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A GENERALIZATION OF THE PRIME RADICAL IN NONASSOCIATIVE RINGS

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In [5] Tsai defined the Brown-McCoy prime radical for Jordan rings in terms of the quadratic operation and proved basic results for the radical. In this paper we give a definition of the prime radical for arbitrary nonassociative rings in terms of a $*$-operation defined on the family of ideals and of a function f of the ring into the family of ideals in the ring. The prime radical for Jordan or standard rings is obtained by a particular choice of the $*$-operation and the function f. We also extend the results for the Jordan case to weakly W-admissible rings which include the generalized standard rings and therefore alternative and standard rings as well as Jordan rings.

1. Let K be any nonassociative ring and let $\mathscr{I}(K)$ denote the family of ideals of K.

DEFINITION 1. We define a $*$-operation as a mapping of $\mathscr{I}(K) \times \mathscr{I}(K)$ into the family of additive subgroups of K such that

($*1$) for A, B, C, and D in $\mathscr{I}(K)$ if $A \subseteqq C$ and $B \subseteqq D$, then $A*B \subseteqq C*D$,

($*2$) $(0)*A = B*(0) = (0)$ for all A, B in $\mathscr{I}(K)$,

($*3$) $\overline{A*B} = \bar{A}*\bar{B}$ for any homomorphic images $\bar{A}$ and $\bar{B}$ of A and B in $\mathscr{I}(K)$.

If K is a Jordan ring, let $U_x \equiv 2R_x^2 - R_{x^2}$ be the quadratic operation and AU_B be the additive subgroup of K generated by xU_y, $x \in A$ and $y \in B$. Then the U-operation satisfies the conditions above. If the characteristic is not 2, it is shown in [5] that $AU_A = AA^2$ and is an ideal of K for A in $\mathscr{I}(K)$.

For any ring K and A, B in $\mathscr{I}(K)$, if we define $A*B$ as the additive subgroup $AB^2 + B^2A + (AB)B + (BA)B$, then $A*B$ also satisfies the conditions in Definition 1. In case K is a standard ring, it is shown in [6] that $A*B$ is an ideal of K for A, B in $\mathscr{I}(K)$. If K is commutative or anticommutative, then $A*B = AB^2 + (AB)B$. In particular, if K is a Lie ring, $A*B$ is an ideal of K. Since A^2 is not in general an ideal of K for A in $\mathscr{I}(K)$, but there are considerably broad classes of nonassociative rings in which $A^3 \equiv AA^2 + A^2A$ is an ideal of K for every ideal A, this example will be particularly interesting.

We recall that a noncommutative Jordan ring K is one satisfying the flexible law $(x, y, x) = 0$ and the Jordan identity $(x, y, x^2) = 0$ for all x, y in K, where $(x, y, z) = (xy)z - x(yz)$. Most of the well known nonassociative rings are included in the class of noncommutative Jordan rings. Recently Thedy [4] defined a considerably broad class of algebras that generalizes many of the well known algebras.

DEFINITION 2. A noncommutative Jordan ring K is called weakly W-admissible if it satisfies

$$[(a, b, c), c] - ([a, c], c, b) = 0 ,$$

and

$$([a, b], d, c] + ([b, c], d, a) + ([c, a], d, b] = p[(a, b, c), d] + q[S(a, b, c), d] + r[d, [b, [a, c]]]$$

for some integers p, q, r such that either $m(p, q, r) \equiv 3 + 2p + 6q - 4r \neq 0$, or $n(p, r) \equiv p + 4r \neq 0$, where $[a, b] = ab - ba$ and $S(a, b, c) = (a, b, c) + (b, c, a) + (c, a, b)$.

Thedy called a noncommutative Jordan algebra over a field W-admissible if it satisfies the identity $[a, (a, a, b)] = 0$ and the two identities above for p, q, r in the field such that either $m(p, q, r) \neq 0$ or $n(p, r) \neq 0$. He proved that if the characteristic is not 2, then any generalized standard ring of Schafer [2] is W-admissible with $p = -2$ and $q = r = 0$. Therefore, weakly W-admissible rings include generalized standard rings and hence alternative and standard rings as well as Jordan rings. In case the characteristic is not 2, it is also shown in [4, p. 192] that in any weakly W-admissible ring K, A^3 is an ideal of K for A in $\mathscr{I}(K)$.

