi)$.*

Proof. Let $S^3(r)$ and $S^2(r)$ be the spheres about $(0, 0)$ of radius r in $C \times C$ and $C \times R$, respectively, and let $\xi: C \times C - \{(0, 0)\} \to C \times R - \{(0, 0)\}$ and $\zeta_r: S^3(r) \to S^2(r)$ be the restrictions of τ. Then ζ_1 is ψ [13, p. 102, (11.6)], ξ is proper, and since $R_2(\tau) = \{(0, 0)\}$, $B_\xi = \phi$ (1.6). Thus (1.9) ξ is a bundle map over $S^2 \times (0, \infty)$, so that ξ is topologically equivalent to $\zeta_1 \times \iota$ [17, p. 53, (11.4)], and the conclusion results.

Outline of the Proof of (1.1) 1.11. For almost all the proof we work in a purely topological context, assuming topological analogs of the hypotheses of (1.1) or less. The lemmas of § 2 show that for each $x \in M^n$ there is a manifold neighborhood U of x such that the restriction map $g = f \mid U$, $g: U \to f(U)$, is proper and $B_g \cap \partial U = \phi$. If $B_g = \phi$, then g is a bundle map (1.9); thus, in general, g can be viewed as a bundle map with singularities. In § 3 we show that g is open. In § 4 we suppose that x is not a point component of $g^{-1}(g(x))$ and deduce (4.5) that g is a bundle map near x, which implies that $x \notin B_f$. (We use § 3 and the 'almost bundle property' of § 2 here.) Hence for each $x \in B_f$, x is a point component of $f^{-1}(f(x))$; this is the situation of [6] and that paper yields the desired conclusion. Differentiability is used only as it is used in [6], i.e. to deduce in (1.1) that $f(B_f)$ is 0-dimensional and nicely embedded.

In several cases lemmas are stated and proved in somewhat greater generality than needed here, for use in [5] and [7].

2. Extended embeddings.

DEFINITION 2.1. A map $g: J^{n-m} \times R^m \to L^{p-m} \times R^m$ is called a *layer* map if for each $t \in R^m$, $g(J^{n-m} \times \{t\}) \subset L^{p-m} \times \{t\}$. (In case g is an embedding it is called an *isotopy.*) The restriction map

$$g \mid (J^{n-m} \times \{t\}): J^{n-m} \times \{t\} \longrightarrow L^{p-m} \times \{t\}$$

is denoted by g_t, and its branch set by $B(g_t)$. Frequently it is convenient to view g_t as a map of J^{n-m} into L^{p-m}.

LEMMA 2.2. *Let $\gamma_i: D^p \to (\text{int } D^{n-p}) \times D^p$ be disjoint embeddings with $\gamma_i(t) = (a_i(t), t)$ $(i = 1, 2, \cdots, k)$. Then there is an isotopy $h: D^{n-p} \times D^p \to D^{n-p} \times D^p$ such that h agrees with the identity map on $(D^{n-p} \times \{0\}) \cup (\partial D^{n-p} \times D^p)$, and $h(\gamma_i(t)) = (a_i(0), t)$.*

Proof. Let $X = \{a_i(0): i = 1, 2, \cdots, k\}$. The γ_i define an isotopy

$$\gamma: X \times D^p \longrightarrow (\text{int } D^{n-p}) \times D^p$$

by sending $(a_i(0), t)$ to $(a_i(t), t)$, which is readily extended to a neighborhood of X by sending (u, t) to $(u - a_i(0) + a_i(t), t)$ for u near $a_i(0)$. The desired ambient isotopy $h: D^{n-p} \times D^p \to D^{n-p} \times D^p$ extending γ is given by Lees' Neighborhood n-Isotopy Extension Theorem [12, p. 530].

DEFINITION 2.3. Let $f: M^n \to N^p$ be a map, let $y \in W \subset N^p$, and let $Q \subset f^{-1}(y)$. Define the embedding $\lambda: Q \times \{y\} \to M^n$ by $\lambda(q, y) = q$ for each $q \in Q$. An embedding $\nu: Q \times W \to M^n$ extending λ with

$$\begin{array}{ccc} Q \times W & \xrightarrow{\nu} & M^n \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle f} \\ W & \xrightarrow{\subset} & N^p \end{array}$$

commutative is called an *extended embedding of Q over W.* For each $w \in W$ let $\nu_w: Q \to f^{-1}(w)$ be the embedding defined by $\nu_w(q) = \nu(q, w)$.

An extended embedding with Q a single point space is called a *cross-section* over W. (In case f is a fiber bundle, it is a cross-section in the usual sense.)

LEMMA 2.4. *Let $f: M^n \to N^p$ be a map, let $n \geqq p$, let $y \in N^p$, and let $p \subset f^{-1}(y) - B_f$ be a compact $(n-p)$-submanifold with bicollared boundary* (e.g. *let P be a closed bicollar of a bicollared compact $(n-p-1)$-submanifold $Q \subset f^{-1}(y) - B_f$*).

