PACIFIC JOURNAL OF MATHEMATICS
Vol. 41, No. 3, 1972

ON THE NON-MONOTONY OF DIMENSION

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In this paper an example is constructed of a compact Hausdorff space X with covering and large inductive dimensions 0, but containing subsets Y_n, with covering dimension n, and large inductive dimension at least n, each $n, 1 \leqq n \leqq \infty$. This result is an interesting contrast with the recent result of D. W. Henderson that there exists an infinite dimensional compact metric space with no n-dimensional compact subsets for $1 \leqq n < \infty$. As a corollary of our results, we show that for each $n, 1 \leqq n \leqq \infty$, there exists a (necessarily non-metric) continuum M_n of covering dimension n, which contains subsets of all positive covering dimensions. Covering dimension is treated in §2, while large inductive dimension is treated in §4. In §3, compactifications of Y_n are discussed and it is shown that the covering dimension of $\beta Y_n = 0$ for $1 \leqq n < \infty$. It is known (see Gillman and Jerison) that for a *normal* space N, cov dim $N =$ cov dim βN.

1. Introduction and construction of examples. For separable metric spaces, cov dim = ind dim = Ind dim, and each of these is *monotone*; that is, $A \subseteqq B$ implies $\dim A \leqq \dim B$. However, it is well-known that although ind dim is always monotone [7, appendix], cov dim and Ind dim need not be monotone. In the appendix of [7], it is shown that if S denotes the Tychonoff Plank, and T denotes a particular (non-normal) subset of S, then ind dim $S =$ cov dim $S =$ Ind dim $S = 0$, while cov dim $T > 0$ and Ind dim $T > 0$. However, cov dim T and Ind dim T are not determined in [7]. In this paper, we will prove, among other things, that the set T (Y_1 in our notation) satitisfies cov dim $T =$ Ind dim $T = 1$.

We establish the following notation.

Let Ω_0 denote the first ordinal whose cardinal is infinite.

Let Ω_n, for each positive integer n, denote the first ordinal greater than Ω_{n-1}, whose cardinality is greater than the cardinality of Ω_{n-1}. (Note that Ω_n is a limit ordinal. See page 100 of [5].)

Let X_i be the set of ordinals from 0 to Ω_i, inclusive, with the order topology; $X_i = [0, \Omega_i]$.

Let $X = \prod_{i=0}^{\infty} X_i$.

Let $p = (\Omega_0, \Omega_1, \cdots, \Omega_i, \cdots) \in X$.

Let $Y_\infty = Y = X - \{p\}$.

Let $Z_n = \prod_{i=0}^{n} X_i$.

Let $p_n = (\Omega_0, \Omega_1, \cdots, \Omega_n) \in Z_n$.

Let $Y_n = Z_n - \{p_n\}$.

The examples of this paper are X and the Y_n's just constructed.

We note that they are simple generalizations of the Tychonoff Plank. (See p. 132 of [9] or p. 154 of [7].) It is known that for any space A, $\operatorname{Ind}\dim A = 0$ iff $\operatorname{cov}\dim A = 0$, and that for a compact Hausdorff space A, $\operatorname{ind}\dim A = 0$ iff $\operatorname{Ind}\dim A = 0$ iff $\operatorname{cov}\dim A = 0$. (See [7, appendix] and [10].) Thus a totally disconnected, compact, Hausdorff space has dimension 0 in all three senses. The spaces Z_n, $1 \leqq n < \infty$, and X, constructed above, are totally disconnected, compact, Hausdorff, and hence, 0-dimensional in all three senses.

All spaces are Hausdorff.

We use the following definitions.

The *small inductive dimension* of a space A, *ind* A, is -1, iff $A = \emptyset$; $\operatorname{ind} A \leqq n$ iff for every point $p \in A$ and open set U containing p, there is an open set V such that $p \in V \subseteqq U$ and $\operatorname{ind}(\operatorname{bdry} V) \leqq n - 1$. We say $\operatorname{ind} A = n$ iff $\operatorname{ind} A \leqq n$ but $\operatorname{ind} A \not\leqq n - 1$.

The *large inductive dimension* of a space A, *Ind* A, is -1 iff $A = \emptyset$; $\operatorname{Ind} A \leqq n$ iff for every closed set $F \subseteqq A$ and open set U containing F, there is an open set V such that $F \subseteqq V \subseteqq U$ and $\operatorname{Ind}(\operatorname{bdry} V) \leqq n - 1$. We say $\operatorname{Ind} A = n$ iff $\operatorname{Ind} A \leqq n$ but $\operatorname{Ind} A \not\leqq n - 1$.

The *covering dimension* of a space A, *cov dim* A, or *dim* A, is -1 iff A is empty; $\dim A \leqq n$ iff every finite open cover of A has a finite open refinement such that at most $(n + 1)$ elements have a nonempty intersection. We say $\dim A = n$ iff $\dim A \leqq n$ but $\dim A \not\leqq n - 1$.

See [7] and [10] for a further discussion of dimension theory.

An *n-dimensional cover* of a space is a cover in which some $(n + 1)$ elements meet, but no $(n + 2)$ elements meet.

An *initial segment* of ordinals in X_n (or $X_n - \{\Omega_n\}$) is the set of all ordinals less than some $\alpha \in X_n$ (or $X_n - \{\Omega_n\}$).

An *end segment* of ordinals is the complement of an initial segment.

A *cofinal set* S in X_n means if $\alpha \in [0, \Omega_n)$ then there exists β such that $\alpha < \beta < \Omega_n$ and $\beta \in S$.

2. The non-monotony of covering dimension. We have seen that $\dim X = 0$. In this section we prove that $\dim Y_n = n$ for $1 \leqq n \leqq \infty$, and that X contains a copy of each Y_n. It is clear that Y_n is not normal for $n \geqq 1$, since it contains, as a closed subset, the nonnormal subset T of the Tychonoff Plank S discussed in §1.

We note that Dowker has given an example of a normal Hausdorff space N with a *normal* subset M such that $\dim N = \operatorname{Ind} N = 0$, but $\dim M = \operatorname{Ind} M = 1$. See [3] or pages 102–3 of [8].