LEMMA 1.1. *Let K be any ring. Then the conditions* $(*2)$ *and* $(*3)$ *imply*

(i) $(A + C)*(B + C) \subseteqq A*B + C$, *and*

(ii) $A*B \subseteqq A \cap B$

for ideals A, B, C of K.

Proof. Consider the quotient ring $\bar{K} = K/C$, then by $(*3)$ $\overline{(A + C)*(B + C)} = \bar{A}*\bar{B} = \overline{A*B}$, and hence (i). Let $\bar{K} = K/A$, then $\overline{A*B} = \bar{A}*\bar{B} = (\bar{0})*\bar{B} = (\bar{0})$ by $(*2)$ and so $A*B \subseteqq A$. Similarly $A*B \subseteqq B$ and $A*B \subseteqq A \cap B$.

DEFINITION 3. Let K be any ring. Then f is defined as a function of K into $\mathscr{I}(K)$ such that for every a in K

(f 1) $a \in f(a)$,

(f 2) if $x \in f(a)$, then $f(x) \subseteqq f(a)$, (f 3) $\overline{f(a)} = f(\bar{a})$, where $\bar{a}$ is a homomorphic image of a.

The principal ideal (a) generated by a in K is an example of $f(a)$. Now let S be a subset of K and define $f(a)$ to be the ideal (a, S) generated by a and S. Then f satisfies the conditions above. A similar function to f has been defined in [1] for the associative case and in [3].

Henceforth we assume that f denotes a function of K into $\mathscr{I}(K)$ satisfying (f 1), (f 2), and (f 3). Then clearly $(a) \subseteqq f(a)$. For an ideal A of K, we denote the ideal $\sum_{a \in A} f(a)$ by $f(A)$. Then $A \subseteqq f(A)$ and $f(A) \subseteqq f(B)$ if $A \subseteqq B$, and also $f((a)) = f(a)$. But in general $f(A) \neq A$ as shown by the example $f(a) = (a, S)$ for a subset S of K. Let $\mathscr{I}'(K)$ denote the family of ideals $f(A)$ for A in $\mathscr{I}(K)$. Then $\mathscr{I}'(K) \subseteqq \mathscr{I}(K)$ and in particular, if f is such that $f(a) = (a)$ for all a in K, then $f(A) = A$ and $\mathscr{I}'(K) = \mathscr{I}(K)$.

2. In this section we give a definition of the prime radical for any ring in terms of the $*$-operation and the function f.

LEMMA 2.1. *Let K be any ring where the $*$-operation and the function f are defined. For an ideal P of K, the following are equivalent*:

(i) *If $f(A)*f(B) \subseteqq P$ for A, B in $\mathscr{I}(K)$, then either $f(A) \subseteqq P$ or $f(B) \subseteqq P$.*

(ii) *If we have $f(A) \cap c(P) \neq \emptyset$ and $f(B) \cap c(P) \neq \emptyset$, then $f(A)*f(B) \cap c(P) \neq \emptyset$.*

(iii) *If a and b are in $c(P)$, then $f(a)*f(b) \cap c(P) \neq \emptyset$.*

Proof. We need only to show that (ii) and (iii) are equivalent. Let a and b be in $c(P)$, then $f(a) \cap c(P) \neq \emptyset$ and $f(b) \cap c(P) \neq \emptyset$. Hence (ii) implies (iii). Now let A and B be ideals of K with $f(A) \cap c(P) \neq \emptyset$ and $f(B) \cap c(P) \neq \emptyset$. Let $a \in f(A) \cap c(P)$ and $b \in f(B) \cap c(P)$. Assuming (iii), we get $f(a)*f(b) \cap c(P) \neq \emptyset$ and by ($*$1) $f(A)*f(B) \cap (P) \neq \emptyset$, thus (ii) holds.