(a) *Then there is an open p-cell neighborhood D of y and an extended embedding $\mu: P \times D \to M^n$ such that $B_f \cap \operatorname{imag} \mu = \phi$ (in the example let $\nu = \mu | Q \times D$).*

(b) *Given ν and Q as in* (a) (or $Q = \phi$), *$\delta > 0$, and a family $\mathscr{J}$ of components of $f^{-1}(y) - Q$ such that $L = \cup\{J: J \in \mathscr{J}\}$ has compact closure, $\bar{L} - L \subset Q$, and either* (i) *$\mathscr{J}$ is finite or* (ii) *each $\operatorname{diam} J < \delta/3$, then there is a p-cell neighborhood E of y in D such that for each $J \in \mathscr{J}$, the component K of $f^{-1}(E)$-imag ν containing J has* (i) *$d(x, J) < \delta/3$ for each $x \in K$, resp.* (ii) $\operatorname{diam} K < \delta$.

Proof of (a). We will assume that the reader has read the proofs of the Theorem and the Remark in [1] and has them at hand. Let M^n, N^p, and y be denoted by W, Y, and y_0, and let the $(n-p)$-manifold P' be the union of P and a bicollar of its boundary. For each $x \in P$, there are closed neighborhoods U of y_0 in Y and V of x in W, and a homeomorphism $h: B(2) \times U \to V$ such that $f \circ h$ is the projection map onto U (since $x \notin B_f$). In fact it is possible to choose the closed neighborhood U and a collection of such embeddings $\{h_j: j = 1, 2, \cdots, k\}$ on $B(2) \times U$ such that $P \subset \bigcup_{j=1}^k \operatorname{int} h_j(B(1))$ and $f^{-1}(y_0) \cap h_j(B(2)) \subset \operatorname{int} P'$. For $y \in U$ define $M_y = f^{-1}(y) - B_f$ (so $P \subset P' \subset M_{y_0}$) and $h_{y,j}: B(2) \to M_y$ by $h_{y,j}(t) = h_j(y, t)$. The proofs of [1] now yield the embedding required for (a).

Proof of (b). Let $W \subset M^n$ be compact with $\bar{L} \subset \operatorname{int} W$; we may suppose that $\delta < d(\bar{L}, \operatorname{bdy} W)$. Let P be a closed bicollar of Q ($P = \phi$ if $Q = \phi$) and $\mu: P \times D \to M^n$ be as in (a); we may suppose that D is sufficiently small and P is a sufficiently small bicollar of Q that $d(x, Q) < \delta/3$ for each $x \in \operatorname{imag} \mu$ (remember that $Q = \mu(Q, y)$). Let L' be $L - \operatorname{int} P$ (since $\bar{L} - L \subset Q$, L' is compact). Each component K of L (resp. of $f^{-1}(D) - \operatorname{imag} \nu$) contains one and only one component K' of L' (resp., of $f^{-1}(D) - \operatorname{int} \operatorname{imag} \mu$). Let $E_r \subset N^p$ ($r = 1, 2, \cdots$) be closed p-cells such that $E_{r+1} \subset E_r$ and $\bigcap_r E_r = \{y\}$. Then it suffices to prove that (*) there exists an integer r such that, for each $J \in \mathscr{J}$, the component H of $f^{-1}(E_r) - \operatorname{int} \operatorname{imag} \mu$ containing $J - \operatorname{int} P$ has $d(x, J - \operatorname{int} P) < \delta/3$ for each $x \in H$.

Suppose the contrary. Then there are components J_r of L', components H_r of $f^{-1}(E_r) - \operatorname{int} \operatorname{imag} \mu$ with $J_r \subset H_r$, $y_r \in H_r$ with $d(J_r, y_r) \geqq \delta/3$, and paths $\Gamma_r \subset H_r$ joining some point $x_r \in J_r$ to y_r. We may suppose that $\Gamma_r \subset W$, that $x_r \to x_0$, and that $y_r \to y_0$. Thus $x_0 \in L'$, $\Gamma =$

$\limsup \Gamma_r$ is connected [23, p. 14, (9.1)], and $\operatorname{diam} \Gamma \geqq \delta/3$. Now Γ is contained in some component J' of L'. In case (i) there are only a finite number of components of L', and since they are compact, all but a finite number of the J_r are in fact J'; since $d(J', y_0) \leqq \delta/3$, a contradiction results. In case (ii), since $\operatorname{diam} J' < \delta/3$ and $\operatorname{diam} \Gamma \geqq \delta/3$, a contradiction results also.