LEMMA 2.1. *Let* $\mathscr{U}$: $U_1, U_2, \cdots, , U_k$ *be a finite open cover of* Y_n. *Let* $B_i = \{(x_0, x_1, \cdots, x_i, \cdots, x_n) \in Y_n \mid x_i \in [0, \Omega_i), x_j = \Omega_j \text{ for } j \neq i\}$. *Then some element of* $\mathscr{U}$ *must contain a cofinal subset of* B_i.

Proof. Suppose not. Since B_i is well-ordered by its ith coordinate, then for each $j, 1 \leqq j \leqq k$, there exists an $\alpha_{i_j} \in [0, \Omega_i]$ such that $(\Omega_0, \Omega_1, \cdots, \alpha_{i_j}, \cdots, \Omega_n)$ is the first element of B_i such that no element following it is in U_j. Let $\alpha = \max \{\alpha_{i_j}\}_{j=1}^k$. Then

$$p = (\Omega_0, \Omega_1, \cdots, \beta_i, \cdots, \Omega_n)$$

is not in any element of $\mathscr{U}$, if β_i is the ith coordinate of p, with $\beta_i > \alpha$. This is a contradiction.

THEOREM 2.1. Y_1 *is one dimensional*.

Proof. Let $\mathscr{U} = \{U_1, U_2\}$ be a cover of Y_1 defined as follows: $U_1 = Y_1 - B_0$, where $B_0 = \{(x, \Omega_1) \mid x \in [0, \Omega_0)\}$, $U_2 = Y_1 - B_1$, where $B_1 = \{(\Omega_0, x) \mid x \in [0, \Omega_1)\}$. We will show that any finite refinement $\mathscr{V}$ of $\mathscr{U}$ must contain some pair of elements which intersect. We first note that some element of $\mathscr{V}$ must contain a cofinal subset of B_0, by Lemma 2.1. We assume that V_1 is such an element.

We show that V_1 must have a limit point in B_1. For each $(\alpha, \Omega_1) \in V_1 \cap B_0$, there exists β_α such that $(\alpha, \beta) \in V_1$ for $\beta \geqq \beta_\alpha$, since V_1 is an open set about (α, Ω_1) and the set $\{(\alpha, x) \mid \alpha \text{ fixed}, x \in [0, \Omega_1]\}$ is homeomorphic to X_1. Then $\operatorname{card} \{\beta_\alpha \mid (\alpha, \Omega_1) \in V_1\} < \operatorname{card} \Omega_1$, and therefore $\gamma = \sup \{\beta_\alpha \mid (\alpha, \Omega_1) \in V_1\} < \Omega_1$. Then (Ω_0, γ) is a limit point of V_1, but not in U_2 and therefore not in V_1. Thus any element of $\mathscr{V}$ containing (Ω_0, γ) must meet V_1. It follows that Y_1 is at least one dimensional.

We must now show that Y_1 is at most one dimensional. Let $\mathscr{U}$ be an arbitrary finite open cover of Y_1. We wish to find a finite refinement which is at most 1-dimensional—i.e. no three elements intersect. Recall that $Y_1 = [0, \Omega_0] \times [0, \Omega_1] - \{(\Omega_0, \Omega_1)\}$.

Now $B_0 = [0, \Omega_0) \times \{\Omega_1\}$ is a closed subset of Y_1 and is homeomorphic to the set of positive integers. Since $\mathscr{U}$ is a finite open cover of Y_1, we note, incidentally, that at least one element of $\mathscr{U}$ contains a cofinal subset of B_0. Let $U_1, U_2, \cdots, U_n$ be all the elements of $\mathscr{U}$ which intersect B_0. If $V_1' = U_1 \cap B_0$, $V_i' = (U_i - \bigcup_{j<i} U_j) \cap B_0$, then $\bigcup_{i=1}^n V_i' = B_0$, and V_i' is open in B_0. For each point $(\alpha_{i_\beta}, \Omega_1) \in V_i'$, there exists $\bar{\gamma}_{i_\beta} \in [0, \Omega_1)$ such that for all $\gamma \geqq \bar{\gamma}_{i_\beta}$, $(\alpha_{i_\beta}, \gamma) \in U_i$. If $\bar{\gamma}_i =$

$\sup\{\bar{\gamma}_{i_\beta} \mid (\alpha_{i_\beta}, \Omega_1) \in V_i'\}$, then $V_i' \times [\bar{\gamma}_i, \Omega_1] \subseteqq U_i$. If $\delta \geqq \max\{\bar{\gamma}_i \mid i=1, \cdots, n\}$, subject to the condition that δ is not a limit ordinal, then $V_i' \times [\delta, \Omega_1] \subseteqq U_i$, $i = 1, \cdots, n$. Let $W_i = V_i' \times [\delta, \Omega_1]$, and we note that W_i is an open subset of u_i. Now $[0, \Omega_0] \times [0, \delta - 1]$ is a compact subset of Y_1, which is also open in Y. Therefore there exists a finite 0-dimensional refinement $\mathscr{U}'$ of $\mathscr{U}$, covering this set. Now the collection of members of $\mathscr{U}'$ together with the $\{W_i\}_{i=1}^n$, forms a 0-dimensional finite open cover of $Y_1 - (\{\Omega_0\} \times [\delta, \Omega_1))$. We will add enough members to this collection in such a way that an at most 1-dimensional cover will result.

If $C = \{\Omega_0\} \times [\delta, \Omega_1)$, then a finite number of the elements of $\mathscr{U}$, say $U_1'', \cdots, U_k''$, cover C. Thus if $D_i = U_i'' \cap C$, then $\{D_i\}_{i=1}^k$ is a finite open cover of C in C. Since C is 0-dimensional, there exists a finite 0-dimensional refinement, say $E_1 \cdots E_s$. Now $[0, \Omega_0] \times E_i$ is open in Y_1. If $N_i = ([0, \Omega_0] \times E_i) \cap U_{E_i}$, where U_{E_i} is any element of $\{U_i\}_{i=1}^n$ which contains E_i, then the collection $\{N_i\}_{i=1}^s$ is 0-dimensional collection.

Let $\mathscr{O}$ be the finite collection of open sets consisting of the members of $\mathscr{U}'$, the members of $\{W_i\}_{i=1}^n$, and the members of $\{N_i\}_{i=1}^s$. Then $\mathscr{O}$ is an at most 1-dimensional cover of Y_1, for the only possible intersections are between an N_i and either an element of $\mathscr{U}'$ or of $\{W_i\}_{i=1}^n$.

It follows that Y_1 is 1-dimensional.