DEFINITION 4. (i) An ideal P of K is called f^*-prime if it satisfies any one of Lemma 2.1. A nonempty subset M of K is called an f^*-system if, for A, B in $\mathscr{I}(K)$, $f(A) \cap M \neq \emptyset$ and $f(B) \cap M \neq \emptyset$ imply $f(A)*f(B) \cap M \neq \emptyset$.

(ii) An ideal P of K is called f^*-semiprime if, for any ideal A of K, $f(A)*f(A) \subseteqq P$ implies $f(A) \subseteqq P$. A nonempty subset M of K is called an sf^*-system if, for A in $\mathscr{I}(K)$, $f(A) \cap M \neq \emptyset$ implies $f(A)*f(A) \cap M \neq \emptyset$.

An ideal P is f^*-prime if and only if $c(P)$ is an f^*-system. Similarly, an ideal P is f^*-semiprime if and only if $c(P)$ is an sf^*-

system. Let K be a Jordan or standard ring. If we define $A*B$ as AU_B or as $AB^2 + B^2A + (AB)B + (BA)B$ and define $f(a)$ as (a) for every a in K, then the defininition of f^*-prime and f^*-semiprime ideals coincide with those in [5] or in [6].

DEFINITION 5. For A in $\mathscr{I}(K)$, $A^* = \{x \in K \mid$ any f^*-system containing x meets $A\}$ is called the f^*-radical of A. Similarly, $A_* = \{y \in K \mid$ any sf^*-system containing y meets $A\}$ is called the sf^*-radical of A.

THEOREM 2.2. *Let A be an ideal of K. Then*

(i) *A^* is the intersection of all the f^*-prime ideals P_i containing A.*

(ii) *A_* is the intersection of all f^*-semiprime ideals containing A.*

(iii) *A_* is an f^*-semiprime ideal of K.*

(iv) *A is f^*-semiprime if and only if $A = A_*$.*

Proof. The proofs are essentially the same as in [5]. But to emphasize use of the $*$-operation and the function f we prove only (i). Let $\bigcap_i P_i$ be the intersection of all the f^*-prime ideals P_i of K containing A. If $a \notin P_i$ for some i, then $a \in c(P_i)$, being an f^*-system, and $c(P_i) \cap A = \emptyset$. Hence $a \notin A^*$ and $A^* \subseteqq \bigcap_i P_i$. Conversely, if $a \notin A^*$, then there exists an f^*-system M with $a \in M$ but $A \cap M = \emptyset$. By Zorn's lemma we find a maximal ideal P such that $P \supseteqq A$ but $P \cap M = \emptyset$. Let B, C be ideals of K such that $f(B) \cap c(P) \neq \emptyset$ and $f(C) \cap c(P) \neq \emptyset$. By the maximality of P, $(f(B) + P) \cap M \neq \emptyset$ and $(f(C) + P) \cap M \neq \emptyset$. Since M is an f^*-system, $\emptyset \neq (f(B) + P)*(f(C) + P) \cap M \subseteqq (f(B)*f(C) + P) \cap M$ by Lemma 1.1 (i), thus $f(B)*f(C) \cap c(P) \neq \emptyset$. Hence P is f^*-prime and $a \notin P$.

LEMMA 2.3. *Let a be an element of K and S be an sf^*-system containing a. Then there exists an f^*-system M such that $a \in M$ and $M \subseteqq S$.*

Proof. Let $a_1 = a$, then $a_1 \in f(a_1) \cap S$ and so $f(a_1)*f(a_1) \cap S \neq \emptyset$. Hence we obtain a set $M = \{a_1, a_2, \cdots, a_n, \cdots\}$ such that $a_{k+1} \in f(a_k) \cap S$ and $M \subseteqq S$. By Lemma 1.1 (ii) we note that $a_{k+1} \in f(a_k)*f(a_k) \subseteqq f(a_k)$ and so $f(a_{k+1}) \subseteqq f(a_k)$. Let $p = \max(i, j)$, then $a_{p+1} \in f(a_p)^*f(a_p) \cap S \subseteqq f(a_i)*f(a_j) \cap S$. Hence $f(a_i)*f(a_j) \cap M \neq \emptyset$ and M is an f^*-system.