LEMMA 2.5. *Let $f: M^n \to N^n$ be a map with $0 \leqq n - p$ and $n - p \neq 4$ or 5, let $y \in N^p$, let $\dim (B_f \cap f^{-1}(y)) \leqq 0$, let $X \subset B_f \cap f^{-1}(y)$ be compact, and let $\varepsilon > 0$. Then there is a compact $(n - p - 1)$-manifold Q (or ϕ), an open p-cell neighborhood D of y in N^p, and an extended embedding $\nu: Q \times D \to M^n$ such that each component K of $f^{-1}(D) - \operatorname{imag} \nu$ meeting X has $\operatorname{diam} K < \varepsilon$, their union has compact closure, and $\operatorname{imag} \nu \cap B_f = \phi$.*

Proof. Let T, $T' \subset f^{-1}(y)$ be compact with $X \subset \operatorname{int} T'$, $T' \subset \operatorname{int} T$ (interior relative to $f^{-1}(y)$), and let $\eta > 0$ be less than both $d(X, \operatorname{bdy} T')$ and $d(T', \operatorname{bdy} T)$. We may suppose that $\varepsilon < \eta$. Let $X' = T' \cap B_f$, and let $U_k = \{x \in T: d(x, X') \leqq 1/k\}$ $(k = 1, 2, \cdots)$.

We will first prove that (1) for k sufficiently large, each component of U_k has diameter less than ε. Suppose the contrary. Then there are a subsequence $\{m(k)\}$, components Γ_k of $U_{m(k)}$, and points $x_k, y_k \in \Gamma_k$ with $d(x_k, y_k) \geqq \varepsilon$. We may suppose that $x_k \to x_0$ and $y_k \to y_0$. Thus $\Gamma = \limsup \Gamma_k$ is connected [23, p. 14, (9.1)], and since $d(x_0, y_0) \geqq \varepsilon$, $\operatorname{diam} \Gamma \geqq \varepsilon$. If k is fixed, then for each $j \geqq k$, $\Gamma_j \subset U_{m(j)} \subset U_{m(k)}$, so that $\Gamma \subset U_{m(k)}$. Thus $\Gamma \subset \bigcap_k U_{m(k)} = X$, contradicting the fact that X is totally disconnected (since $X \subset B_f \cap f^{-1}(y)$). Thus (1) is true.

If a generalized continuum fails to be locally connected, it fails at (at least) a subcontinuum of points [23, p. 19, (12.3)]. Since $f^{-1}(y) - B_f$ is an $(n - p)$-manifold (or ϕ) and $B_f \cap f^{-1}(y)$ is totally disconnected, each component Λ of $f^{-1}(y)$ is locally connected. Hence (2) if U is open in Λ, then each component of U is open in Λ.

Let k be the number given by (1), and (3) let V be a component of U_k meeting X. Then $\operatorname{diam} V < \varepsilon$, so that $V \subset T'$. Let $\operatorname{bdy} V$ refer to the boundary of V in the component Λ_V of $f^{-1}(y)$ containing V, and let $x \in \operatorname{bdy} V$. Since $V \subset U_k$, $d(x, X') \leqq 1/k$; suppose $d(x, X') = a < 1/k$. The component W of $\{u \in \Lambda_V: d(x, u) < 1/k - a\}$ containing x is open in Λ_V (by (2)), $W \subset U_k$, and thus $W \subset \operatorname{int} V$ (relative to Λ_V), contradicting the choice of x; thus (4) $d(x; X) = 1/k$ for each $x \in \operatorname{bdy} V$.

Let Δ be the closure of the union of $\operatorname{bdy} V$ for V satisfying (3) (actually the union is closed). For each $x \in \Delta$, $d(x, X') = 1/k$, and since

$\Delta \subset T'$ and $T' \cap B_f = X'$, $\Delta \cap B_f = \phi$. Let L be the $(n-p)$-manifold $\operatorname{int} T - B_f$ (possibly empty). Siebenmann and Kirby have shown (see [15, p. 949] that a topological manifold with dimension not 4 or 5 has a handle decomposition, so in particular there are compact $(n-p)$-manifolds with boundary $\{L_j\}$ ($j = 1, 2, \cdots, k$ or $j = 1, 2, \cdots$) such that $L_j \subset \operatorname{int} L_{j+1}$ and $L = \bigcup_j L_j$. Since Δ is compact, there is a j such that $\Delta \subset \operatorname{int} L_j$. Since ∂L_j is collared, there is a compact $(n-p)$-manifold P^n with boundary (or ϕ) such that $\Delta \subset \operatorname{int} P^n$, $P^n \subset \operatorname{int} L_j$, (and so $P^n \subset L$), and ∂P^n is bicollared in L. Let $Q = \partial P^n$.

Since each component Y of $f^{-1}(p) - Q$ meeting X is contained in some V satisfying (3), diam $Y < \varepsilon$. Let μ and ν be the extended embeddings given by (2.4a and b); the conclusion results.

COROLLARY 2.6. *Let $f: M^n \to N^p$ be a map with $0 \leqq n - p$ and $n \neq 4,5$, let $x \in M^n$, and let* $\dim(B_f \cap f^{-1}(f(x))) \leqq 0$. *Then there is a connected (not neccessarily compact) manifold $K^n \subset M^n$ with boundary such that $x \in \operatorname{int} K^n (= K^n - \partial K^n)$, the closure $\bar{K}^n$ of K^n in M^n is compact, there is an open p-cell $D \subset N^p$ with $f(K^n) \subset D$, and the restriction map $g: K^n \to D$ is proper with $B_g \cap \partial K^n = \phi$.*

For example, let $f: R^2 \to R$ be projection on the first factor, let $x = (0, 0)$, and let $K^2 = (-1, 1) \times [-1, 1]$.

