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# ON THE ABSOLUTE MATRIX SUMMABILITY OF A FOURIER SERIES

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**In this paper, the author gives sufficient conditions for a Fourier series at an arbitrary but fixed point to be absolutely matrix summable.**

**1. Introduction.** Let $\sum_0^\infty u_n$ be an infinite series with partial sums $s_n$, and let $A = (a_{nk})$ be a triangular infinite matrix of real numbers (see Hardy [2]). The series $\sum u_n$ is said to be absolutely summable $A$, or summable $|A|$, if

$$\sum_1^\infty |\tau_n - \tau_{n-1}| < \infty ,$$

where

$$\tau_n = \sum_{k=0}^n a_{nk} s_k .$$

Let $f(t)$ be a Lebesgue-integrable function of period $2\pi$, with Fourier series

$$\frac{1}{2}a_0 + \sum_1^\infty (a_n \cos nt + b_n \sin nt) \equiv \sum_0^\infty A_n(t) . \tag{1.1}$$

With a fixed point $x$, we set

$$\phi(t) = \phi_x(t) = \frac{1}{2}[f(x+t) + f(x-t)] , \tag{1.2}$$

$$\Phi(t) = \int_0^t |\phi(u)| du . \tag{1.3}$$

We establish the following theorem for the absolute matrix summability of the Fourier series (1.1) of $f(t)$ at $t = x$.

THEOREM. *Let $A = (a_{nk})$ be a triangular infinite matrix of real numbers such that $\Delta a_{nk} = a_{nk} - a_{n,k+1}$ is monotonic with respect to $n \geqq k$ for each fixed $k \geqq 0$.*

*Let $\alpha(t)$ be a positive function such that $t^r/\alpha(t)$, for some $r$ with $0 < r < 1$, is nondecreasing for $t \geqq t_0$. Suppose that*

$$\sum_{n=1}^\infty \frac{n|a_{nn}|}{\alpha(n)} < \infty , \tag{1.4}$$

$$|\Delta a_{m,0}| + \sum_{n=1}^{m-1} \frac{n|\Delta a_{mn}|}{\alpha(n)} = O(1) \qquad as\ m \to \infty . \tag{1.5}$$

*Further, let*

$$\Phi(t) = O\left[\frac{t}{\alpha(1/t)}\right] \qquad as\ t \to 0+ . \tag{1.6}$$

*If all of the above conditions hold, then the Fourier series* (1.1) *of* $f(t)$ *at* $t = x$ *is summable* $|A|$.

We shall require the following lemmas.

LEMMA 1. *If* $\alpha(t)$ *is defined as in the theorem, then*

$$\int_{t_0}^{t} \frac{du}{\alpha(u)} = O\left[\frac{t}{\alpha(t)}\right] \qquad for\ all\ t \geqq t_0 . \tag{2.1}$$

*Proof.*

$$\int_{t_0}^{t} \frac{du}{\alpha(u)} = \int_{t_0}^{t} \frac{u^r}{\alpha(u)} \cdot \frac{du}{u^r}$$
$$\leqq \frac{t^r}{\alpha(t)} \int_{t_0}^{t} \frac{du}{u^r} \leqq \frac{t^r}{\alpha(t)} \cdot \frac{t^{-r+1}}{1-r} = O\left[\frac{t}{\alpha(t)}\right] .$$

LEMMA 2. *If* $A = (a_{nk})$ *is defined as in the theorem and if*

$$\sum_{n=0}^{\infty} |t_n| \cdot |a_{nn}| < \infty , \tag{2.2}$$

$$\sum_{n=0}^{m-1} |t_n| \cdot |\Delta a_{mn}| = O(1) \qquad as\ m \to \infty , \tag{2.3}$$

*where*

$$t_n = \sum_{k=0}^{n} s_k ,$$

*then* $\sum u_n$ *is summable* $|A|$.

*Proof.* By Abel's transformation,

$$\tau_n - \tau_{n-1} = \sum_{k=0}^{n} (a_{nk} - a_{n-1,k}) s_k$$
$$= \sum_{k=0}^{n-1} (\Delta a_{nk} - \Delta a_{n-1,k}) t_k + a_{nn} t_n .$$

Now

$$\sum_{n=1}^{m} \sum_{k=0}^{n-1} |\Delta a_{nk} - \Delta a_{n-1,k}| \cdot |t_k|$$
$$= \sum_{k=0}^{m-1} |t_k| \cdot \left( \sum_{n=k+1}^{m} |\Delta a_{nk} - \Delta a_{n-1,k}| \right) = \sum_{k=0}^{m-1} |t_k| \cdot |\Delta a_{mk} - a_{kk}| .$$

Thus,

$$\sum_{n=1}^{m} |\tau_n - \tau_{n-1}| \leqq \sum_{n=0}^{m-1} |t_n| \cdot |\Delta a_{mn}| + 2 \sum_{n=0}^{m} |t_n| \cdot |a_{nn}| = O(1)$$

as $m \to \infty$, by (2.2) and (2.3).

This completes the proof of the lemma.