THEOREM 2.2. *Y_n has dimension $\geqq n$.*

Proof. The proof is by induction on the method of proof of Theorem 2.1. Let $\mathscr{U}$: $U_0, U_1, \cdots, U_n$ be an open cover of Y_n defined by $U_i = Y_n - \bigcup_{j \neq i} B_{n,j}$, where $B_{n,j} = \{(x_0, x_1, \cdots, x_j, \cdots, x_n) \in Y_n \mid x_j \in [0, \Omega_j)$ and $x_k = \Omega_k$ for $k \neq j\}$. Let $\mathscr{V}$ be any finite refinement of $\mathscr{U}$. By Lemma 2.1, for each i, $0 \leqq i \leqq n$, there exists $V_i \in \mathscr{V}$ such that V_i contains a cofinal subset of $B_{n,i}$. We will show that $\bigcap_{i=0}^n V_i \neq \varnothing$, by showing that $\bigcap_{i=0}^{n-1} V_i$ has an end segment of $B_{n,n}$ as limit points. We note that Y_{n-1} is homeomorphic to

$$\{(x_0, x_1, \cdots, x_{n-1}, \Omega_n) \in Y_n \mid (x_0, x_1, \cdots, x_{n-1}) \in Y_{n-1}\} \subseteqq Y_n ,$$

and that $B_{n-1,i}$ is homeomorphic to $B_{n,i}$ for $0 \leqq i \leqq n-1$. We identify these in the remainder of the proof, without explicitly referring to the homeomorphism between them.

We assume, as an inductive hypothesis, that for any collection of V_i's chosen as above, $\bigcap_{i=0}^{n-1} V_i \neq \varnothing$. For $n = 2$, this is proved in Theorem 2.1. Let $V_i' = Y_{n-1} \cap V_i$, let $V' = \bigcap_{i=0}^{n-1} V_i'$, and let $V = \bigcap_{i=0}^{n-1} V_i$.

Now card $Y_{n-1} < \text{card}\,\Omega_n$, and therefore card $V' < \text{card}\,\Omega_n$. For $x \in V'$, let $\alpha_x \in [0, \Omega_n)$ be an ordinal such that if $\alpha > \alpha_x$, then $(x, \alpha) \in V$. We know that α_x exists since V is open in Y_n containing the point (x, Ω_n). Card $\{\alpha_x | x \in V'\} < \text{card}\,\Omega_n$. Therefore if $\bar{\alpha} = \sup\{\alpha_x\}$, then $\bar{\alpha} < \Omega_n$ and $V' \times [\bar{\alpha}, \Omega_n] \subseteqq V$.

Now let $y = (\Omega_0, \Omega_1, \cdots, \Omega_{n-1}, \beta) \in B_{n,n}$ such that $\beta \geqq \bar{\alpha}$. We wish to show that y is a limit point of V. We accomplish this by showing that if $(\beta_0, \beta_1, \cdots, \beta_{n-1}, \beta) \in Y_n$, $\beta_i < \Omega_i$, then there exist $\gamma_0, \gamma_1, \cdots, \gamma_{n-1}$ such that $\gamma_i > \beta_i$ and $(\gamma_0, \gamma_1, \cdots, \gamma_{n-1}, \beta) \in V$. To this end, let η_i be a non-limit ordinal $> \beta_i$ such that $\eta_i \in V_i \cap B_{n,i}$, let

$$W_i = \left([\eta_i, \Omega_i] \times \prod_{\substack{j=0 \\ j \neq i}}^{n} x_j\right) \cap V_i \quad \text{for}$$

$0 \leqq i \leqq n-1$, and let $W_i' = W_i \cap Y_{n-1}$. Then W_i' is a neighborhood of a cofinal set in $B_{n,i} \subseteqq Y_{n-1}$, and therefore, by the inductive hypothesis, $W_i' = \bigcap_{i=0}^{n-1} W_i' \neq \emptyset$. We note that $W' \subseteqq V'$, and therefore there exists a point

$$(\gamma_0, \gamma_1, \cdots, \gamma_{n-1}) \in W' \subseteqq V' .$$

Then $(\gamma_0, \gamma_1, \cdots, \gamma_{n-1}, \beta) \in V' \times [\bar{\alpha}, \Omega_n] \subseteqq V$.

Thus we have shown that each point of $\{(\Omega_0, \Omega_1, \cdots, \Omega_{n-1})\} \times [\bar{\alpha}, \Omega_n)$ is a limit point of V. Therefore any neighborhood V_n of a cofinal set in $B_{n,n}$ must meet V and $\bigcap_{i=0}^{n} V_i \neq \emptyset$. It follows that $\dim Y_n \geqq n$.

THEOREM 2.3. *Y_n has dimension $\leqq n$.*

Proof. Let $\mathscr{U}$ be a finite open cover of Y_n. We wish to find a finite refinement of $\mathscr{U}$ which is an at most n-dimensional cover of Y_n. Let $B_{n,i}$ be defined as in the proof of Theorem 2.2, and again we don't distinguish in notation between $B_{n-1,i}$ in Y_{n-1}, and its homeomorphic copy $B_{n,i}$ in Y_n for $1 \leqq i \leqq n-1$. We also identify Y_i with $(Y_i \times \prod_{j=i+1}^{n} \{\Omega_j\})$ in Y_n.

We assume, as an inductive hypothesis, that Y_{n-1} has an at most $(n-1)$ dimensional refinement of any given cover. Let $\mathscr{U}'$ be the elements of $\mathscr{U}$ intersected with Y_{n-1}. Then $\mathscr{U}'$ is a cover of Y_{n-1}, and by hypothesis, has an at most $(n-1)$-dimensional refinement. Let $\mathscr{V}'$: $V_1', \cdots, V_k'$ be such a refinement of $\mathscr{U}'$. Let $V_i' \subseteqq U_i'$. Note that it is possible that $U_i' = U_j'$ for $i \neq j$. Note also that card $Y_{n-1} <$ card Ω_n. Fix i. For each $x \in V_i'$, there exists α_{x_i} such that for $\alpha > \alpha_{x_i}$, $(x, \alpha) \in U_i$. Let $\bar{\alpha}_i = \sup\{\alpha_{x_i} | x \in V_i'\}$ and we see that $\bar{\alpha}_i < \Omega_n$. Let $\bar{\alpha} \geqq \sup\{\bar{\alpha}_i | i = 1, \cdots, k\}$ such that $\bar{\alpha}$ is a non-limit ordinal. Then $V_i = V_i' \times [\bar{\alpha}, \Omega_n] \subseteqq U_i \in \mathscr{U}$, for each i, $1 \leqq i \leqq k$. Thus $Y_{n-1} \times [\bar{\alpha}, \Omega_n]$ is a subset of Y_n which is covered by an at most $(n-1)$-dimensional finite collection, $\mathscr{V}$: $V_1, \cdots, V_k$, which refines $\mathscr{U}$. The points of $[B_{n,n} \cup (Y_{n-1} \times [0, \bar{\alpha}))]$ are the only points not yet covered. Now $Z_{n-1} \times [0, \bar{\alpha})$ is a zero dimensional clopen subset of Y_n, and therefore there exists a finite, zero dimensional refinement of $\mathscr{U}$ covering it, say $\mathscr{O}$.

