

## ON ELEMENTARY IDEALS OF POLYHEDRA IN THE 3-SPHERE

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Let  $K$  be a polygonal simple closed curve (a knot) in a 3-sphere  $S^3$ . For each nonnegative integer  $d$  the  $d$ th elementary ideal  $E_d$  of  $K$  in the integral group-ring over an infinite cyclic group is defined by R. H. Fox. The ideal  $E_d$  of  $K$  is a topological invariant of the position of  $K$  in  $S^3$ . This method has been applied to various more general settings, for instance, links in  $S^3$ ,  $S^{n-2}$  in  $S^n$  ( $n > 2$ ) and etc. In this paper the  $d$ th elementary ideals  $E_d(l)$  are associated to each  $(n-2)$ -dimensional integral cycle  $l$  on a polyhedron  $L$  in an  $n$ -sphere  $S^n$  ( $n > 2$ ) that does not separate  $S^n$ . The collection of  $E_d(l)$  for all possible  $l$  on  $L$  forms a topological invariant of the position of  $L$  in  $S^n$ .

In §2 we prove theorems of the  $d$ th elementary ideal  $E_d(l)$  associated with an  $(n-2)$ -dimensional integral  $l$  on a polyhedron  $L$  in  $S^n$  that does not separate  $S^n$  ( $n > 2$ ). In §3 we will consider to the case of polyhedra in  $S^3$ . After studying an example of a  $\theta$ -curve in  $S^3$  in §4, we reconsider knots and links from this point of view in §5. In §6 we will give a remark on a Torres' formula for a link in  $S^3$  ([7]) from this point of view.

Our discussion will be based on Fox's free differential calculus ([1], [2], [3]), though other methods, especially the covering space technique, would also be helpful. We need some minor adjustment of free differential calculus that will be given in §1.

The method of the paper is essentially different from that of [4], though a close relation between them will be observed.

1. From free differential calculus. Let  $G$  be a group with a presentation  $(x_1, \dots, x_m; r_1, \dots, r_n)$  ( $n \leq \infty$ ). Let  $H$  be a multiplicative abelian group and  $\psi$  a homomorphism of  $G$  into  $H$ . Let  $\phi$  be the canonical homomorphism of the free group  $F(x_1, \dots, x_m)$  onto  $G$ . These homomorphism  $\phi$  and  $\psi$  are naturally extended to ring homomorphisms of the integral group-ring  $JF$  onto  $JG$  and of  $JG$  into  $JH$ , respectively. Using Fox's free differential calculus ([1], [2], [3]), we have an  $n \times m$  matrix  $A(G, \psi) = (r_{ij})$ , where

$$r_{ij} = \left( \frac{\partial r_i}{\partial x_j} \right)^{\psi\phi} \in JH.$$

Generally, let  $(r_{ij})$  be an  $n \times m$  matrix ( $n \leq \infty$ ) over a commutative

ring  $R$  with unity. Then for each nonnegative integer  $d$  the  $d$ th elementary ideal  $E_d$  of the matrix  $(r_{i,j})$  is the ideal of  $R$  generated by all minors of order  $m - d$ , if  $m - d > 0$  and  $n - (m - d) \geq 0$ . If  $m - d > 0$ , but  $n - (m - d) < 0$ , let  $E_d = (0)$ . If  $m - d \leq 0$ , let  $E_d = (1)$ .

It was proved in [3] that the  $d$ th elementary ideal  $E_d$  of the matrix  $A(G, \alpha)$  is an invariant of the group  $G$ , where  $\alpha$  is the abelianizer of  $G$ . Since there is a unique homomorphism  $\sigma$  of the abelianization of  $G$  into  $H$  with  $\psi = \sigma\alpha$ , the  $d$ th elementary ideal  $E_d(G, \psi)$  of the matrix  $A(G, \psi)$  is an invariant of  $G$  with respect to  $\psi$ .