Therefore, as in [5], we have

THEOREM 2.4. *For any ideal A of K, $A^* = A_*$. A^* is called the f^*-prime radical of A.*

DEFINITION 6. The f^*-prime radical, $R^*(K)$, of K is the f^*-prime radical of the ideal (0). A ring K is said to be f^*-semisimple if $R^*(K) = (0)$.

LEMMA 2.5. *Let $\bar{K}$ be a homomorphic image of K. If M is an f^*-system of K, then so is $\bar{M}$ in $\bar{K}$.*

Proof. Let $\bar{A}, \bar{B}$ be ideals of $\bar{K}$ such that $f(\bar{A}) \cap \bar{M} \neq \emptyset$ and $f(\bar{B}) \cap \bar{M} \neq \emptyset$, where A and B are ideals in K containing the kernel. Recalling (f 3) and $A \subseteqq f(A)$, these imply $f(A) \cap M \neq \emptyset$ and $f(B) \cap M \neq \emptyset$. Since M is an f^*-system, by ($*$3) and (f 3) we see that $f(\bar{A}) * f(\bar{B}) \cap \bar{M} \neq \emptyset$.

Therefore, by Lemma 2.3 we easily see that any homomorphic image of an f^*-prime ideal containing the kernel is also f^*-prime. Hence we obtain

THEOREM 2.6. *Let K be a ring and $R^*(K)$ be the f^*-prime radical of K, then $R^*(K/R^*(K)) = (0)$, that is, $K/R^*(K)$ is f^*-semisimple.*

DEFINITION 7. A ring K is called an f^*-prime ring if (0) is an f^*-prime ideal in K.

Clearly, an f^*-prime ring is f^*-semisimple. Since any homomorphic image of an f^*-prime ideal is f^*-prime, if P is an f^*-prime ideal in K then K/P is an f^*-prime ring. Let $\bar{K} = K/P$ be an f^*A prime ring and let $f(A) * f(B) \subseteqq P$, then $f(\bar{A}) * f(\bar{B}) \subseteqq (\bar{0})$ and so $f(A) \subseteqq P$ or $f(B) \subseteqq P$, thus P is f^*-prime in K. Hence P is an f^*-prime ideal of K if and only if K/P is an f^*-prime ring. Therefore, as for Jordan rings, we obtain

THEOREM 2.7. *A ring K is isomorphic to a subdirect sum of f^*-prime rings if and only if K is f^*-semisimple.*

3. Throughout this section we assume that the $*$-operation satisfies the following additional condition:

($*$4) $A * A = A^3$ and $A * A$ is an ideal of K for A in $\mathscr{I}(K)$.

We recall that if K is a weakly W-admissible or Lie ring then $A * B = AB^2 + B^2A + (AB)B + (BA)B$ satisfies ($*$4).

THEOREM 3.1. *Let A be an ideal of a ring K and $r \in A_*$. Then a power of r belongs to A. Furthermore if K is power-associative, then the f^*-radical $R^*(K)$ is a nil ideal in K.*

Proof. Let M be the multiplicatively closed system generated by r in K. Then it follows from ($*4$) that M is an sf^*-system containing r. Hence $M \cap A \neq \emptyset$. If K is power-associative and $r \in R^*(K)$, then $r^k \in (0)$ for some k and so $R^*(K)$ is nil.

Therefore, the f^*-radical $R^*(K)$ is contained in the nil radical $N(K)$ (the maximal nil ideal in K).

Let $\mathscr{I}'(K)$ denote the set of ideals $f(A)$ for A in $\mathscr{I}(K)$. Then $\mathscr{I}'(K) \subseteq \mathscr{I}(K)$.

