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A REMARK ON TONELLI'S THEOREM ON INTEGRATION IN PRODUCT SPACES

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This paper is concerned to show a connection between the validity of Tonelli's theorem on integration in the product of two measure spaces and the semifiniteness of the product measure. The classical Tonelli theorem is usually stated in a sigma-finite setting. It is shown in this paper, among other things, that in a product measure space, where one of the measures is sigma-finite and other one semifinite (not necessarily sigma-finite), Tonelli's theorem is valid only if the product measure is semifinite and on the other hand, if the product of any two measures is semifinite, then Tonelli's theorem is valid.

1. Let (X, U, β_1) and (Y, V, β_2) be any two arbitrary measure spaces where U and V are sigma-algebras of subsets of X and Y, respectively, and β_1 and β_2 are two nonnegative measures on U and V respectively. Let $U \times V$ be the smallest sigma-algebra containing all the measurable rectangles of $X \times Y$. The product measure $\beta_1 \times \beta_2$ (we call it β^*, for simplicity) is the restriction to $U \times V$ of the outer measure induced by the measure β on the algebra W consisting of the measurable rectangles of $X \times Y$ and their finite disjoint unions where for every measurable rectangle $P \times Q$, $\beta(P \times Q) = \beta_1(P)\beta_2(Q)$. (See [4], p. 254). β_1 is called semifinite if given A in U with $\beta_1(A) = \infty$, we can find B in U, $B \subset A$ and $0 < \beta(B) < \infty$. This definition, which at first glance seems to be less restricted than semifiniteness as defined in [4], p. 220, is actually equivalent to Royden's definition, as Lemma 1 in the next section shows. Every sigma-finite measure is semifinite, but not conversely. (For example, consider any non-sigma-finite regular Borel measure on a locally compact space or a counting measure on an uncountable set). The product measure $\beta_1 \times \beta_2$ may not be semifinite even when β_1 is sigma-finite and β_2 semifinite, as Example 1 shows. For the purpose of reference, let us state the following two well-known Theorems in a form, which is slightly different from that given in [2] or [4].

FUBINI'S THEOREM. *Let $f(x, y)$ be β^*-integrable on $U \times V$. Then both the iterated integrals of f are well-defined and*

$$\int f d\beta^* = \iint f d\beta_1 d\beta_2 = \iint f d\beta_2 d\beta_1 .$$

The proof of this theorem follows easily from Theorem C, p. 147 in [2], if we observe the following points. First, since f is β^*-integrable on $U \times V$, the set

$$E = \{(x, y) \in X \times Y: f(x, y) \text{ is nonzero}\}$$

is sigma-finite with respect to β^*. Second, it follows from Proposition 6 on p. 256 in [4] that we can find a measurable rectangle $P \times Q$ such that $E \subset P \times Q$ where P and Q are sigma-finite with respect to β_1 and β_2 respectively.

TONELLI'S THEOREM. *Let β_1 and β_2 be both sigma-finite. Let $f(x, y)$ be a nonnegative $U \times V$ measurable function. If one of the iterated integrals of f is well-defined and finite, then the other one is also and*

$$\int f d\beta^* = \int\int f d\beta_1 d\beta_2 = \int\int f d\beta_2 d\beta_1 .$$

This Theorem is precisely Theorem B in [2], p. 147.

The main purpose of our note is to present the following version of Tonelli's Theorem, which certainly is more informative than the classical version and tells us more about product integration of nonnegative measurable functions. The proof of this Theorem is given in the next section.

THEOREM 1. *Suppose one of β_1 and β_2 is sigma-finite and the other one is semifinite. Then Tonelli's Theorem is valid if and only if β^* is semifinite. The sigma-finiteness assumption can be replaced by semifiniteness if we assume one of the following two conditions:*

(a) *for all $U \times V$ measurable sets A, the function $\beta_1(A^y)$ is measurable, where $A^y = \{x: (x, y) \in A\}$;*

(b) *for all $U \times V$ measurable sets A, the function $\beta_1(A_x)$ is measurable, where $A_x = \{y: (x, y) \in A\}$.*

Berberian in [1] defines product measure in a different way. In §3, we show that for his product measure, Fubini's theorem does not hold for non-sigma-finite β_1 and β_2 and only a part of Tonelli's theorem holds. This answers, at least partly, a question of Berberian [2, Problem 4, p. 144].