It remains to cover the set of points of $\{(\Omega_0, \Omega_1, \cdots, \Omega_{n-1})\} \times [\bar{\alpha}, \Omega_n)$. Call this set A. The elements of $\mathscr{U}$ intersected with A is a cover of A, say $\mathscr{U}''$, and A is zero dimensional. Thus there exists a zero dimensional finite refinement of $\mathscr{U}''$ which covers A, say $\mathscr{W}$: $W_1, W_2, \cdots, W_s$. Now fix i. For each $x \in W_i \subseteqq U_i$, there exists a clopen product neighborhood, $H_{i_x} = H_0 \times H_1 \cdots \times H_{n-1} \times W_{i_x}$, where $W_{i_x} \subseteqq W_i$ and such that $H_{i_x} \subseteqq U_i$. Let $H_i' = \bigcup_{y \in W_i} H_{i_y}$. Then $x \in H_i'$, $H_i' \subseteqq U_i$, and H_i' is open in Y_n. Further $H_i' \cap H_j' = \emptyset$ for $i \neq j$, and $H_i' \cap O_j = \emptyset$ for any $O_j \in \mathscr{O}$. Let $\mathscr{M} = \{H_1', H_2', \cdots, H_s'\}$. Then since at most n elements of $\mathscr{V}$ have $\neq \emptyset$ intersection, the addition of $\mathscr{M}$ to $\mathscr{V}$ will make at most $n + 1$ elements intersect. Thus the cover defined by the elements of $\mathscr{V}$, $\mathscr{O}$, and $\mathscr{M}$ is an at most n-dimensional cover of Y_n. It follows that Y_n is at most n-dimensional.

THEOREM 2.4. *Y_n has dimension n.*

Proof. Clear from Theorem 2.2 and 2.3.

THEOREM 2.5. *Y is infinite dimensional and contains subsets of dimension n, for every positive integer n.*

Proof. It is clear that Y contains n-dimensional subsets for all n, since Y_n is homeomorphic to $\{(x_0, x_1, \cdots, x_n, \cdots) \in Y | x_i = \Omega_i \text{ for all } i > n\} \subseteqq Y$. We identify this set with Y_n in our notation. Since Y_n is closed in Y, it follows that $\dim Y = \infty$.

THEOREM 2.6. *X is a 0-dimensional, compact, Hausdroff space, and contains subsets of dimension n, for $1 \leqq n \leqq \infty$.*

Proof. By Theorem 2.5, Y is ∞-dimensional and contains subsets of dimension n for $1 \leqq n \leqq \infty$. But $Y \subseteqq X$. The theorem follows.

COROLLARY 2.6.1. *For each $n, 1 \leqq n \leqq \infty$, there exists an n-dimensional continuum with subsets of dimension $k, 1 \leqq k \leqq \infty$.*

Proof. X_i is a totally ordered, totally disconnected compact Hausdorff space, and can be imbedded in a generalized arc, A_i, preserving the order of X_i, in such a way that if $x < y$ in X_i and there does not exist z in X_i with $x < z < y$, then A_i contains a real arc from x to y.

Now $X = \prod_{j=0}^{\infty} X_i$ and is a compact totally disconnected Hausdorff space with a lattice-like structure, intuitively like the ordered ∞-tuples of integers in R^{∞}. We fill in the lattice structure, to make a 1-dimensional grid, in the following way. Let $M = \prod_{i=1}^{\infty} A_i$ and let M_1 be the subset of M defined by $M_1 = \{(\alpha_1, \alpha_2, \cdots, \alpha_i, \cdots) \in M \mid$ there exists at most one coordinate α_k such that $\alpha_k \in A_k - X_k\}$. We show that $\dim M_1 = 1$. Clearly $\operatorname{ind} M_1 = 1$, since, if $x \in M_1$ and U is open with $x \in U$, then there exists a neighborhood V of x such that $V \subseteqq U$, and $Bd\,V$ is homeomorphic to a subset of $\prod_{i=0}^{\infty} X_i$. Thus $Bd\,V$ is 0-dimensional, and $\operatorname{ind} M_1 \leqq 1$. But M_1 contains arcs. Thus $\operatorname{ind} M_1 = 1$. Now, by Theorem B page 198 of [10], since M_1 is compact Hausdorff, $\dim M_1 \leqq \operatorname{ind} M_1$. It follows that $\dim M_1 = 1$, since clearly $\dim M_1 > 0$.

Let $I_i = [0, 1]$ and let $M_n = M_1 \times \prod_{i=1}^{n-1} I_i$. By Theorem 4.7 of [2] the product of an n-dimensional compact space with the unit interval has dimension $\leqq n + 1$. Now M_1 contains an interval, so that $\dim M_2 = 2$, and inductively, $\dim M_n = n$. Since $X \subseteqq M_n$, the corollary follows.

3. On compactifications of Y_n and Y. In [4, pg. 124], it is shown that the one point compactification of $Y_1 = \beta Y_1 = Z_1$. In this section, we show that the one points compactification of $Y_n = \beta Y_n = Z_n$, for $1 \leqq n < \infty$. Thus for each $n, 1 \leqq n < \infty$, we have an example of a space of covering dimension n, whose Stone-Čech compactification is of dimension 0. On pages 244–245 of [4], it is proved that $\dim X = \dim \beta X$, if X is normal. Thus we see the necessity of the hypothesis of normality. We note that in [4, pg. 248], it is pointed out that Y_1 is a space of *positive* dimension whose Stone-Čech compactification has dimension 0.