Proof. We may suppose that $x \in B_f$. Apply (2.5) where $X = \{x\}$, and let K^n be the closure in $f^{-1}(D)$ of K. Thus K^n is a manifold with boundary, and $\partial K^n \subset \operatorname{imag} \nu$ (which may be empty). Let Y be a compact subset of the open 2-cell D. Since bdy K^n (bdy taken relative to M^n) is the disjoint union of ∂K^n and a subset of $f^{-1}(\text{bdy } D)$, $g^{-1}(Y) = f^{-1}(Y) \cap K^n = f^{-1}(Y) \cap \bar{K}^n)$ (closure in M^n), and so is compact; thus g is proper.

3. Open maps.

DEFINITION 3.1. For $f: M^n \to N^p$ and $x \in M^n$, let $\Gamma(x) = \Gamma_f(x)$ be the component of $f^{-1}(f(x))$ containing x. If, for every neighborhood U of x, $f(x) \in \operatorname{int} f(U)$, then f is *open at* x.

LEMMA 3.2. *Let $f: M^n \to N^p$ be a map with $n \geqq p$, and let $x \in M^n$ with* $\dim(f^{-1}(f(x)) \cap B_f) \leqq 0$ *and $\Gamma(x) \neq \{x\}$. Then f is open at x.*

Proof. Let U be an open neighborhood of x in M^n. For $n = p$, $f^{-1}(y) - B_f$ is discrete, so the hypotheses cannot be satisfied. Thus $n > p$, and there is $z \in (\Gamma(x) \cap U) - B_f$, so that $f(x) = f(z) \in \operatorname{int} f(U)$.

Since U is arbitrary, f is open at x.

LEMMA 3.3. *Let $f: M^n \to N^p$ be a map, and let $x \in M^n$ with $\Gamma(x) = \{x\}$. Then there is an open p-cell D about $f(x)$ such that, if K is the component of $f^{-1}(D)$ containing $x, f: K \to D$ is a proper map.*

Proof. Use the proof of [6, p. 74, (1.14)].

LEMMA 3.4. *Let $f: M^n \to N^p$ be a map with $n \geqq p$ and $\dim(f^{-1}(y) \cap B_f) \leqq 0$ for every $y \in N^p$, and let E_f be the set of points at which f fails to be open.*

(a) *Then either $E_f = \phi$ (so that f is open) or $\dim f(E_f) \geqq p - 1$.*

(b) *In particular, if $\dim f(B_f) \leqq p - 2$, then f is open.*

Proof. Let $x \in E_f$. By (3.2) we may as well assume that (1) $\Gamma(x) = \{x\}$; let $g: K \to D$ be the proper map given by (3.3). (In case $n - p \neq 4,5$, we could use (2.6) instead.) Then $E_g = E_f \cap K$.

We observe that (2) if $V \subset K$ is open, then $\operatorname{int} g(V) \neq \phi$, i.e. $\dim g(V) = p$ [11, p. 46]. If $V \not\subset B_g$, the conclusion is immediate. If $V \subset B_g$, let $U \subset V$ be a closed n-cell. The map $g|U: U \to g(U)$ is light (from the dimension hypothesis), so that [11, p. 91, VI. 7] $\dim g(U) \geqq n$. Since $n \geqq p$, $\dim g(V) = \dim g(U) = n = p$.

Now suppose that $g(K) \neq D$ and $\dim g(E_g) \leqq p - 2$. Since g is proper, $g(K)$ is closed, so that $D - g(K)$ is a nonempty open subset of D. By (2) $\operatorname{int} g(K) \neq \phi$. Thus $D - \operatorname{bdy} g(K)$ is not connected, and hence [11, p. 48] $\dim \operatorname{bdy} g(K) = p - 1$. There is a $z \in K$ with $g(z) \in (\operatorname{bdy} g(K)) - g(E_g)$, and since g is open at z, $g(z) \in \operatorname{int} g(K)$, so a contradiction results. Thus (3) $\dim g(E_g) \leqq p - 2$ implies $g(K) = D$.

It is immediate from (2) that (4) $\dim g(E_g) \leqq p - 2$ implies $Cl[K - g^{-1}(g(E_g))] = K$.

From (1), (3), and (4) if $\dim g(E_g) \leqq p - 2$, then g satisfies the hypotheses of [23, p. 149, (7.81)] at x, and so $g(E_g)$ locally separates D at x. A contradiction of [11, p. 48] results, and hence $\dim g(E_g) \geqq p - 1$, yielding conclusion (a).

Since $E_f \subset B_f$, conclusion (b) follows from (a).

4. 0-regular maps.