Finally, we take this opportunity to record our thanks to the referee for his useful comments and also for pointing out that Example 1, Lemma 2, with a slightly different proof, and Proposition 1 (only its first sentence) appeared in [3].

2. First, we consider an Example showing that β^* need not be

semifinite even when β_1 is sigma-finite and β_2 is semifinite.

EXAMPLE 1. Let $X = Y = [0, \infty)$ and $U = V =$ the Lebesgue measurable subsets of $[0, \infty)$. Let β_1 be the Lebesgue measure and β_2 be the counting measure. Let D be the diagonal of $X \times Y$. Then $\iint I_D(x, y)d\beta_1 d\beta_2 = 0$ and $\iint I_D(x, y)d\beta_2 d\beta_1 = \infty$, where I_D is the characteristic function of D. Hence $\beta^*(D)$ is ∞, since otherwise, by Fubini's Theorem, the iterated integrals will be equal. If $D_1 \subset D$, $D_1 \in U \times V$ and $\beta^*(D_1) < \infty$, then by Fubini's Theorem,

$$\beta^*(D_1) = \iint I_{D_1}(x, y)d\beta_1 d\beta_2 = 0 .$$

Hence β^* is not semifinite.

The proof of Theorem 1 will follow from the following four Lemmas.

LEMMA 1. *Let β_1 be semifinite. Then given $B \in U$ with $\beta_1(B) = \infty$ and any positive integer n, there exists $C \in U, C \subset B$ and $n < \beta_1(C) < \infty$.*

Proof. Let T be the family of all collections Q of β_1 measurable subsets of B with finite positive β_1-measure such that any two distinct sets in Q are disjoint. Then clearly T is nonempty, since β_1 is semifinite. We partially order the collections Q in T by inclusion. Every linearly odered subset of T has a upper bound, namely the union of all the collections in this subset. Therefore, by Zorn's Lemma, this set T has a maximal element, say Q_0. If Q_0 is a countable collection, then we must have $\beta_1(\bigcup_{A \in Q_0} A) = \infty$; for, otherwise, we can find $D \subset B - \bigcup_{A \in Q_0} A$ with $0 < \beta_1(D) < \infty$ and then $Q_0 \cup \{D\}$ will contradict the maximality of Q_0. The Lemma then follows clearly when Q_0 is countable. Now let Q_0 be uncountable. Let us define $Q_{0n} = \{A \in Q_0: \beta_1(A) > 1/n\}$. Then for some positive integer m, Q_{0m} is uncountable. From this observation, the Lemma is clear again.

LEMMA 2. *Let β^* be semifinite. Then Tonelli's Theorem is valid.*

Proof. Suppose $f(x,y)$ is a nonnegative $U \times V$-measurable function such that $\iint f(x, y)d\beta_1 d\beta_2$ is well-defined and equal to some nonnegative-number $k < \infty$. We claim that the support of f is β^*-sigma-finite. If our claim is false, we can find a positive integer n such that if

$$A_n = \{(x, y): f(x, y) > 1/n\} ,$$

then $\beta^*(A_n) = \infty$. Since β^* is semifinite, we can find $B \subset A_n$, $B \in$

$U \times V$ and $2kn < \beta^*(B) < \infty$, by Lemma 1. Then by Fubini's Theorem,

$$2k < \beta^*(B)/n = \iint 1/n \,.\, I_B(x, y)d\beta_1 d\beta_2 \leqq \iint f d\beta_1 d\beta_2$$

which is a contradiction. Hence, the support of f is sigma-finite so that we can find a measurable rectangle $P \times Q$ containing the support of f such that P and Q are sigma-finite with respect to β_1 and β_2 respectively. (This can be done using Proposition 6 on p. 256 in [4]). Now the Lemma follows from the classical Tonelli Theorem.

