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ON THE REDUCTION OF RANK OF LINEAR DIFFERENTIAL SYSTEMS

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The *rank* of a linear differential system in the neighborhood of a pole of the system is defined to be one less than the order of the pole of the coefficient matrix of the system. H. L. Turrittin has shown that arbitrary rank can be reduced to rank one at the expense of increasing the dimension of the system in proportion to the amount of reduction. Can this procedure lead to extraneous solutions of the rank-reduced system which differ in behavior from solutions of the given system? This question is answered by a transformation of the rank-reduced system to a block-diagonal form, exhibiting the precise relation between solutions of the two systems. In particular, if the original system has a regular singularity at the pole in question, then so does the rank-reduced system.

An application of this gives some new necessary conditions for the regular singularity of a linear differential system (§3).

Poincaré [7] devised a method which reduced the rank of an nth order (scalar) differential equation. His method is based upon differentiation of the given equation. In §4 the differentiation of a system of equations is discussed. It is shown by an example that this procedure does not necessarily propagate the regular singular property from the original system.

The reduction of rank procedure was originally applied by Poincaré, Horn, and others in the study of Laplace integral representations for solutions. The reduced rank allowed a simplification of the expression for the kernel in the Laplace transform. It has also been used in questions of factorial series representation of solutions. (See Turrittin [8]; pp. 272–274 and Wasow [9]; pp. 340–344 for references to applications.)

2. Block-diagonalization of the rank-reduced system.

Define (following Turrittin) the *rank* r of the n-dimensional vector system of linear differential equations

$$z^q y'(z) = A(z)y(z) \tag{2.1}$$

at 0 as $r = q - 1$. Here, $A(z) = \sum_0^\infty A_\nu z^\nu$ is an $n \times n$ matrix of functions which are analytic in some neighborhood of 0, y is an n-dimensional column vector, $A(0) \neq 0$, and q is a positive integer. Turrittin's result ([8]; p. 271) states that if $q \geqq 3$, then associated with (2.1) is the system of rank one

$$t^2 \dot{Y}(t) = \mathscr{A}(t) Y(t) , \tag{2.2}$$

where $\mathscr{A}(t)$ is an nr-dimensional matrix of functions which are analytic in a t-neighborhood of 0. System (2.2) is called the (Turrittin) rank-reduced system associated with (2.1). The scope of the relation between (2.1) and (2.2) is our main consideration.

For the reader's convenience, the following computation is included which leads to the construction of $\mathscr{A}(t)$. This is a small variation in Turrittin's argument ([8]; pp. 271–272).

The main idea involves a change of the independent variable in (2.1). We let $t = z^r$ and $y(t^{1/r}) = v(t)$. Then v satisfies the system

$$\dot{v} = \frac{1}{r} t^{1/r-1-q/r} A(t^{1/r}) v = \frac{1}{r} t^{-2} A(t^{1/r}) v . \tag{2.3}$$

The expansion of the coefficient of this system in powers of $t^{1/r}$ has leading term $r^{-1}t^{-2}A(0)$. In order to construct a system which has a coefficient which is meromorphic in t at 0 and has rank one, we introduce the nr-dimensional vector

$$Y(t) = \begin{bmatrix} v(t) \\ t^{-1/r}v(t) \\ t^{-2/r}v(t) \\ \vdots \\ t^{-(r-1)/r}v(t) \end{bmatrix} .$$

Differentiating $Y(t)$ and using (2.3),

$$\dot{Y}(t) = \frac{1}{r} \begin{bmatrix} t^{-2}A(t^{1/r})v \\ (t^{-2}A(t^{1/r}) - t^{-1}I)vt^{-1/r} \\ (t^{-2}A(t^{1/r}) - 2t^{-1}I)vt^{-2/r} \\ \vdots \\ (t^{-2}A(t^{1/r}) - (r-1)t^{-1}I)vt^{-(r-1)/r} \end{bmatrix} .$$

I denotes the $n \times n$ identity matrix throughout this paper.