DEFINITION 4.1. Let X and Y be metric spaces, let $f: X \to Y$ be open, and let $x \in X$. The map f is *0-regular at* x if, for every $\varepsilon > 0$, there is a $\delta > 0$ such that if $y \in Y$ and $u, v \in S(x, \delta) \cap f^{-1}(y)$, then there is an arc from u to v in $S(x, \varepsilon) \cap f^{-1}(y)$. If f is proper, onto, and 0-regular at each $x \in X$, then f is *0-regular*.

LEMMA 4.2. *If M^n and N^p are manifolds with boundary, $f\colon M^n \to N^p$ is a map, and $x \in M^n - B_f$, then f is 0-regular at x.*

The proof is immediate from (1.5).

LEMMA 4.3. *Let X be a separable metric space, let $A \subset X$ be closed, let $X - A = M^n$ and N^p be manifolds with boundary, let $n > p$, let $f\colon X \to N^p$ be proper, open, and onto, and let $B = B(f \mid M^n) \cup A$ (1.5). Suppose that $f(B)$ is nowhere dense,* $\dim (B \cap f^{-1}(y)) \leqq 0$ *for each $y \in N^p$, and $f^{-1}(z)$ is connected for each $z \in N^p - f(B)$.*

(a) *Then $f^{-1}(y)$ is path connected for each $y \in N^p$.*

(b) *If f is 0-regular, $F \approx S^1$ or $[0, 1]$, and $f^{-1}(z) \approx F$ for each $z \in N^p - f(B)$, then f is a bundle map with fiber F.*

Proof. Each $y \in N^p$ has a compact neighborhood V, and $f^{-1}(V)$ is compact. There are $z_k \in V - f(B)$ with $z_k \to y$, and $f^{-1}(z_k) \to f^{-1}(y)$ [23, p. 10, and p. 130, (4.32)], so that $f^{-1}(y)$ is connected [23, p. 14]. Since $f^{-1}(y)$ is locally connected except possibly at most 0-dimensional set $B \cap f^{-1}(y)$, it is locally connected [23, p.19, (12.3)], and so path connected [23, p. 38, (5.2)].

Under the hypotheses of (b) $f^{-1}(z_k) \to f^{-1}(y)$ 0-regularly [21, p. 482], and thus ([21, p. 484, Theorem 2] and [22, p. 341, Theorem 5.1]) $f^{-1}(y) \approx F$. By [8, p. 115, Theorem 7] $f \mid f^{-1}(V)$ is bundle map, and (b) results. This lemma is used in (4.5) and [7], and its considerable generality is required for the latter applications.

DEFINITION 4.4. Given $f\colon M^n \to N^p$, the *singular set* A_f (see [6]) is defined as follows: $x \in M^n - A_f$ if and only if there are open neighborhoods U of $\Gamma(x)$ and V of $f(x)$ such that $f \mid U\colon U \to V$ is topologically equivalent to the projection map $\pi\colon V \times \Gamma(x) \to V$. Thus (1.5) $B_f \subset A_f$. See (4.6).

LEMMA 4.5. *Let $f\colon M^n \to N^p$ be a proper map with $n \geqq p$, ∂M^n possibly nonempty, $B_f \subset$* int M^n, $\dim f(B_f) \leqq p - 2$, *and* $\dim (f^{-1}(y) \cap B_f) \leqq 0$ *for each $y \in N^p$.*

(a) *If $\Gamma(\bar{x}) \neq \{\bar{x}\}$ (see (3.1)), then there are open neighborhoods U of $\Gamma(\bar{x})$ and V of $f(\bar{x})$ such that $f \mid U\colon U \to V$ is a proper, open, monotone (onto) map.*

(b) *If, in addition,* $n - p \neq 4,5$, *then* $f|U$ *is* 0*-regular.*

(c) *If, in addition,* $n = p + 1$, *then* $f|U$ *is a bundle map (thus* $A_f \cap U = \phi$).

(d) *If* $\Gamma(x) \neq \{x\}$ *for every* $x \in M^n$ *and* M^n *and* N^p *are connected, then* $f = \psi \circ \phi$, *where* ϕ *is monotone and* ψ *is a k-to-one covering map.*

Proof. By (3.4) f is open. The hypotheses that $\Gamma(\bar{x}) \neq \{\bar{x}\}$ and $\dim (f^{-1}(f(\bar{x})) \cap B_f) \leqq 0$ imply that $n > p$. If $p \leqq 1$, then $B_f = \phi$, and the conclusions follow from (4.2), [18, p. 63, (2.3)], and [19, p. 661, (2.1)]. Thus we may suppose that $n > p \geqq 2$.

Let $x \in \Gamma(\bar{x}) - B_f$. There is a cross section μ at x over an open p-cell $V \subset N^p$. Let U be the component of $f^{-1}(V)$ containing $\Gamma(\bar{x})$ and let $g: U \to V$ be the restriction of f. Since $\dim (g^{-1}(y) \cap (B_g) \leqq 0$, each $\dim g^{-1}(y) \leqq n - p$; thus by [11, p. 91, Theorem VI 7] $\dim (g^{-1}(gB_g)) \leqq n - 2$, so that $U' = U - g^{-1}(g(B_g))$ is path connected.