LEMMA 3.1. *Let f be continuous from X_n (or $X_n - \{\Omega_n\}$) to the reals, $n \geqq 1$. Then there exists an end segment $E = [\alpha, \Omega_n)$ (or $[\alpha, \Omega_n]$) such that f is constant on E.*

Proof. This is proved for X_1 on page 75 of [4]. It is clear that the same proof works for X_n.

THEOREM 3.1. *The Stone-Čech compactification of Y_n is Z_n, $n \geqq 1$.*

Proof. For $n = 1$, this is proved on page 124 of [4].

In general, $Y_n = \prod_{i=0}^{n} X_i - \{(\Omega_0, \Omega_1, \cdots, \Omega_n)\}$. By Theorems 6.4, 6.5 of [4] it is sufficient to show that every (bounded) continuous function from Y_n to the reals, has a continuous bounded extension to Z_n. It follows from the argument below that any continuous real function on Y_n is necessarily bounded.

We assume, as an inductive hypothesis, that every real continuous function on Y_{n-1} can be extended to Z_{n-1}. Let $f\colon Y_n \to R$ be continuous. For each $x \in Y_{n-1}$, $W_x = \{(x, \alpha) \mid \alpha \in X_n\}$ is homeomorphic to X_n, and therefore $f \mid W_x$ is constant on an end segment, by Lemma 3.1; that is, there exists α_x such that $f(x, \alpha) = f(x, \alpha_x)$ for all $\alpha \geqq \alpha_x$. Further, if $W = \{(\Omega_0, \Omega_1, \cdots, \Omega_{n-1}, \alpha) \mid \alpha \in [0, \Omega_n)]$, then f is constant on an end segment of W, also, say for $\alpha \geqq \alpha_0$. Now card $Z_{n-1} <$ card Ω_n. Let $A = [\{\alpha_0\} \cup \{\alpha_x \mid x \in Y_{n-1}\}]$, and it follows that card $A <$ card Ω_n. Therefore $\sup A < \Omega_n$. Let $\bar{\alpha} \geqq \sup A$ such that $\bar{\alpha}$ is a non-limit ordinal. Then $[Z_{n-1} \times (\bar{\alpha}, \Omega_n]] - \{(\Omega_0, \Omega_1, \cdots, \Omega_n)\}$ has the property that on each $\{x\} \times (\bar{\alpha}, \Omega_n]$, f is constant. It follows that for $x \in Y_{n-1}$, $f((x, \Omega_n)) = f((x, \beta))$, for each $\beta \geqq \bar{\alpha}$. Now, since $f \mid Y_{n-1} \times \{\Omega_n\}$ can be extended to $(\Omega_0, \cdots, \Omega_{n-1}, \Omega_n)$, say it has value k, then since this extension is unique, it has same value as f on $(\Omega_0, \cdots, \Omega_{n-1}, \beta)$ for each $\beta \geqq \bar{\alpha}$. Thus we can extend f to Z_n, by defining $f((\Omega_0, \cdots, \Omega_n)) = k$.

It follows that Z_n is the Stone-Čech compactification of Y_n.

QUESTION. Is X the Stone-Čech compactification of Y?

THEOREM 3.2. *For each* n, $1 \leqq n < \infty$, *there exists a (non-normal) space of dimension* n, *whose Stone-Čech compactification has dimension* 0.

Proof. Our spaces Y_n satisfy the conditions of the theorem, by Theorems 2.4, 2.5, and 3.1.

4. The non-monotony of large inductive dimension.

In §2, we proved that $\dim Y_n = n$, $1 \leqq n \leqq \infty$. In this section, we show that $\operatorname{Ind} Y_n \geqq n$, $1 < n < \infty$, $\operatorname{Ind} Y_1 = 1$, and $\operatorname{Ind} Y_\infty = \infty$. Thus $\operatorname{Ind} X = 0$, but X contains subsets Y_n whose large inductive dimension is at least n, $1 \leqq n \leqq \infty$.

LEMMA 4.1. *If* F_1 *and* F_2 *are disjoint closed sets in* $[0, \Omega_n)$, $0 < n < \infty$, *then one of them is bounded.*

Proof. (See 5.12 (b) p. 74 of [4]. Proof is included here for completeness.) Suppose neither is bounded. Let $\{\alpha_i\}_{i=1}^\infty$ be a sequence of elements of $[0, \Omega_n)$ chosen inductively so that (1) $\alpha_i < \alpha_{i+1}$, (2) $\alpha_i \in F_1$ for i odd, and (3) $\alpha_i \in F_2$ for i even. If $\gamma = \operatorname{lub}\{\alpha_i\}$ then $\gamma < \Omega_n$ and γ is a limit point of both F_1 and F_2. Therefore $\gamma \in F_1 \cap F_2$, and this is a contradiction.

LEMMA 4.2. $\operatorname{Ind}[0, \Omega_n) = 0$, $0 \leqq n < \infty$.

Proof. Case 1. $n > 0$. Let F be closed in $[0, \Omega_n)$, and let U be a neighborhood of F. We wish to find an open set V such that $F \subseteqq V \subseteqq U$ and $\operatorname{Bd} V = \varnothing$.

Suppose F is bounded, there exists a clopen cover of F by subsets V_α of U. Since F is bounded then it must be compact, and therefore there exist $V_{\alpha_1}, \cdots, V_{\alpha_k}$ contained in U such that $F \subseteqq V = \bigcup_{i=1}^k V_{\alpha_i}$. Then V is open, $F \subseteqq V \subseteqq U$, and $\operatorname{Bd} V = \varnothing$.

If F is not bounded then, by Lemma 4.1, $C(U)$ is bounded, and by a similar argument, we find the required set V.

Thus, in either event, there exists an open set V such that $F \subseteqq V \subseteqq U$ and $\operatorname{Bd} V = \varnothing$.

Case 2. $n = 0$. Clear, since the nonnegative integers are discrete.