If $\mathscr{A}(t) = (A_{ij}(t))$, $(1 \leqq i, j \leqq r)$ denotes the partition of $\mathscr{A}(t)$ into n-dimensional blocks, then Y satisfies (2.2) if and only if the following system of matrix equations is satisfied by the A_{ij};

$$\begin{aligned} &\frac{1}{r}[t^{-(i-1)/r}A(t^{1/r}) - (i-1)t^{-(i-1)/r+1}I] \\ &= \sum_{j=1}^{r} A_{ij}(t)^{-(j-1)/r} , \quad \text{for} \quad 1 \leqq i \leqq r . \end{aligned} \tag{2.4}$$

The following choice for the $A_{ij}(t)$ accomplishes this:

$$A_{ij}(t) = \frac{1}{r}\Big[\sum_{k=0}^{\infty} A_{rk+i-j}\, t^k - (i-1)\delta_{ij} tI\Big]\,, \quad 1 \leqq i, j \leqq r\,. \tag{2.5}$$

(δ_{ij} denotes the Kronecker delta and the convention is made that $A_\nu = 0$ for $\nu < 0$.) It can be verified directly by substitution that the matrices defined by (2.5) satisfy (2.4). The essential features to take note of are the identities

$$\sum_{\nu=0}^{\infty} A_\nu t^{\nu-(i-1)/r} = \sum_{j=1}^{r}\sum_{k=0}^{\infty} A_{rk+i-j} t^{k-(j-1)/r}\,, \tag{2.6}$$

for each i, $1 \leqq i \leqq r$, which amount to rearrangements of the power series on the left hand side. From (2.5) it is clear that $\mathscr{A}(t)$ is analytic in some t-neighborhood of 0. This completes the construction.

The rest of this section is devoted to the block-diagonalization of (2.2).

First note the following relations between the block entries of $\mathscr{A}(t)$:

(i) $A_{ij}(t) = A_{i+1,j+1}(t)$ for each i, j such that $1 \leqq i < j \leqq r-1$ and $1 \leqq j < i \leqq r-1$. (i) follows directly from inspection of (2.5).

(ii) $A_{ii}(t) = (1/r)[\sum_{k=0}^{\infty} A_{rk}t^k - (i-1)tI]$ for $1 \leqq i \leqq r$, which likewise follows immediately.

(iii) $A_{ij}(t) = tA_{r-k,j-i-k}(t)$ for each k, $0 \leqq k \leqq j-i-1$, where $i < j$. To prove (iii), use (2.5) to get

$$\begin{aligned} A_{r-k,j-i-k}(t) &= \frac{1}{r}\sum_{l=0}^{\infty} A_{rl+r-j+i} t^l = \frac{t^{-1}}{r}\sum_{l=0}^{\infty} A_{r(l+1)+i-j} t^{l+1} \\ &= \frac{t^{-1}}{r}\sum_{l=0}^{\infty} A_{rl+i-j} t^l = t^{-1} A_{ij}(t) \end{aligned}$$

since $A_{i-j} = 0$.

Therefore according to (2.5) and (i), (ii) and (iii) if

$$\mathscr{A}_\nu(t) = \frac{1}{r}\sum_{k=0}^{\infty} A_{rk+\nu} t^k\,, \quad 0 \leqq \nu \leqq r-1\,, \tag{2.7}$$

then

$$\mathscr{A}(t) = \begin{bmatrix} \mathscr{A}_0(t) & t\mathscr{A}_{r-1}(t) & & \cdots & t\mathscr{A}_1(t) \\ \mathscr{A}_1(t) & \mathscr{A}_0(t) - (tr)^{-1}I & & & \vdots \\ \mathscr{A}_2(t) & \mathscr{A}_1(t) & & & \\ \vdots & \vdots & & & t\mathscr{A}_{r-1}(t) \\ \mathscr{A}_{r-1}(t) & \cdots & & \mathscr{A}_1(t) & \mathscr{A}_0(t) - \left(\frac{r-1}{r}\right)t^{-1}I \end{bmatrix}.$$