Let $\alpha: U' \to V - g(B_g)$ be the restriction of g; since α is a proper map with $B_\alpha = \phi$, α can be factored $\alpha = \gamma \circ \beta$, where β is a monotone map and γ is a covering map [18, p. 63, (2.3)]. Now $\beta \circ \mu | (V - g(B_g))$ is a global cross-section of γ, so that [16, p. 77 (6) and (7)] γ is a homeomorphism. Thus α is monotone, and by [18, p. 64, (2.5); the proof is still valid for $\partial M^n \neq \phi$ and $B_f \subset \operatorname{int} M^n$] g is monotone onto. Conclusion (a) results.

Now suppose that $\Gamma(x) \neq \{x\}$ for every $x \in M^n$. Since f is open, and thus quasi-monotone, there is a natural number k such that each $f^{-1}(y)$ has at most k components, and for $y \in N^p - f(B_f)$, $f^{-1}(y)$ has exactly k components [18, p. 64, (2.5)]. Let $y \in N^p$, and let Γ_i ($i = 1, 2, \cdots, h \leqq k$) be the components of $f^{-1}(y)$. Let U_i and V_i be as given by (a) for Γ_i, and let V be a p-cell neighborhood of y such that $V \subset \bigcap_i V_i$ and the Γ_i are in distinct components of $f^{-1}(V)$. Since f is quasi-monotone, each component of $f^{-1}(V)$ meets some Γ_i, so there are exactly h components W_i, where $\Gamma_i \subset W_i$. Since $f|W_i: W_i \to V$ is monotone, for each $z \in V - f(B_f)$, $f^{-1}(z)$ has h components; thus $h = k$. From [18, p. 63, (2.1)] (d) follows.

For $x \in U - B_g$, g is 0-regular at x (4.2). Thus, to prove (b) it suffices to prove that g (or equivalently f) is 0-regular at each $x \in B_g$. For $x \in B_g$ and $\varepsilon > 0$, let K be as given in (2.5) for g and $X = \{x\}$, and let $\zeta: \bar{K} \to D$ be the restriction of g. Thus $\bar{K} \subset S(x, \varepsilon) \subset U$. Since g has a global cross-section, $\Gamma_g(u) \neq \{u\}$ for every $u \in U$; thus $\Gamma_\zeta(u) \neq \{u\}$, and by (d) $\zeta = \psi \circ \phi$, where ϕ is monotone and ψ is a covering map. Since D is simply connected, ψ is a homeomorphism, so that ζ is monotone.

By (4.3) (a) $\zeta^{-1}(y)$ is path connected for each $y \in D$. Choose $\delta > 0$ such that $S(x, \delta) \subset \operatorname{int} K$. Since

$$S(x, \delta) \cap f^{-1}(y) \subset \zeta^{-1}(y) \subset S(x, \varepsilon) \cap f^{-1}(y) ,$$

f is 0-regular at x. Conclusion (b) results.

If $n = p + 1$, then $g|(U - g^{-1}(g(B_g)))$ is bundle map (1.9) with fiber a compact, connected 1-manifold F, i.e., $E \approx S^1$ or $[0, 1]$, and conclusion (c) follows from (4.3(b)).

LEMMA 4.6. *Let $f: M^n \to N^p$ be a proper map with $\Gamma(x) \subset B_f$ (3.1) for every $x \in B_f$. Then $B_f = A_f$ (4.4).*

Proof. If $x \notin A_f$, then $x \notin B_f$. If $x \notin B_f$, then from the hypothesis $\Gamma(x) \cap B_f = \phi$. There is an open p-cell neighborhood V of $f(x)$ such that the component U of $f^{-1}(V)$ containing $\Gamma(x)$ is disjoint from B_f. Since $f|U: U \to V$ is also proper, it is a bundle map (1.9), and since V is contractible, the bundle is trivial. Thus $x \notin A_f$.

PROPOSITION 4.7. *Let $f: M^n \to N^p$ be C^n with $n = p$ or $p + 1$, $B_f \neq \phi$,* $\dim f(B_f) \leqq p - 2$, *and* $\dim (f^{-1}(y) \cap B_f \leqq 0$ *for each $y \in N^p$. Then* $\dim B_f = 2p - n - 2$ *and there is a closed subset $X \subset B_f$ such that* $\dim X < 2p - n - 2$ *and, for each $x \in B_f - X$, f at x is locally topologically equivalent to the layer map*

$$c(\psi) \times \iota: D^{2(n-p+1)} \times R^{2p-n-2} \to D^{n-p+2} \times R^{2p-n-2}$$

(see (1.2)), *where $c(\psi) = \sigma$* (see (1.1)) *if $n = p$, and $c(\psi) = \tau$,* i.e. (1.10) *the cone map of the Hopf fibration $\psi: S^3 \to S^2$, if $n = p + 1$.*

Proof. Let X be the set of all $x \in B_f$ such that f at x is not locally equivalent to $c(\psi)$; then X is closed.