LEMMA 3. *Let β_1 be sigma-finite and β_2 semifinite. If Tonelli's Theorem is valid, then β^* is semifinite.*

Proof. Let A be a $U \times V$ measurable set such that $\beta^*(A)$ is infinite. Since β_1 is sigma-finite, $\beta_1(A^y)$ is a measurable function of y. (This follows from the proof of the classical Tonelli theorem). Therefore, by hypothesis,

$$\iint I_A(x, y)d\beta_1 d\beta_2 = \int_S \beta_1(A^y)d\beta_2 = \infty \ ,$$

where $S = \{y\colon \beta_1(A^y) > 0\}$.

We separate the proof into two distinct cases.

Case 1. Suppose $\beta_2(S) < \infty$.

Subcase (i). Suppose $\beta_2(B) = 2p > 0$, where

$$B = \{y\colon \beta_1(A^y) = \infty\}.$$

Since β_1 is sigma-finite, there are sets C_i in U such that $C_i \subset C_{i+1}$, $X = \bigcup_{i=1} C_i$ and $\beta_1(C_i) < \infty$ for every positive integer i. Now $\beta_1(C_i \cap A^y)$ is a measurable function of y since by the sigma-finiteness of β_1, the function $\int I_A(x, y) I_{C_i}(x) d\beta_1$ is measurable. Since for any arbitrary positive number k,

$$B \subset \bigcup_{i=1}^{\infty} \{y\colon \beta_1(C_i \cap A^y) > k\} \ ,$$

there exist an i and D in V such that

$$p < \beta_2(D) < \infty \quad \text{and} \quad D \subset \{y\colon \beta_1(C_i \cap A^y) > k\} \ .$$

Let $E = A \cap (C_i \times D)$. Then by Fubini's theorem,

$$0 < kp < \int_D \beta_1(C_i \cap A^y)d\beta_2 = \beta^*(E) \leqq \beta_1(C_i)\beta_2(D) < \infty \ .$$

Subcase (ii). Suppose $\beta_2(B) = 0$, where B is as above. Now if $C_n = \{y: n < \beta_1(A^y) \leqq n + 1\}$, then $S = \bigcup_{n=0}^{\infty} C_n$. Therefore,

$$\sum_{n=0}^{\infty} n\beta_2(C_n) \leqq \int_S \beta_1(A^y)d\beta_2 = \infty = \sum_{n=0}^{\infty} (n+1)\beta_2(C_n) .$$

Since $\beta_2(S) < \infty$ and β_2 is semifinite, given an arbitrary positive number m, we can find D_n in V such that $D_n \subset C_n$ and $\beta_2(D_n) < \infty$ and for some i,

$$m < \sum_{n=0}^{i} n\beta_2(D_n) < \infty .$$

Let $E = A \cap (X \times \bigcup_{n=0}^{i} D_n)$. Now, we have

$$\int \beta_1(E^y)d\beta_2 = \int \beta_1(A^y)I_{\bigcup_{n=0}^{i} D_n}(y)d\beta_2 < \infty .$$

Hence, since Tonelli's Theorem is valid, $\beta^*(E) < \infty$. Since $\beta^*(E) > m$, semifiniteness of β^* follows.

Case 2. Suppose $\beta_2(S) = \infty$.