Let $\mathscr{D}(t) = \text{diag}\{I, t^{-1/r}I, t^{-2/r}I, \cdots, t^{-(r-1)/r}I\}$ and $Y(t) = \mathscr{D}(t)U(t)$. Then $U(t)$ satisfies

$$\dot{U}(t) = [\mathscr{D}^{-1}(t)t^{-2}\mathscr{A}(t)\mathscr{D}(t) - \mathscr{D}^{-1}(t)\dot{\mathscr{D}}(t)]U(t) = \hat{\mathscr{A}}(t)U(t)\,, \tag{2.8}$$

where $\hat{\mathscr{A}}(t) = \begin{pmatrix} \hat{\mathscr{A}}_0(t) & \hat{\mathscr{A}}_{r-1}(t) & \cdot & \cdot & \cdot & \hat{\mathscr{A}}_1(t) \\ \hat{\mathscr{A}}_1(t) & \hat{\mathscr{A}}_0(t) & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \hat{\mathscr{A}}_{r-1}(t) \\ \hat{\mathscr{A}}_{r-1}(t) & \cdot & & \cdot & \cdot & \hat{\mathscr{A}}_1(t)\hat{\mathscr{A}}_0(t) \end{pmatrix}$, and

$$\hat{\mathscr{A}}_i(t) = t^{-2+i/r}\mathscr{A}_i(t)\,, \quad 0 \leqq i \leqq r-1\,. \tag{2.9}$$

The matrix $\hat{\mathscr{A}}(t)$ is called a *block-circulant matrix*. It is well-known (see, for example, [6]; p. 66) that circulant matrices can be diagonalized by a similarity transform using an orthogonal matrix. A block-circulant can be block-diagonalized in essentially the same manner (see [1]).

Let

$$\mathscr{P} = \begin{pmatrix} 0 & \cdot & \cdot & \cdot & 0 & I \\ I & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & I & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & I & 0 \end{pmatrix}.$$

Then $\hat{\mathscr{A}}(t)$ can be expressed in terms of $\mathscr{P}$ as

$$\hat{\mathscr{A}}(t) = \sum_0^{r-1} \hat{\mathscr{A}}_i \cdot X\mathscr{P}^i\,. \tag{2.10}$$

The symbol $\cdot X$ is used to designate the left Kronecker product with respect to the block decomposition of $\mathscr{P}^i$. (See [5]; pp. 81–82.)

Let $\varepsilon = \exp(2\pi i/r)$ and $F_{ij} = \varepsilon^{ij}I$ for $0 \leqq i, j \leqq r-1$. Form the matrix

$$\mathscr{F} = (F_{ij})\,, \quad 0 \leqq i, j \leqq r-1$$

and let $\mathscr{F}_1$ denote the first n columns of $\mathscr{F}$, $\mathscr{F}_2$ denote the next n columns of $\mathscr{F}$, etc. Then $\mathscr{P}\mathscr{F}_j = \varepsilon^{-(j-1)}\mathscr{F}_j$ for each j, $1 \leqq j \leqq r$, and hence by induction

$$\mathscr{P}^i\mathscr{F}_j = \varepsilon^{-i(j-1)}\mathscr{F}_j \quad \text{for each} \quad i, 0 \leqq i \leqq r-1\,, \tag{2.11}$$

and each j, $1 \leqq j \leqq r$.