Let $x \in B_f$, and let K be the neighborhood of x and $g: K \to D$ be the proper map given by (2.6). If $n = p$, then for each $y \in D$, $g^{-1}(y) - B_f$ is discrete and $\dim (g^{-1}(y) \cap B_f) \leqq 0$, so that g is light, i.e. each $\Gamma(u) = \{u\}$. If $n = p + 1$ it follows from (4.5(c)) that for each $u \in B_g$, $\Gamma(u) = \{u\}$. There is an open p-cell neighborhood $U \subset D$ of $f(x)$ sufficiently small that the component V of $f^{-1}(U)$ containing x has $\bar{V} \subset \operatorname{int} K$. Since $\bar{K}$ is compact, the restriction map $h: V \to U$ is proper, so that (4.6) $B_h = A_h$.

Since $\dim f(B_f) \leqq p - 2$ and $B_f \neq \phi$, $p \geqq 2$. In case $n = p$, h is light, and since $\dim h(B_h) \leqq p - 2$, the Jacobian determinant of h is (locally) nonnegative or nonpositive [3, p. 94, (2.3) and p. 98, (1.7)].

In both cases by [6, p. 83, (4.1)] there is a closed set $Y_h \subset h(A_h)$ such that $\dim Y_h < \dim h(A_h)$ and, for each $x \in A_h - h^{-1}(Y_h)$, h at x is locally topologically equivalent to $c(\psi) \times \iota$. Thus $\dim h(A_h) = 2p - n - 2$ and $\dim A_h \geqq 2p - n - 2$. Since $h|A_h$ is light, $\dim A_h \leqq \dim h(A_h)$ [11, p. 91, Theorem VI 7], so that $\dim A_h = 2p - n - 2$, and $\dim (A_h \subset h^{-1}(Y_h)) \leqq \dim Y_h \leqq 2p - n - 2$. Since $A_h = B_h$ and $V \cap X \subset A_h \cap h^{-1}(Y_h)$, $\dim (V \cap X) < 2p - n - 2$; since $x \in B_f$ was arbitrary and V is a neighborhood of x, $\dim X < 2p - n - 2$.

REMARK 4.8. Theorem (1.1) is a Corollary of (4.7). (Use the Rank Theorem (1.6).)

The next result (4.9) is a topological analog of (1.1). If $f\colon M^{p+1} \rightarrow N^p$ is continuous with $p \geqq 2$ and B_f discrete, then f satisfies the hypotheses of (4.9); in this case the result was proved by Timourian [19].

PROPOSITION 4.9. *Let* $f\colon M^n \rightarrow N^p$ *be a map with* $n = p$ *or* $p + 1$ *and* $p \geqq 2$, *let* $\dim B_f \leqq 0$, *and let* $\dim f(B_f) \leqq 0$. *If* $p \geqq 3$ *suppose in addition that for each* $y \in f(B_f)$ *and neighborhood* W *of* y, *there is an open p-cell* U *such that* $y \in U \subset W$ *and* $U - f(B_f)$ *is simply connected. Then at each* $x \in M^n$, f *is locally topologically equivalent to one of the maps* ρ, σ, *or* τ *of* (1.1).

Proof. By the first two paragraphs of the proof of (4.7), for each $x \in B_f$ there is a neighborhood V of x such that the restriction $h\colon V \rightarrow U$ of f is proper and $B_h = A_h$. By [6, p. 75, (2.3)] at each $u \in B_h$, h at u is locally topologically equivalent to σ or to $c(\psi)$, where ψ is the Hopf map, i.e. to σ or τ by (1.10).

PROPOSITION 4.10. *Let* $f\colon J^{n-m} \times R^m \rightarrow L^{p-m} \times R^m$ *be a* C^{n-p+1} *layer map with* $n - p = 0$, 1, *or* 2, *and* $\dim (B_f \cap f^{-1}(y, t)) \leqq 0$ *for each* $t \in R^m$. *Then* $B_f = Cl[\bigcup_t \{B(f_t)\}\colon t \in R^m]$.

By Sard's Theorem (e.g. [2, p. 156]) $\dim (f_t(R_{p-m-1}(f(t))) \leqq p - m - 1$, and by the Rank Theorem (1.6) $(L^{p-m} \times \{t\}) - f(B_f)$ is dense in $L^{p-m} \times \{t\}$ for each $t \in R^{p-m}$. Our proof uses only this last statement, rather than C^{n-p+1}.

Proof. If $(x, s) \notin B_f$, then there is a layer embedding $\lambda\colon (D^{n-p} \times D^{p-m}) \times D^m \rightarrow J^{n-m} \times R^m$ witn $(x, s) \in \operatorname{int} \operatorname{imag} \lambda$, $D^{p-m} \subset L^{p-m}$, $\pi\colon D^{n-p} \times (D^{p-m} \times D^m) \rightarrow L^{p-m} \times R^m$ projection, and $f \circ \lambda = \pi$. For each $(v, t) \in \operatorname{int} \operatorname{imag} \lambda$, $\lambda_t\colon D^{n-p} \times D^{p-m} \rightarrow J^{n-m}$ is an embedding with $v \in \operatorname{int} \operatorname{imag} \lambda_t$ and $f_t \circ \lambda_t = \pi_t$. Thus $(v, t) \notin B(f_t)$, so that $(x, s) \notin Cl[\bigcup_t B(f_t)]$.