Now let us consider the following diagram:

$$G_1 \xrightarrow{\psi} G_2 \xrightarrow{\sigma} H,$$

where  $G_1$  and  $G_2$  are groups,  $H$  a multiplicative abelian group, and  $\psi$  is a homomorphism of  $G_1$  onto  $G_2$  and  $\sigma$  of  $G_2$  into  $H$ . Then we have the following theorem.

**THEOREM 1.**  $E_d(G_1, \sigma\psi) \subset E_d(G_2, \sigma)$ .

*Proof.* Let  $(x_1, \dots, x_m; r_1, \dots, r_n) (n \leq \infty)$  be a presentation of  $G_1$ . Then  $x_1^\psi, \dots, x_m^\psi$  are generators of  $G_2$ , since  $\psi$  is a homomorphism of  $G_1$  onto  $G_2$ , and  $r_1^\psi, \dots, r_n^\psi$  are some of the relators of  $G_2$ . From this it follows that one of  $A(G_1, \sigma\psi)$  is a submatrix of one of  $A(G_2, \sigma)$ , where the both matrices have the same number of columns. The theorem is now clear.

A trivializer of a group  $G$  is a homomorphism of  $G$  onto the trivial group that consists of only one element. Any trivializer will be denoted by the same notation  $o$  in this paper. Further, the group-ring  $Jo(G)$  will be identified with  $J$ .

For the convenience of the readers we refer to the following two theorems from free differential calculus ([2]).

**THEOREM A (Fox).** *Let  $G$  be a group and  $o$  a trivializer of  $G$ . Then a matrix  $A(G, o)$  is elementarily equivalent (see [1], [3]) to the following matrix:*

$$\begin{bmatrix} \tau_1 & & & & & & 0 \\ & \ddots & & & & & \\ & & \ddots & & & & \\ & & & \tau_n & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \\ 0 & & & & & & \\ & & & \mu & & & \\ & & & & & & 0 \end{bmatrix}$$

where  $\tau_1, \dots, \tau_n$  are torsion numbers and  $\mu$  the Betti number of the

abelianization of  $G$ . (We assume that the abelianization of  $G$  is finitely generated.)

**THEOREM B (Fox).** *Let  $H$  be a multiplicative free abelian group of rank  $\mu$  ( $\mu \geq 1$ ) with generators  $t_1, \dots, t_\mu$  and let  $i$  be the identity of  $H$ . Then we have*

$$\begin{cases} E_0(H, i) = (0) , \\ E_d(H, i) = (t_1 - 1, \dots, t_\mu - 1)^{\mu-d}, \text{ if } 1 \leq d < \mu , \\ E_d(H, i) = (1), \text{ if } d \geq \mu . \end{cases}$$

2. **On polyhedra in  $S^n$ .** Let  $L$  be a polyhedron in an  $n$ -sphere  $S^n$  ( $n \geq 3$ ) that does not separate  $S^n$ , and let  $G_L$  be the fundamental group of  $S^n - L$ . Let  $l$  be an  $(n - 2)$ -dimensional cycle with integral coefficients on  $L$ . There is a homomorphism  $\psi$  of the group  $G_L$  into the multiplicative infinite cyclic group  $H$  generated by  $t$  such that for each  $g \in G_L$ ,

$$g^\psi = t^{\text{link}(g,l)} ,$$

where  $\text{link}(g, l)$  is the linking number between  $g$  and  $l$  in  $S^n$ . Since the  $d$ th elementary ideal  $E_d(G_L, \psi)$  is an invariant of the group  $G_L$  with respect  $\psi$ , it is a topological invariant of  $S^n - L$  with respect to  $l$  on  $L$ , and from this it follows that  $E_d(G_L, \psi)$  is a topological invariant of the position of  $l$  on  $L$  in  $S^n$ . We will denote it by  $E_d(l)$ . If two  $(n - 2)$ -cycles  $l$  and  $l'$  are homologous on  $L$ , then the corresponding  $d$ th elementary ideals  $E_d(l)$  and  $E_d(l')$  are the same. The collection of  $E_d(l)$  for all possible  $(n - 2)$ -cycles  $l$  on  $L$  forms a topological invariant of the position of  $L$  in  $S^n$ .