Now consider the product

$$\hat{\mathscr{A}}\mathscr{F} = (\hat{\mathscr{A}}\mathscr{F}_1, \hat{\mathscr{A}}\mathscr{F}_2, \cdots, \hat{\mathscr{A}}\mathscr{F}_r) .$$

$$\hat{\mathscr{A}}\mathscr{F}_j = \left(\sum_{i=0}^{r-1} \hat{\mathscr{A}}_i \cdot X \mathscr{P}^i\right)\mathscr{F}_j = \sum_{i=0}^{r-1} (\hat{\mathscr{A}}_i \cdot X \mathscr{P}^i \mathscr{F}_j)$$

using the property $(A \cdot X B)\,(C \cdot X D) = (AC) \cdot X\,(BD)$ of the Kronecker product (see Theorem 43.4; [5] p. 82). Therefore using (2.11)

$$\hat{\mathscr{A}}\mathscr{F}_j = \left(\sum_{i=0}^{r-1} \hat{\mathscr{A}}_i \varepsilon^{-i(j-1)}\right) \cdot X \mathscr{F}_j .$$

Defining

$$\mathscr{B}_j = \sum_{i=0}^{r-1} \hat{\mathscr{A}}_i \varepsilon^{-i(j-1)} , \quad 1 \leqq j \leqq r , \tag{2.12}$$

then

$$\hat{\mathscr{A}}\mathscr{F} = (\mathscr{B}_j F_{ij}) \quad 1 \leqq i, j \leqq r .$$

But since each block F_{ij} is a scalar multiple of I, then

$$\hat{\mathscr{A}}\mathscr{F} = (\mathscr{F}_{ij}\mathscr{B}_j) = \mathscr{F} \operatorname{diag}\{\mathscr{B}_1, \cdots, \mathscr{B}_r\} .$$

Furthermore, $\mathscr{F}$ is nonsingular because the columns of $\mathscr{F}$ are pairwise orthogonal. The inner product of any two distinct columns from the same $\mathscr{F}_j$ is clearly zero, as is the inner product of the jth column of $\mathscr{F}_i$ with the kth column of $\mathscr{F}_l$, $j \neq k$. It only remains to check the inner product of the jth column of $\mathscr{F}_i$ with the jth column of $\mathscr{F}_k$, $i \neq k$. This is computed to be

$$\sum_{l=0}^{r-1} \varepsilon^{(i-k)l} = (1 - \varepsilon^{r(i-k)})/(1 - \varepsilon^{(i-k)}) = 0 .$$

Therefore $\mathscr{F}^{-1}\hat{\mathscr{A}}\mathscr{F} = \operatorname{diag}\{\mathscr{B}_1, \cdots, \mathscr{B}_r\} \stackrel{\text{def}}{=} \mathscr{B}(t)$ and if $U(t) = \mathscr{F}\, W(t)$, then $W(t)$ satisfies

$$\dot{W}(t) = \mathscr{B}(t) W(t) . \tag{2.13}$$

This proves

THEOREM 1. *The linear transformation*

$$Y(t) = \mathscr{D}(t)\mathscr{F}\, W(t) \tag{2.14}$$

transforms the rank-reduced system (2.2) *into the block-diagonal system* (2.13).

As an immediate consequence, this gives an alternate construction for $\mathscr{A}(t)$ as

$$\mathscr{A}(t) = (\dot{\mathscr{D}}\mathscr{D}^{-1} + \mathscr{D}\mathscr{F}\mathscr{B}\mathscr{F}^{-1}\mathscr{D}^{-1})t^{-2} .$$

The fact that $\mathscr{A}(t)$, so defined, has a convergent power series expansion in powers of t is a consequence of our previous calculations, notably (2.6).

A fundamental solution matrix for (2.13) can be easily obtained as the direct sum of fundamental solutions for the subsystems

$$\dot{w}_j(t) = \mathscr{B}_j(t)w_j(t) \quad (1 \leqq j \leqq r) . \tag{2.14}$$

It is natural to restrict t to lie in some sector which the coefficient $\mathscr{B}_j(t)$ is single-valued.