LEMMA 4.3. *If $\mathscr{U}$ is a finite open cover of $[0, \Omega_n)$, $1 \leqq n < \infty$, then some element of $\mathscr{U}$ contains an end segment of $[0, \Omega_n)$.*

Proof. Suppose no element of $\mathscr{U}$ contains an end segment. Then at least two elements of $\mathscr{U}$ must contain disjoint cofinal sets in $[0, \Omega_n)$. Let $U_1, \cdots, U_k$ be those elements of $\mathscr{U}$ which contain pairwise disjoint cofinal sets $A_1, \cdots, A_k$ respectively, in $[0, \Omega_n)$ with $A_i \cap U_j = \varnothing$, $j \neq i$, $i, j = 1, \cdots, k$, and let $a_{1,1}, a_{2,1}, \cdots, a_{k,1}$ be elements of $A_1 \cdots, A_k$ respectively, such that $a_{i,1} > x$ for all $x \notin \bigcup_{j=1}^k U_j$. Let $b_1 \in [0, \Omega_n)$ such that $b_1 > a_{i,1}$, $1 \leqq i \leqq k$, let $a_{1,2}, a_{2,2}, a_{3,2}, \cdots, a_{k,2}$ elements of $A_1, \cdots, A_k$ respectively, such that $a_{i,2} > b_1$, and let $b_2 \in [0, \Omega_n)$ such that $b_2 > a_{i,2}$, $1 \leqq i \leqq k$.

Continue the above process inductively, and let $b = \sup\{b_i\}$. Then $b < \Omega_n$ and b is a limit point of A_i, $1 \leqq i \leqq k$. Further, if $b \in U_j$ then $j \in \{1, 2, \cdots, k\}$ and U_j contains points of *each* A_i, $1 \leqq i \leqq k$. But $A_i \cap U_j = \varnothing$ for $i \neq j$ when $1 \leqq i \leqq k$, and this is a contradiction.

LEMMA 4.4. *If F_1 and F_2 are disjoint closed sets in $\prod_{i=0}^{n-1} [0, \Omega_i] \times [0, \Omega_n)$, then at least one of F_1 and F_2 is bounded away from the corner point of Y_n in the n^{th} direction; that is, there exist non-limit ordinals $\alpha_0, \alpha_1 \cdots, \alpha_n$ such that $\prod_{i=0}^{n-1} [\alpha_i, \Omega_i] \times [\alpha_n, \Omega_n)$ misses, say, F_2. (The corner point of Y_n is $(\Omega_0, \cdots, \Omega_n)$ even though this point does not belong to Y_n.)*

Proof. Let $B_n = \{(\Omega_0, \cdots, \Omega_{n-1}, x) \mid x \in [0, \Omega_n)\}$ and we see by Lemma 4.1, not both $F_1 \cap B_n$ and $F_2 \cap B_n$ can be unbounded in B_n. Thus we assume that there exists a non-limit ordinal α_n such that if $\alpha \geqq \alpha_n$ then $(\Omega_0, \Omega_1, \cdots, \Omega_{n-1}, \alpha) \notin F_2$.

Now, suppose the lemma is false. Then for each point $(\beta_0, \beta_1, \cdots, \beta_n)$ with $\beta_n > \alpha_n$, and $\beta_i < \Omega_i$ for all i, there exists another point $(\beta'_0, \beta'_1, \cdots, \beta'_n)$ in F_2 such that $\beta_i < \beta'_i \leqq \Omega_i$ for $0 \leqq i \leqq n-1$ and $\beta_n < \beta'_n < \Omega_n$. Thus after a countable number of choices, with each point closer, in at least one coordinate to $(\Omega_0, \cdots, \Omega_n)$ than the preceding point (closer in 0^{th} coordinate whenever possible), we exhaust all possibilities for β_0. This is true because there are only a countable number of points in X_0. Then if $\beta_{i,1}$ denotes $\sup\{\beta'_i \mid$ for each choice of $\beta'_0 \leqq \Omega_0\}$ it follows that $(\Omega_0, \beta_{1,1}, \beta_{2,1}, \cdots, \beta_{n,1})$ is a point or limit point of F_2 and therefore must be in F_2. Further, each $\beta_{i,1} \leqq \Omega_i$, $1 \leqq i < n$, and $\beta_{n,1} < \Omega_n$.

We continue the process inductively and note that after at most $|\Omega_1|$ choices, we obtain a point $(\Omega_0, \Omega_1, \beta_{2,2} \cdots, \beta_{n,2})$ as a point or limit point of F_2, and therefore in F_2, and such that $\beta_{i,2} \leqq \Omega_i$, $2 \leqq i < n$, with $\beta_{n,2} < \Omega_n$. After n steps, we obtain a point $(\Omega_0, \Omega_1, \cdots, \Omega_{n-1}, \beta_{n,n})$ in B_n as a point or limit point of F_2 and therefore in F_2. But $\beta_{n,n} > \alpha_n$ and this is a contradiction. Thus F_2 must be bounded away from the corner point of Y_n in the n^{th} direction.

LEMMA 4.5. *If* $B \subseteqq A$ *and* $U \subseteqq A$, *then* $\mathrm{Bd}_B(U \cap B) \subseteqq \mathrm{Bd}_A(U)$.

Proof. Let $x \in Bd_B(U \cap B)$. Then each neighborhood of x contains points of $U \cap B$ and of $C(U) \cap B$ and therefore contains points of both U and $C(U)$. It follows that $x \in \mathrm{Bd}_A(U)$.

REMARK. The following two lemmas are well known among dimension theorists. However, we include them here since they may not have been explicitly stated in other published work. The proof is the same as the proof for the metric case [10, II. 1.A].

LEMMA 4.6. *If* $\mathrm{Ind}\, A = 0$ *and* B *is a closed subset of* A, *then* $\mathrm{Ind}\, B = 0$.

LEMMA 4.7. *If* $\mathrm{Ind}\, A = k$ *and* B *is a closed subset of* A, *then* $\mathrm{Ind}\, B \leqq k$.

REMARK. The next proposition is not used in the rest of the paper, but is included for its (possible) interest. Also, the proof of this proposition is referred to in Theorem 4.1.

PROPOSITION 4.1.[1] *If* $S = (\prod_{i=0}^{n-1} X_i) \times [0, \Omega_n)$, then $\dim S = \mathrm{Ind}\, S = 0$.

[1] The referee has pointed out that this theorem follows easily from a theorem of Morita (*Topogical completeness and M-spaces*, Sci. Reports Tokyo Kyoiku Daigaku. Sec. A **10** (1970) 271-288) that $\dim X \times Y \leqq \dim X + \dim Y$ if X is paracompact, locally compact, and Y is pseudoparacompact.