3. *Proof of the Theorem.* We write

$$s_n(x) = \sum_{0}^{n} A_k(x),\ t_n(x) = \sum_{0}^{n} s_k(x) .$$

By (1.6), there exists $\delta (0 < \delta < 1)$ such that

$$\Phi(t) \leqq K \frac{t}{\alpha(1/t)} \qquad \text{for } 0 < t \leqq \delta , \tag{3.1}$$

where $K$ is a positive constant (not necessarily the same at each occurrence). Now, for $n > \delta^{-1}$,

$$\begin{aligned} \pi t_n(x) &= \int_0^{\pi} \phi(t) \left[ \frac{\sin (n+1)(t/2)}{\sin (t/2)} \right]^2 dt \\ &= \int_0^{n^{-1}} + \int_{n^{-1}}^{\delta} + \int_{\delta}^{\pi} = I_1 + I_2 + I_3 , \text{ say.} \end{aligned} \tag{3.2}$$

We observe that

$$\left[ \frac{\sin (n+1) \cdot (t/2)}{\sin (t/2)} \right]^2 = \begin{cases} O(n^2) & \text{for } \sin t/2 \neq 0 \text{ and } n \geqq 1 , \\ O(1/t^2) & \text{for } 0 < t \leqq \pi . \end{cases} \tag{3.3}$$

So, by (3.1),

$$|I_1| \leqq K n^2 \int_0^{n^{-1}} |\phi(t)|\, dt \leqq K \frac{n}{\alpha(n)} . \tag{3.4}$$

Further, assuming $t^r/\alpha(t)$ nondecreasing for $t \geqq \delta^{-1}$,

$$\begin{aligned} |I_2| &\leqq K \int_{n^{-1}}^{\delta} \frac{|\phi(t)|}{t^2} dt \\ &= K \left\{ \left[ \frac{\Phi(t)}{t^2} \right]_{n^{-1}}^{\delta} + 2 \int_{n^{-1}}^{\delta} \frac{\Phi(t)}{t^3} dt \right\} \\ &\leqq K \left[ \frac{\Phi(\delta)}{\delta^2} + \int_{n^{-1}}^{\delta} \frac{dt}{t^2 \alpha(1/t)} \right] \\ &= K \left[ \frac{\Phi(\delta)}{\delta^2} + \int_{\delta^{-1}}^{n} \frac{du}{\alpha(u)} \right] \\ &\leqq K \frac{n}{\alpha(n)} \qquad \text{as } n \to \infty , \text{ by (2.1).} \end{aligned} \tag{3.5}$$

Obviously,

$$I_3 = O(1) . \tag{3.6}$$

From (3.2), (3.4)-(3.6), it follows that

$$t_n(x) = O\left[\frac{n}{\alpha(n)}\right] \qquad \text{as } n \to \infty . \tag{3.7}$$

Hence

$$\sum_n^\infty |t_k(x)| \cdot |a_{kk}| = O\left[\sum_n^\infty \frac{k}{\alpha(k)} | a_{kk} |\right] = o(1) \tag{3.8}$$

$$\text{as } n \to \infty , \text{ by (1.4).}$$

Moreover,

$$\begin{aligned}\sum_0^{m-1} |t_n(x)| \cdot |\Delta a_{mn}| &= |t_0(x)| \cdot |\Delta a_{m\,0}| + O\left[\sum_1^{m-1} \frac{n}{\alpha(n)} \cdot |\Delta a_{mn}|\right] \\ &= O(1) \qquad \text{as } m \to \infty , \text{ by (1.5).}\end{aligned} \tag{3.9}$$

Now the theorem follows from Lemma 2.

4. NOTE. Let $A = (a_{nk})$ be a triangular infinite matrix of real numbers such that $a_{nn} \geqq 0$ for all $n \geqq 0$ and $\Delta a_{nk}$ is nondecreasing with respect to $n \geqq k$ for each fixed $k \geqq 0$. Let $\alpha(t)$ be defined as in the theorem, and let

$$\Delta a_{m,0} + \sum_{n=1}^{m} \frac{n(\Delta a_{mn})}{\alpha(n)} = O(1) \qquad \text{as } m \to \infty . \tag{4.1}$$

Then, if the condition (1.6) holds, the Fourier series (1.1) of $f(t)$ at $t = x$ is summable $|A|$.

*Proof.* Let

$$\tau_n(x) = \sum_{k=0}^{n} a_{nk} s_k(x) .$$

Then

$$\begin{aligned}&\sum_{n=1}^{m} |\tau_n(x) - \tau_{n-1}(x)| \\ &\leqq \sum_{n=1}^{m} \sum_{k=0}^{n} |\Delta a_{nk} - \Delta a_{n-1\,k}| \cdot |t_k(x)| \\ &= \sum_{k=1}^{m} |t_k(x)| \left(\sum_{n=k}^{m} |\Delta a_{nk} - \Delta a_{n-1,k}|\right) + |t_0(x)| \sum_{n=1}^{m} |\Delta a_{n\,0} - \Delta a_{n-1\,0}| \\ &= \sum_{k=1}^{m} |t_k(x)| (\Delta a_{mk}) + |t_0(x)| (\Delta a_{m\,0} - a_{0\,0})\end{aligned} \tag{4.2}$$

$$\leqq |t_0(x)|(\Delta a_{m,0}) + O\left[\sum_{k=1}^{m} \frac{k}{\alpha(k)} (\Delta a_{mk})\right], \qquad \text{by (3.7)}$$

$$= 0(1) \qquad \text{as } m \to \infty \text{ , by (4.1).}$$

So the required result follows.

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