Subcase (i). Suppose $\beta_2(B) = 0$, where B is as in Case 1. By the semifiniteness of β_2, we can find $G \subset S$, $G \in V$ and $0 < \beta_2(G) < \infty$ such that $\int_G \beta_1(A^y)d\beta_2 > 0$. This is possible since for every y in S, $\beta_1(A^y) > 0$. Now if

$$G_n = \{y \in G: \beta_1(A^y) < n\} ,$$

then using the monotone convergence Theorem, we can find n such that $0 < \int_{G_n} \beta_1(A^y)d\beta_2 < \infty$. Let $E = A \cap (X \times G_n)$. Then since Tonelli's Theorem is valid, $\beta^*(E) < \infty$. Clearly,

$$\beta^*(E) = \iint I_E(x, y)d\beta_1 d\beta_2 = \int_{G_n} \beta_1(A^y)d\beta_2 > 0 .$$

Subcase (ii). Suppose $\beta_2(B) > 0$, where B is as before. Then since β_2 is semifinite, we can find $C \subset B$, $C \in V$ and

$$0 < \beta_2(C) < \infty \quad \text{and} \quad \int_C \beta_1(A^y)d\beta_2 = \infty .$$

Now, if we replace the set B in the proof of Case 1, Subcase (i), by the set C above, the proof of this Subcase follows.

LEMMA 4. *Let β_1 and β_2 be semifinite. Suppose for every $U \times V$ measurable set A, the function $\beta_1(A^y)$ is a measurable function of*

y. Then the validity of Tonelli's Theorem implies the semifiniteness of β^*.

Proof. Let $S = \{y: \beta_1(A^y) > 0\}$. We separate the proof into two distinct cases. (Here A is as in Lemma 3).

Case 1. Suppose $\beta_2(B) > 0$, where

$$B = \{y: \beta_1(A^y) = \infty\} .$$

Let $C \subset B$, $C \in V$ and $0 < \beta_2(C) < \infty$. Let $A_0 = A \cap (X \times C)$, and let the measure β_3 be defined by $\beta_3(M) = \beta_2(M \cap C)$ for every M in V. If $\beta_1 \times \beta_3(A_0) < \infty$, then by Fubini's Theorem,

$$\infty > \beta_1 \times \beta_3(A_0) = \int \beta_1(A_0^y) d\beta_3 = \int_C \beta_1(A^y) d\beta_3 = \infty$$

atnd this is a contradiction. Therefore, $\beta_1 \times \beta_3(A_0) = \infty$. Now we observe that $\int \beta_3(A_{0x}) d\beta_1 = \infty$, since, otherwise, we have

$$\iint I_A(x, y) I_C(y) d\beta_2 d\beta_1 < \infty$$

and therefore, by the validity of Tonelli's Theorem,

$$\iint I_A(x, y) I_C(y) d\beta_1 d\beta_2 < \infty$$

which means that $\int_C \beta_1(A^y) d\beta_2 < \infty$, which is a contradiction. If $F = \{x: \beta_3(A_{0x}) > 0\}$, then $\beta_1(F) > 0$. Since β_1 is semifinite, we can find $D \in U$, $D \subset F$ and $0 < \beta_1(D) < \infty$. Then

$$0 < \int_D \beta_3(A_{0x}) d\beta_1 < \infty ,$$

which means that

$$0 < \iint I_A(x, y) I_C(y) I_D(x) d\beta_2 d\beta_1 < \infty .$$

Hence if $E = A \cap (D \times C)$, then by the validity of Tonelli's Theorem, $0 < \beta^*(E) < \infty$.

Case 2. Suppose $\beta_2(B) = 0$, where B is as above. The proof in this case follows exactly as in the corresponding situation in Lemma 3.

3. In this section, we consider a question of Berberian. In [1, p. 129], the product measure $\beta_1 \times \beta_2$, where (X, U, β_1) and (Y, V, β_2)

are any two arbitrary measure spaces as defined in § 1, has been defined to be the unique measure π on $U \times V$ having the following two properties:

(i) for every finite measurable rectangle $P \times Q$,

$$\pi(P \times Q) = \beta_1(P)\beta_2(Q)$$

and (ii) for every $A \in U \times V$,

$$\pi(A) = \sup\{\pi(A \cap (P \times Q))\},$$

where the supremum is taken over all finite measurable rectangles $P \times Q$. (Here 'finiteness' means that each side of the rectangle has finite measure). When β_1 and β_2 are sigma-finite, π coincides with the usual product measure β^*. In fact, Proposition 2 below tells us more than this. In [1, p. 142-3], the proofs of Fubini's and Tonelli's Theorems are given in the sigma-finite case. On p. 144, in Problem 4, Berberian asks the following question:

"What part, if any, of the Fubini theory survives for the product of arbitrary (not necessarily sigma-finite) measures? Does it help to assume that the measures are semifinite?"