Suppose that $(x, s) \notin Cl[\bigcup_t B(f_t)]$, but $(x, s) \in B_f$. Choose $\eta > 0$ such that $S((x, s), \eta) \cap Cl[\bigcup_t B(f_t)] = \phi$, and let K be the set given by (2.5) for f, $X = \{(x, s)\}$, and $\varepsilon = \eta$, over (we may suppose) $D = U \times V$, where U and V are open $(p - m)$ – and m-cells, respectively. Let $\bar{K}$ be the closure of K in $f^{-1}(U \times V)$, and let $f\colon \bar{K} \to U \times V$ and $g_t\colon \bar{K} \cap (J^{n-m} \times \{t\}) \to U \times \{t\}$ be restrictions of f. Each is a proper map. Since $B(g_t) = \phi$, each g_t is a bundle map (1.9): call its fiber F_t.

For $u \in V$ there exists y with $(y, u) \in (U \times \{u\}) - g(B_g)$. Choose open k-cell Γ and m-cell Δ neighborhoods of y and u, respectively, such that $(\Gamma \times \Delta) \cap g(B_g) = \phi$. Since $g \mid g^{-1}(\Gamma \times \Delta)$ is a bundle map (1.9), F_t is independent of t for $t \in \Delta$. Since u is arbitrary and V is connected, F_t is independent of t for $t \in V$, i.e., $g^{-1}(y, t)$ is independent of y and t.

By (3.2) g is open, and thus [23, p. 152, (8.1) and (8.11)] quasi-monotone. Since the number of components of $g^{-1}(y, t)$ is independent of y and t, $g = \psi \circ \phi$, where ϕ is monotone and ψ is a covering map [18, p. 63, (2.1)]. Since $U \times V$ is simply connected, ψ is a homeomorphism, so that g is monotone, i.e., F_t is connected. In case $n - p = 0$ each g_t is a homeomorphism, so that the open map g is also one-to-one and onto—thus g is a homeomorphism, contradicting the choice of (x, s). In case $n - p = 1$ or 2 to obtain a contradiction it suffices [10, p. 527, Theorem B and p. 530, Corollary 2] to prove that g is 0-regular.

Given $(z, u) \in B_g$ and $\varepsilon > 0$, let $T \subset g^{-1}(g(z, u)) - \partial K$ be a closed $(n - p)$-cell with $(z, u) \in \operatorname{int} T$, $\partial T \cap B_g = \phi$, and $\dim T < \varepsilon$. Let $M^n = K$, $Q = \partial T$, and let $\nu\colon Q \times D \to K$ be the extended embedding given by (2.4 (a)); we may suppose (2.4 (b)) that the component X of $g^{-1}(D) - \operatorname{imag} \nu$ containing int T has $\dim X < \varepsilon$ and $X \cap \partial K = \phi$. Since each $g^{-1}(y, t)$ is a compact connected $(n - p)$ – manifold with nonempty boundary ($n - p = 1$ or 2) and $\nu(Q \times \{t\}) \approx \partial T \approx S^{n-p-1}$, it follows from the cohomology sequence with compact supports of this pair that $g^{-1}(y, t) - \nu(Q \times \{t\})$ has either one or two components. Since $g^{-1}(y, t) \cap X$ contains any component it meets, there are two and $g^{-1}(y, t) \cap X$ is one of them. Choose $\delta > 0$ such that $S((z, u), \delta) \subset X$. Then for every $(y, t) \in U \times V$,

$$g^{-1}(y, t) \cap S((z, u), \delta) \subset g^{-1}(y, t) \cap X \subset g^{-1}(y, t) \cap S((z, u), \varepsilon)\ ,$$

so that g is 0-regular at (z, u). Since g is 0-regular at each $X \notin B_f$ (4.2), g is 0-regular.

While (4.10) is not used in this paper, [7] refers to (4.10) and (4.11), they will be used elsewhere, and the proof of (4.10) fits naturally into this section. For these reasons, they are given here.

REMARK 4.11. The hypothesis $\dim(B_f \cap f^{-1}(y, t)) \leqq 0$ is (surprisingly) essential. There is a proper layer map $f: ([-2, 2] \times R) \times R \to R \times R$ such that each f_t is topologically equivalent to the projection map, f_0 is the projection map, and $B_f = ([-1, 1] \times R) \times \{0\}$. For example each $f^{-1}(0, t)$ $(t \neq 0)$ might be the union of the three subsets of $([-2, 2] \times R) \times \{t\}$ defined by:

(1) $x = \sin t^{-1}y$ for $|y| \leqq 3\pi t/2$, (2) $y = 3\pi t/2$ for $-2 \leqq x \leqq -1$ and (3) $y = -3\pi t/2$ for $1 \leqq x \leqq 2$. Open maps similar to this have been defined in [14, p. 9] and [9, p. 341].

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