This can also be accomplished by changing the variable in (2.14) from t to z^r and letting $w_j(z^r) = u_j(z)$. Then (2.14) becomes

$$\begin{aligned} u_j'(z) &= rz^{r-1}\mathscr{B}_j(z^r)u_j(z) \\ &= \left[rz^{r-1}\sum_{i=0}^{r-1}\hat{\mathscr{A}}_i(z^r)\varepsilon^{-i(j-1)}\right]u_j(z) \\ &= \left[rz^{-r-1}\sum_{i=0}^{r-1}\frac{1}{r}\sum_{k=0}^{\infty}A_{rk+i}z^{rk+i}\varepsilon^{-i(j-1)}\right]u_j(z) , \end{aligned}$$

and performing the rearrangement (2.6) of the power series

$$z^q u_j'(z) = \left(\sum_{\nu=0}^{\infty} A_\nu(z\varepsilon^{-(j-1)})^\nu\right)u_j(z) = A(z\varepsilon^{-(j-1)})u_j(z) \tag{2.15}$$

(since ε is an r^{th} root of unity).

Now change the variable again from z to $\varepsilon^{(j-1)}\zeta$ and let $u_j(\varepsilon^{(j-1)}\zeta) = v(\zeta)$. Then (2.15) becomes

$$\zeta^q \frac{d}{d\zeta}v(\zeta) = A(\zeta)v(\zeta) . \tag{2.16}$$

If $\Phi(z)$ denotes a fundamental solution matrix for (2.1), then $\Phi(\zeta)$ is a fundamental solution matrix for (2.16) and so $\Phi(\varepsilon^{-(j-1)}t^{1/r})$ is a fundamental solution matrix for (2.14). This gives

THEOREM 2. *A fundamental solution matrix for* (2.2) *is*

$$\psi(t) = \mathscr{D}(t)\,\mathscr{F}\,\mathrm{diag}\,\{\Phi(t^{1/r}), \Phi(\varepsilon^{-1}t^{1/r}), \cdots, \Phi(\varepsilon^{-(r-1)}t^{1/r}\} , \tag{2.17}$$

where $\Phi(z)$ denotes a fundamental solution matrix for the system (2.1).

This representation collectively relates the behavior of a full set of nr-linearly independent solution vectors of (2.2) to a fundamental solution for (2.1). Turrittin's characterization of the connection between the solutions of the two systems presented as Theorem 48.1 in Wasow ([9]; p. 341). In our notation, it states that if a solution vector Y of (2.2) is partitioned into n vectors $Y^{(0)}, Y^{(1)}, \cdots, Y^{(n-1)}$†, then

$$y = \sum_{j=0}^{r-1} z^j Y^{(j)}(z^r) \tag{2.18}$$

is a solution vector of (2.1) and all solution vectors of (2.1) can be represented in this way. From (2.18) it can be shown that the regular singularity (see next section for the definition) of (2.2) implies the regular singularity of (2.1). However, it is not apparent that the converse is true since the summation of the components of Y could mask properties of the entries. However, using (2.17), it is shown in the next section that the regular property of (2.2) is inherited from (2.1).

3. Application to regular singular systems.

The system (2.1) is said to have a regular singularity at 0 (or equivalently that 0 is a regular singular point) if there exists a fundamental solution matrix $\Phi(z)$ for (2.1) having the property

$$||\Phi(z)||^{\dagger\dagger} = 0(|z|^K) \tag{3.1}$$

for some real number K as $|z| \rightarrow 0$, $|\arg z| < \infty$. Of course, if one fundamental solution has this property, then all do and this becomes a property of $A(z)$ which imposes severe restrictions on the coefficients in the expansion of $A(z)$. For example, it is well-known that the leading coefficient A_0 must be nilpotent. Lutz ([4]; p. 314) has proven that it is also necessary that

$$\text{trace } (A_0^k A_1) = 0 \tag{3.2}$$

for each $k = 0, 1, 2, \cdots, n-1$ provided $q \geqq 3$. Lately, Harris ([3]; p. 2) has shown that in case $q = 2$, the conditions (3.2) are necessary for the values of $k = 1, 2, \cdots, n-1$.