In Example 2 and Proposition 4, we answer this question, at least, partly.

Let β^* and β_* be the outer and inner measure (see [4], p. 254 and p. 274) induced by the measure β (defined in § 1). First, we state a few results (omitting their proofs which would be obvious to the serious reader) showing some connections between π, β^* and β_*.

PROPOSITION 1. *If $B \in U \times V$ and $\beta^*(B) < \infty$, then $\beta^*(B) = \pi(B)$. If β_1 and β_2 are semifinite, then $\beta_* = \pi$ on $U \times V$.*

PROPOSITION 2. *If β^* is semifinite, then $\beta^* = \pi$.*

PROPOSITION 3. *There exists a measure β' on $U \times V$, taking only the values 0 and ∞ such that $\beta^* = \pi + \beta'$.*

Now we consider Fubini's Theorem for the product measure π. The following Example gives a negative answer even when β_1 and β_2 are semifinite.

EXAMPLE 2. Consider β_1, β_2, U, V and D as in Example 1 in § 2. We note that if $\beta^*(P \times Q) < \infty$, then Q is a finite set if $\beta_1(P) > 0$. This means that $\beta^*(D \cap (P \times Q)) = 0$ for every finite rectangle $P \times Q$. Then $I_D(x, y)$ is clearly π-integrable since $\pi(D) = 0$. But the iterated integrals of $I_D(x, y)$ are not equal.

However, as the following Theorem shows, we can have a partial converse of Fubini's Theorem for measure π.

PROPOSITION 4. *Suppose* $f(x, y)$ *is a nonnegative* $U \times V$*-measurable function and either* $\iint f d\beta_2 d\beta_1$ *or* $\iint f d\beta_1 d\beta_2$ *is well-defined and finite. Then* f *is* π*-integrable.*

Proof. Suppose $\iint f d\beta_1 d\beta_2$ is well-defined and equals k which is finite. Assume that $\int f d\pi = \infty$. We will get a contradiction to this assumption to prove the proposition.

By the monotone convergence theorem, there is a nonnegative simple function $g(x, y) = \sum_{i=1}^n c_i I_{A_i}(x, y)$ such that

$$g \leqq f \quad \text{and} \quad 2k < \int g d\pi .$$

By the property (ii) of π, we can find a finite measurable rectangle $P \times Q$ such that for each i, $1 \leqq i \leqq n$, $\pi(A_i)$ is so close to $\pi(B_i)$, where $B_i = A_i \cap (P \times Q)$, that

$$k < \int h d\pi < \infty ,$$

where $h(x, y) = \sum_{i=1}^n c_i I_{B_i}(x, y)$. Now we define for every P' in U, Q' in V and A in $U \times V$, the measures

$$\pi_0(A) = \pi(A \cap (P \times Q)) ,$$
$$\beta_3(P') = \beta_1(P' \cap P) \quad \text{and}$$
$$\beta_4(Q') = \beta_2(Q' \cap Q) .$$

Since B_3 and β_4 are both finite measures and since π_0 coincides with $\beta_3 \times \beta_4$ (defined as in § 1) on all the $U \times V$ measurable sub-rectangles of $P \times Q$, by the uniqueness of the product measure in this case (see p. 257 in [4]), π_0 is the product measure $\beta_3 \times \beta_4$. Since the function h is π-integrable, it is also π_0-integrable and so by Fubini's Theorem, we have

$$k < \int h d\pi = \int h d\pi_0 = \iint h d\beta_3 d\beta_4$$

which contradicts that $\iint f d\beta_1 d\beta_2 = k$. This proves the Proposition.

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