Proof. Let $\mathscr{U}$ be a finite open cover of S. Each $\{x\} \times [0, \Omega_n)$ is homeomorphic to $[0, \Omega_n)$, and therefore, by Lemma 4.3, some end segment of $\{x\} \times [0, \Omega_n)$, say $\{x\} \times [\alpha_x, \Omega_n)$, is a subset of every element of $\mathscr{U}$ which it meets. Since card $(\prod_{i=0}^{n-1} X_i) <$ card $[0, \Omega_n)$, let $\alpha' = \sup\{\alpha_x \mid x \in \prod_{i=0}^{n-1} X_i\}$ and let $\alpha > \alpha'$ be a non-limit ordinal. Then for each x, there is a j such that $(\{x\} \times [\alpha, \Omega_n)) \subseteqq U_j$ Further, if $\{x\} \times [\alpha, \Omega_n)$ meets several U_j's, then it is a subset of each of them. Consider $(\prod_{i=0}^{n-1} X_i) \times \{\alpha\}$. This is a compact zero dimensional space, and $\mathscr{U}' = \{U_j \cap (\prod_{i=0}^{n-1} X_i \times \{\alpha\}) \mid U_j \in \mathscr{U}\}$ is a finite open cover of this space. Thus there exists a zero dimensional refinement $\mathscr{V}'$ of $\mathscr{U}'$. For each $V' \in \mathscr{V}'$, $V' \times [\alpha, \Omega_n) \subseteqq$ some U_j, and the collection $\mathscr{V} = \{V' \times [\alpha, \Omega_n) \mid V' \in \mathscr{V}'\}$ is a zero dimensional cover of $(\prod_{i=0}^{n-1} X_i) \times [\alpha, \Omega_n)$ refining $\mathscr{U}$.

Now since $\prod_{i=0}^{n-1} X_i \times [0, \alpha - 1]$ is a zero dimensional compact Hausdorff space, there exists a finite zero dimensional refinement $\mathscr{W}$ of $\mathscr{U}$ covering this set. Then the collection $\mathscr{V} \cup \mathscr{W}$ is a finite zero dimensional refinement of $\mathscr{U}$. It follows that $\dim S = 0$.

Now in the appendix of [7], it is shown that for any space S, $\dim S = 0$ iff $\operatorname{Ind} S = 0$. Thus the lemma is proved.

THEOREM 4.1. $\operatorname{Ind} Y_1 = 1$.

Proof. If $B_0 = \{(x, \Omega_1) \mid x \in [0, \Omega_0)\}$, $B_1 = \{(\Omega_0, x) \mid x \in [0, \Omega_1)\}$, and $U = Y_1 - B_1$, then B_0 is a closed subset of Y_1 and $B_0 \subseteqq U$, with U open in Y_1. As in the proof of Theorem 2.1, we see that for any open set V such that $B_0 \subseteqq V \subseteqq U$, $\operatorname{Bd} V$ must contain $a \neq \varnothing$ end segment of B_1. Thus $\operatorname{Ind} Y_1 \geqq 1$.

We now show $\operatorname{Ind} Y_1 \leqq 1$. Let F be closed in Y_1 and let U be and neighborhood of F. F and $\operatorname{Bd} U$ are disjoint closed sets in Y_1.

If F contains a cofinal set in B_1, then by Lemma 4.1, $\operatorname{Bd} U$ misses an end segment of B_1 and by Lemma 4.4, there exist non-limit ordinals α_0 and α_1 such that $[\alpha_0, \Omega_0] \times [\alpha_1, \Omega_1)$ misses $\operatorname{Bd} U$. Thus $\operatorname{Bd} U \subseteqq K \cup ([\alpha_0, \Omega_0] \times \{\Omega_1\})$, where K is the totally disconnected compact open subset of Y_1 defined by $K = Z_1 - [\alpha_0, \Omega_0] \times [\alpha_1, \Omega_1]$. Then $K \cap \operatorname{Bd} U$ can be covered by a finite number of clopen sets $V_1, \cdots, V_s$, each of which is a subset of K and misses F, so that $V = U - \bigcup_{i=1}^{s} V_i$ is open, contains F, and $\operatorname{Bd} V = \varnothing$ or $\operatorname{Bd} V \subseteqq [\alpha_0, \Omega_0] \times \{\Omega_1\}$.

If F contains no cofinal set in B_1, then by an argument like that of the above paragraph, $F \subseteqq L \cup ([\beta_0, \Omega_0) \times \{\Omega_1\})$ where L is a totally disconnected compact open subset of Y_1 defined by $L = Z_1 - [\beta_0, \Omega_0] \times [\beta_1, \Omega_1]$, and β_0 and β_1 are some non-limit ordinals. Then $L \cap F$ can be covered by a finite number of clopen sets $W_1, \cdots, W_t$, each of which is a subset of $U \cap L$, so that if $W = \bigcup_{i=1}^{t} W_i$, then $\operatorname{Bd} W = \varnothing$. From this we see that if $F \subseteqq L$, there exists an open set W such that

$F \subseteqq W \subseteqq U$ and $\mathrm{Bd}\, W = \emptyset$. Otherwise $F \cap (Y_1 - L) \subseteqq [\beta_0, \Omega_0] \times \{\Omega_1\}$. Let $F' = F \cap (Y_1 - L)$ and let $U' = U \cap (Y_1 - L)$. It suffices to show that there exists W' open such that $F' \subseteqq W' \subseteqq U'$ and $\mathrm{Bd}\, W'$ is at most 0-dimensional (in the large inductive sense).

To this end, we note that U' is a neighborhood of F', $F' \subseteqq ([\beta_0, \Omega_0) \times \{\Omega_1\})$ and, since there are only countably many elements in F', by the method used in Proposition 4.1, there exists a non-limit ordinal γ such that if $W' = ((F' \cap B_0) \times [\gamma, \Omega_1])$, then $W' \subseteqq U'$. Since $[0, \Omega_0)$ is discrete, W' is open, so that W' is open, contains F' and $\mathrm{Bd}\, W' \subseteqq B_1$ or $\mathrm{Bd}\, W' = \emptyset$. Thus $\mathrm{Bd}\, W'$ is at most zero dimensional. The theorem follows.

COROLLARY 4.1.1. *If* $A = [\alpha, \Omega_0] \times [\beta, \Omega_1] - \{(\Omega_0, \Omega_1)\}$ *then* $\mathrm{Ind}\, A = 1$.