In order to apply Harris' result to the system (2.2), it is helpful to first prove

THEOREM 3. *The linear differential system* (2.1) *has* 0 *as a regular singularity if and only if the rank-reduced system* (2.2) *has* 0 *as a regular singularity.*

Proof. To show that (2.1) and (2.2) are regular singular together at 0, use the representation (2.17). If $\Phi(z)$ satisfies property (3.1), then $||\Phi(\varepsilon^{-j}t^{1/r})|| = O(|t|^K)$ as $t \rightarrow 0$, $|\arg t| < \infty$ for all $j = 0, 1, \cdots, r-1$

† The super-scripts do not indicate successive derivatives.

†† Any matrix norm may be used here.

(where K is used generically to denote some real number). Since $\|\mathscr{D}(t)\mathscr{T}\| = O(|t|^K)$ as $t \to 0$, $|\arg t| < \infty$, then $\psi(t)$ likewise has this property. To prove the converse, just solve for $\Phi(t^{1/r})$ as the first diagonal block of $\mathscr{T}^{-1}\mathscr{D}^{-1}(t)\psi(t)$ and use the same argument.

Next, we investigate the product $\mathscr{A}_0{}^k\mathscr{A}_1$ which appears in Harris' necessary conditions for the regular singularity of (2.2).

LEMMA. *Let $T[X_0, X_1, \cdots, X_{r-1}]$ denote the lower regular block-triangular matrix having entries X_0 on its diagonal blocks, X_1 on its first sub-diagonal blocks, etc. Then*

$$\mathscr{A}_0{}^k = \frac{1}{r^k} T[S_0^{(k)}, S_1^{(k)}, \cdots, S_{r-1}^{(k)}] , \tag{3.3}$$

where

$$S_j^{(k)} = \sum_{i_1+i_2+\cdots+i_k=j} A_{i_1}A_{i_2}\cdots A_{i_k} , \quad 0 \leqq j \leqq r-1 ,$$

and $k \geqq 1$. (The summation above is taken over all nonnegative values of the indices with the indicated sum.)

Proof. (3.3) is proven by induction on k. Note that

$$S_j^{(1)} = \sum_{i_1=j} A_{i_1} = A_j , \quad 0 \leqq j \leqq r-1 .$$

From (2.5) $A_{ij}(0) = (1/r)A_{i-j}$, hence $\mathscr{A}_0 = (1/r)T[A_0, A_1, \cdots, A_{r-1}]$ and (3.3) is verified for $k = 1$. Assuming (3.3) is true for k, multiply on the left by $\mathscr{A}_0$ to obtain

$$\mathscr{A}_0{}^{k+1} = T[X_0, X_1, \cdots, X_{r-1}] ,$$

where $X_j = \sum_{l=0}^{j} A_{j-l}S_l^{(k)}$. Then by the induction hypothesis

$$\begin{aligned} X_j &= \sum_{l=0}^{j} A_{j-l} \sum_{i_1+i_2+\cdots+i_{k+1}=j} A_{i_1}A_{i_2}\cdots A_{i_k} \\ &= \sum_{i_1+i_2+\cdots+i_{k+1}=j} A_{i_1}A_{i_2}\cdots A_{i_{k+1}} = S_j^{(k+1)} \end{aligned}$$

and the induction is completed.

As a consequence of (3.3), note that if A_0 is nilpotent of order h (i.e., $A_0^h = 0$, but $A_0^{h-1} \neq 0$, $2 \leqq h \leqq n$), then $\mathscr{A}_0$ is nilpotent of order at most rh. This follows from the fact that if $k \geqq rh$, then in each of the index sets $(i_1, i_2, \cdots, i_k)$, there must be at least h consecutive 0's since

$$\sum_1^k i_j \leqq r-1 .$$

It then follows from Theorem 3 and Harris' result that

$$(3.4)\qquad \operatorname{trace}(\mathscr{A}_0{}^k\mathscr{A}_1) = 0\,, \quad 1 \leqq k \leqq rh - 1 \leqq rn - 1\,,$$

are necessary conditions for the regular singularity of (2.1). Again, from (2.5)

$$\mathscr{A}_1 = \frac{1}{r}(A_{q-1+i-j} - (i-1)\delta_{ij}I)\,, \quad 1 \leqq i, j \leqq q-1\,.$$