**THEOREM 2.** *Let  $L$  be a polyhedron in an  $n$ -sphere  $S^n$  ( $n \geq 3$ ) that does not separate  $S^n$ . Let  $p_{n-2}(L)$  be the  $(n - 2)$ -dimensional Betti number of  $L$  and  $\tau_1, \tau_2, \dots, \tau_r$  are  $(n - 3)$ -dimensional torsion numbers with  $\tau_i | \tau_{i+1}$  ( $i = 1, 2, \dots, r - 1$ ). Then for each  $(n - 2)$ -dimensional cycle  $l$  on  $L$  the  $d$ th elementary ideal  $E_d(l)$  of  $l$  on  $L$  in  $S^n$  satisfies the following conditions:*

$$\begin{cases} (E_d(l))^\circ = (0), \text{ if } d < p_{n-2}(L) , \\ (E_d(l))^\circ = \tau_1 \tau_2 \dots \tau_m, \text{ where } m = r - (d - p_{n-2}(L)) , \\ \qquad \qquad \qquad \text{if } p_{n-2}(L) \leq d < p_{n-2}(L) + r , \\ (E_d(l))^\circ = (1), \text{ if } d \geq p_{n-2}(L) + r . \end{cases}$$

*Proof.* By the Alexander duality theorem we have

$$p_1(S^n - L) = p_{n-2}(L)$$

and

$$T_1(S^n - L) \approx T_{n-3}(L) ,$$

where  $T_i(K)$  is the  $i$ -dimensional torsion group of a complex  $K$ . Since the abelianizations of  $\pi(S^n - L)$  is the 1-dimensional homology group of  $S^n - L$ , the theorem is clear by Theorem A.

Especially if  $l_0$  is the  $(n - 2)$ -cycle on  $L$  such that the coefficients of  $l_0$  on every  $(n - 2)$ -simplexes of  $L$  are 0, then for every  $g \in G_L$  we have  $\text{link}(g, l_0) = 0$ . Hence the homomorphism  $\psi$  of  $G_L$  into  $H$  with respect to  $l_0$  is a trivializer. Therefore, we have the following theorem.

**THEOREM 3.** *Let  $L$  be a polyhedron in  $S^n (n \geq 3)$  that does not separate  $S^n$  and  $l_0$  the  $(n - 2)$ -cycle on  $L$  with coefficients 0 on every  $(n - 2)$ -simplexes on  $L$ . Then we have*

$$E_d(l_0) = (E_d(l))^\circ ,$$

where  $l$  is an  $(n - 2)$ -cycle on  $L$ .

Now let  $\mu$  be the number of components of  $L$  (i.e.  $\mu = p_0(L)$ ) and  $L_i (i = 1, 2, \dots, \mu)$  the  $i$ th component of  $L$ . Then an  $(n - 2)$ -cycle  $l$  on  $L$  can be expressed as  $\sum_{i=1}^{\mu} l_i$ , where  $l_i (i = 1, 2, \dots, \mu)$  is an  $(n - 2)$ -cycle on  $L$  defined by

$$\begin{cases} l_i | L_i = l | L_i , \\ l_i | L_j = 0 \quad (i \neq j) . \end{cases}$$

Let  $H_0$  be a multiplicative free abelian group of rank  $\mu$ , i.e.  $H_0 = \prod_{i=1}^{\mu} H_i$ , where  $H_i = (t_i: ) (i = 1, 2, \dots, \mu)$ . Define a homomorphism  $\psi_0$  of  $G_L$  into  $H_0$  by

$$g^{\psi_0} = \prod_{i=1}^{\mu} t_i^{\text{link}(g, l_i)} .$$

Then, as before, we have the  $d$ th elementary ideal  $E_d(G_L, \psi_0)$  in  $JH_0$ , that is a topological invariant of the position of  $l$  on  $L$  in  $S^n$ .  $E_d(G_L, \psi_0)$  will be denoted by  $E_d[l]$ .