Proof. Clear from the proof of Theorem 4.1.

COROLLARY 4.1.2. *Let* $Y_1' = [0, \Omega_1] \times [0, \Omega_2] - \{(\Omega_1, \Omega_2)\}$ *and let* $Y_1'(\alpha, \beta) = [\alpha, \Omega_1] \times [\beta, \Omega_2] - (\Omega_1, \Omega_2)$. *Then* $\mathrm{Ind}\, Y_1' = 1$ *and* $\mathrm{Ind}\, Y_1'(\alpha, \beta) = 1$.

Proof. A simple modification of the proof of Theorem 4.1 shows this.

DEFINITION. Let $Y_{n-1}' = \prod_{i=1}^{n} [0, \Omega_i] - \{(\Omega_1, \Omega_2, \cdots, \Omega_n)\}$. By a *corner* of Y_{n-1}' we mean a set of the form $\prod_{i=1}^{n} [\alpha_i, \Omega_i] - \{(\Omega_1, \cdots, \Omega_n)\}$, where α_i is a non-limit ordinal, $i = 1, \cdots, n$. In the proof below, we identify Y_{n-1}' with the set of points of Y_n whose first coordinate is Ω_0.

THEOREM 4.2. $\mathrm{Ind}\, Y_n \geqq n$, $1 \leqq n < \infty$.

Proof. If $B_0 = \{(x, \Omega_1, \Omega_2, \cdots, \Omega_n) \in Y_n \mid x \in [0, \Omega_0)\}$ and

$$B_i = \{(\Omega_0, \cdots, \Omega_{i-1}, x, \Omega_{i+1}, \cdots, \Omega_n) \in Y_n \mid x \in [0, \Omega_i)\}$$

then B_0 is closed and $U = Y_n - Y_{n-1}'$ is a neighborhood of B_0. We assume as an inductive hypothesis that any corner of Y_{n-1}' has large inductive dimension at least $n - 1$, and note that for $n = 2$, this is just Corollary 4.1.2. We will show that any neighborhood V of B_0 such that $V \subseteqq U$ must contain a corner of Y_{n-1}' on its boundary. As in the proofs of §2, we do not distinguish in notation between Y_{n-1}' and its copy in Y_n.

To this end, we first prove that if V is any neighborhood of B_0 such that $V \subseteqq U$, then there exist non-limit ordinals $\alpha_1, \alpha_2, \cdots, \alpha_n$ such that $B_0 \times \prod_{i=1}^{n} [\alpha_i, \Omega_i] \subseteqq V$. By an analogous argument to the proof of Theorem 2.1, there is a non-limit ordinal α_1 such that $B_0 \times$

$[\alpha_1, \Omega_1] \subseteqq U$. Inductively assume that there exist non-limit ordinals $\alpha_1, \cdots, \alpha_{k-1}$ such that $B_0 \times \prod_{i=1}^{k-1} [\alpha_i, \Omega_i] \subseteqq V$. Now each point of $B_0 \times \prod_{i=1}^{k-1} [\alpha_i, \Omega_i]$ is the last point of a "vertical set" over it in Y_n with the vertical set homeomorphic to $[0, \Omega_k]$, and card $(B_0 \times \prod_{i=1}^{k-1} [\alpha_i, \Omega_i]) < \Omega_k$. Thus for each of these "vertical sets", there exists $\beta < \Omega_k$ such that the end segment below β lies in V, since V is a neighborhood of the last point of this segment. If we let α_k be greater than or equal to $\sup\{\beta \mid \beta$ is obtained as described for each point of $B_0 \times \prod_{i=1}^{k-1} [\alpha_{i_k}, \Omega_i]\}$ with α_k a non-limit ordinal, then $\alpha_k < \Omega_k$ and $B_0 \times \prod_{i=1}^{k} [\alpha_i, \Omega_i] \subseteqq V$. Our assertion about the existence of $\alpha_1, \cdots, \alpha_n$ follows.

Thus we see that each point of

$$A = [\{\Omega_0\} \times \prod_{i=1}^{n} [\alpha_i, \Omega_i] - \{(\Omega_0, \Omega_1, \cdots, \Omega_n)\}]$$

is a limit point of $B_0 \times \prod_{i=1}^{n} [\alpha_i, \Omega_i]$, so that $A \subseteqq \mathrm{Bd}\, V$. But $\mathrm{Ind}\, A = n - 1$, and it follows that any neighborhood V of B_0 such that $V \subseteqq U$ has at least $(n - 1)$ dimensional (Ind) set on its boundary. Thus $\mathrm{Ind}\, Y_n \geqq n$.

COROLLARY 4.2.1. *Let* $T = \prod_{\substack{i=0 \\ i \neq i_0}}^{n} [0, \Omega_i] - \{(\Omega_0, \cdots, \hat{\Omega}_{i_0}, \cdots, \Omega_n)\}$ *and let* $T' = \prod_{i=0}^{n} [\alpha_i, \Omega_i] - \{(\Omega_0, \cdots, \hat{\Omega}_{i_0}, \cdots, \Omega_n)\}$. *Then each of* T *and* T' *has large inductive dimension at least* $n - 1$.

Proof. A modification of the proof of Theorem 4.2 works.

THEOREM 4.3. $\mathrm{Ind}\, Y = \infty$.

Proof. $Y_n \subseteqq Y$ homeomorphically and we identify Y_n with its image in Y. Since $\mathrm{Ind}\, Y_n \geqq n$, and Y_n is closed in Y, $\mathrm{Ind}\, Y = \infty$.

THEOREM 4.4. *X is a totally disconnected compact Hausdorff space with* $\mathrm{Ind}\, X = 0$, *but X contains subsets* Y_n, $1 \leqq n \leqq \infty$, *with* $\mathrm{Ind}\, Y_n \geqq n$; $\mathrm{Ind}\, Y_1 = 1$, *and* $\mathrm{Ind}\, Y_\infty = \infty$.

Proof. Follows from Theorems 4.2 and 4.3.

QUESTION. Is $\mathrm{Ind}\, Y_n = n$ for $1 < n < \infty$?

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Received February 4, 1971 and in revised form August 17, 1971. Presented to the American Mathematical Society meeting in Athens, Georgia, on Nov. 21, 1970. The author would like to express her appreciation to the referee for his kind comments and criticisms.

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