The jth diagonal block of the product $\mathscr{A}_0{}^k\mathscr{A}_1$ is calculated to be

$$\frac{1}{r^{k+1}}\left(\sum_{m=0}^{j-1} S_m^{(k)} A_{q-1-m} - (j-1)A_0^k\right).$$

But since the trace is a linear function and $\operatorname{trace}(A_0^k) = 0$, then the conditions (3.4) can be expressed in a straight-forward manner as

THEOREM 4. *Necessary conditions for the regular singularity of* (2.1) *at* 0 *are*:

$$(3.5)\qquad \operatorname{trace}\left(\sum_{j=1}^{q-1}\sum_{m=0}^{j-1}\left(\sum_{i_1+i_2+\cdots+i_k=m} A_{i_1}A_{i_2}\cdots A_{i_k}\right)A_{q-1-m}\right) = 0$$

for $1 \leqq k \leqq rn - 1$.

Further simplifications in the form of (3.5) can be made by using the identity: $\operatorname{trace}(AB) = \operatorname{trace}(BA)$. For example, it can be shown that the condition corresponding to $k = 1$ can be rewritten in a simpler way as:

$$\operatorname{trace}\left(\sum_{i=0}^{q/2-1} A_iA_{q-1-i}\right) = 0 \quad \text{when } q \text{ is } \textit{even}\,,$$

and

$$\operatorname{trace}\left(\sum_{i=0}^{(q-1)/2-1} A_iA_{q-1-i} + (1/2)A^2_{(q-1)/2}\right) = 0$$

when q is odd.

The conditions given in Theorem 4 are generally not sufficient for regular singularity. One reason for this is that conditions (3.5) involve restrictions only on $A_0, A_1, \cdots, A_{q-1}$ and it is known that the necessary and sufficient conditions may depend non-trivially on the first $n(q-1)$ coefficients in the expansion for $A(z)$.

4. Differentiation of a linear differential system. Poincaré ([7]; pp. 328–335) was concerned with (scalar) nth order linear differential equations of the form

$$(4.1)\qquad P_ny^{(n)} + P_{n-1}y^{(n-1)} + \cdots + P_1y' + P_0y = 0$$

in a neighborhood of $x = \infty$, where the P_i denote polynomials in x. His definition of rank appears different from the one used by Turrittin. They can be made comparable by writing (4.1) as a first order system in the following manner. Let $y_1 = y$, $y_2 = xy_1'$, etc. and make the change of variable $z = x^{-1}$ to change the singular point to the origin. The linear differential system obtained in this manner will have the same rank as the nth order scalar differential equation.

To make his reduction of (4.1) to rank one, Poincaré differentiates a function formed from a solution of (4.1) and obtains a system of differential equations of order up to and including n^p, where p denotes the rank of (4.1) in Poincaré's sense. An important feature of this construction is that the rank in invariant with respect to differentiation of the equation. After some algebraic manipulations (which have been called "impracticable" by Birkhoff) and a change of variable, he is able to reduce the rank of a linear differential equation of order n^p to rank one.

Rank zero (in Poincaré's sense) is equivalent to the following necessary and sufficient conditions for regular singularity to due Fuchs ([3]; p. 122):

A differential equation of the form $u^{(n)} + \sum_{j=1}^{n} a_j(z)u^{(n-j)} = 0$ has a regular singularity at 0 if and only if $a_j(z) = z^{-j}p_j(z)$, where $p_j(z)$ is analytic at 0, for each $j (1 \leqq j \leqq n)$ and not all the $a_j(z)$ are analytic at 0.