THEOREM 3.2. *A ring K is f^*-semisimple if and only if $\mathscr{I}'(K)$ contains no nonzero nilpotent ideal.*

Proof. It follows from Theorem 2.2 (iv) that K is f^*-semisimple if and only if the ideal (0) is f^*-semiprime. If $f(A)$ is a nonzero nilpotent ideal for A in $\mathscr{I}(K)$, there exist positive integers $u = 3^t$ and $v = 3^{t-1}$ such that $f(A)^u = (0)$ but $f(A)^v \neq (0)$. But then since $f(A)^v * f(A)^v \subseteq f(A)^{3v} = f(A)^u = (0)$, (0) is not f^*-semiprime. Conversely, if (0) is not f^*-semiprime, then there exists an ideal $f(A) \neq (0)$ such that $f(A) * f(A) = f(A)^3 = (0)$, thus $f(A)$ is nilpotent.

COROLLARY 3.3. *The f^*-radical $R^*(K)$ contains all the nilpotent ideals in $\mathscr{I}'(K)$.*

Proof. Let $f(A)$ be a nilpotent ideal in $\mathscr{I}'(K)$ and $\bar{K} = K/R^*(K)$, then $\overline{f(A)} = f(\bar{A}) \in \mathscr{I}'(\bar{K})$, and $f(\bar{A})$ is nilpotent in $\bar{K}$. Since $\bar{K}$ is f^*-semisimple, by Theorem 3.2 $f(\bar{A}) = (\bar{0})$, thus $f(A) \subseteq R^*(K)$.

THEOREM 3.4. *If K is a ring and $\mathscr{I}'(K)$ contains a maximal nilpotent ideal $S'(K)$, then $R^*(K) = S'(K)$.*

Proof. By Corollary 3.3, $S'(K) \subseteq R^*(K)$. Let $\bar{K} = K/S'(K)$, then $\mathscr{I}'(\bar{K})$ contains no nonzero nilpotent ideal and by Theorem 3.2 $R^*(\bar{K}) = (0)$. If $r \notin S'(K)$, then $\bar{r} \neq \bar{0}$ and so there exists an f^*-prime ideal $\bar{P}$ in $\bar{K}$ with $\bar{r} \notin \bar{P}$. From ($*3$) and (f 3) it follows that the inverse image P of $\bar{P}$ is an f^*-prime ideal in K. But since $\bar{r} \notin \bar{P}$, $r \notin P$ and so $r \notin R^*(K)$, thus $R^*(K) \subseteq S'(K)$.

Now suppose that $f(a) = (a)$ for every element a in K. Then $\mathscr{I}(K) = \mathscr{I}'(K)$. Hence by Theorem 3.2 K is f^*-semisimple if and only if K has no nonzero nilpotent ideal, and $R^*(K)$ contains all nilpotent ideals of K. In this case the ideal $S'(K)$ is a maximal nilpotent ideal $S(K)$ in K and by Theorem 3.4 $R^*(K) = S(K)$.

Let K now be a finite dimensional W-admissible or Lie algebra over a field. Let $f(a) = (a)$ for all a in K. If K is W-admissible, then it is shown in [4] that the nil radical $N(K)$ is nilpotent and so the unique maximal nilpotent ideal $S(K)$. Hence by Theorem 3.4 $R^*(K) = N(K) = S(K)$. If K is a Lie algebra, it is well known that K has a maximal nilpotent ideal $S(K)$ and hence $R^*(K) = S(K)$.

REFERENCES

1. K. Murata, Y. Kurata and H. Marubayashi, *A generalization of prime ideals in rings*, Osaka J. Math., **6** (1969), 291-301.

2. R. D. Schafer, *Generalized standard algebras*, J. of Algebra, **12** (1969), 386-417.

3. M. F. Smiley, *Application of a radical of Brown and McCoy to non-associative rings*, Amer. J. Math., **72** (1950), 93-100.

4. A. Thedy, *Zum Wedderburnschen Zerlegungssatz*, Math. Z., **113** (1970), 173-195.

5. C. Tsai, *The prime radical in a Jordan ring*, Proc. Amer. Math. Soc., **19** (1968), 1171-1175.

6. L. J. Zettel, *Radicals in standard rings*, Ph. D. Thesis, Michigan State University, 1970.

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