Then Poincaré's argument on the invariance of the rank of the differentiated equations shows that the regular singular property is preserved by differentiation of the equation.

It is not true, however, that differentiation of a system of the form (2.1) necessarily preserves the regular singular property. Let $\hat{A}(x) = z^{-q}A(z)$ and differentiate

$$Y' = \hat{A}(z)\,Y \tag{4.2}$$

once to obtain

$$Y'' = \hat{A}'(z)\,Y + \hat{A}^2(z)\,Y\,,$$

which is written in first-order system form as

$$\begin{pmatrix} Y \\ Y' \end{pmatrix}' = \begin{pmatrix} 0 & I \\ \hat{A}' + \hat{A}^2 & 0 \end{pmatrix} \begin{pmatrix} Y \\ Y' \end{pmatrix}. \tag{4.3}$$

n solution vectors of (4.3) are regular singular at 0 simultaneously with (4.2), since if $\Phi(z)$ is a fundamental solution matrix for (4.2), then $\begin{pmatrix} \Phi \\ \Phi' \end{pmatrix}$ is a set of n linearly independent solution vectors of (4.3). The other n linearly independent solutions of a fundamental set for (4.3), the so-called extraneous solutions, may have irregular singular behavior at 0.

To see when this may happen, let

$$Y = \begin{pmatrix} I & 0 \\ \hat{A} & I \end{pmatrix} W .$$

Then W satisfies the differential equation

$$W' = \begin{pmatrix} \hat{A} & I \\ 0 & -\hat{A} \end{pmatrix} W$$

and clearly if $y' = \hat{A}y$ has a regular singularity at 0, then (4.3) has a regular singularity at 0 if and only if

$$w' = -\hat{A}(z)w \tag{4.4}$$

has a regular singularity at 0.

The following example shows that (4.4) may have 0 as an irregular singular point even though 0 is a regular singular point of (4.2). Consider

$$Y' = \hat{A}(z)\,Y = \begin{bmatrix} z^{-3} - z^{-2} & z^{-3} + z^{-2} \\ -z^{-3} + 3z^{-2} - 4z^{-1} + 6 & -z^{-3} + z^{-2} \end{bmatrix} Y . \tag{4.5}$$

To show that (4.5) has a regular singularity at 0, transform it so that it is equivalent to a second order scalar equation.

If $y' = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} y$ and $a_{12} \not\equiv 0$, then the transformation $v = \begin{pmatrix} 1 & 0 \\ a_{11} & a_{12} \end{pmatrix} y$ yields the system $v' = \begin{pmatrix} 0 & 1 \\ -c & -b \end{pmatrix} v$, where $-b = a_{11} + a_{22} + a_{12}^{-1}a'_{12}$ and $c = a_{11}a_{22} - a_{12}a_{21} + a_{11}a_{12}^{-1}a'_{12} - a'_{11}$. Hence $u = v_1$ satisfies the second order equation

$$u'' + bu' + cu = 0 . \tag{4.6}$$

Applying this procedure to (4.5), obtain an equation of the form (4.6) with $b = O(z^{-1})$ and $c = O(z^{-2})$ as $z \to 0$. Therefore the Fuchs' conditions are satisfied and so (4.5) has a regular singularity at 0.

However, if the procedure is applied to (4.4) with $\hat{A}(z)$ given by (4.5), then an equation of form (4.6) is obtained with $b = O(z^{-1})$ and $c = 4z^{-3} + O(z^{-2})$ as $z \to 0$. The Fuchs' conditions are not satisfied, hence (4.4) is not regular singular at 0. This is called irregular singular behavior at 0 and is characterized by exponential-type growth of solutions in sectors as $z \to 0$. The four-dimensional system (4.3) with $\hat{A}(z)$ given by (4.5) has a solution space consisting of a two-dimensional subspace of solutions which are regular singular at 0 and a two-dimensional subspace which are irregular singular at 0.

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