Let  $\sigma$  be a homomorphism of  $H_0$  onto  $H = (t: )$  defined by  $t_i^\sigma = t$ . Since  $g^\psi = t^{\text{link}(g, l)}$ , we have  $\sigma\psi_0 = \psi$ . Since  $\sigma$  is an homomorphism of  $H_0$  onto  $H$ , we have the following theorem.

**THEOREM 4.** *We have*

$$\begin{cases} (E_d[l])^\sigma = E_d(l) \text{ and} \\ (E_d[l])^\circ = (E_d(l))^\circ . \end{cases}$$

Now assume that  $p_{n-2}(L) \geq 1$ . We have a sequence of homomorphism

$$\begin{array}{ccccccc} & & & \xrightarrow{\sigma_0} & & & \\ & & & \nearrow & & & \\ G_L & \xrightarrow{\alpha} & H_L & \xrightarrow{\alpha'} & H' & \xrightarrow{\sigma'} & H_0, \\ & & & \searrow & & & \\ & & & & & & \\ & & & \xrightarrow{\psi_0} & & & \end{array}$$

where  $\alpha$  is the abelianizer of  $G_L$ ,  $H_L \cong H'$  (a free abelian group of rank  $p_{n-2}(L)$ )  $\times$  (the torsion subgroup of  $H_L$ ) and  $\alpha'$  is the projection of  $H_L$  onto  $H'$ ,  $\psi_0 = \sigma_0\alpha$ , and  $\sigma_0 = \sigma'\alpha'$ . Then, by Theorems 1 and B, we have

$$\begin{aligned} & E_d(G_L, \alpha'\alpha) \subset E_d(H', i) \\ & = \begin{cases} (0), & \text{if } d = 0, \text{ and} \\ (1 - s_1, \dots, 1 - s_{p_{n-2}(L)})^{p_{n-2}(L)-d}, & \text{if } 1 \leq d < p_{n-2}(L), \end{cases} \end{aligned}$$

in  $JH'$ , where we assume that  $H'$  is generated by  $s_1, \dots, s_{p_{n-2}(L)}$ . In [2] it is proved that

$$(1 - s_1^\sigma, \dots, 1 - s_{p_{n-2}(L)}^\sigma) \subset (1 - t_1, \dots, 1 - t_\nu).$$

Hence we have the following theorem.

**THEOREM 5.** *Assume that  $p_{n-2}(L) \geq 1$ . Then we have*

$$\begin{cases} E_0[l] = (0), \text{ and} \\ E_d[l] \subset (1 - t_1, \dots, 1 - t_\nu)^{p_{n-2}(L)-d}, \text{ if } 1 \leq d < p_{n-2}(L). \end{cases}$$

**THEOREM 6.** *Assume that  $p_{n-2}(L) \geq 1$ . Then we have*

$$\begin{cases} E_0(l) = (0), \text{ and} \\ E_d(l) \subset (1 - t)^{p_{n-2}(L)-d}, \text{ if } 1 \leq d < p_{n-2}(L). \end{cases}$$

**REMARK.** The first formula in Theorem 2 follows from Theorem 6.

### 3. On polyhedra in $S^3$ .

**THEOREM 7<sup>1</sup>.** *Let  $M^3$  be a 3-dimensional manifold and  $L$  a polyhedron in  $M^3$  that does not separate  $M^3$ . Then  $\pi(M^3 - L)$  has a presentation with deficiency  $p_1(M^3 - L) - p_2(M^3 - L)$ .*

*Proof.* Let  $K$  be a connected 2-dimensional polyhedron. Then  $\pi(K)$  has a presentation with  $\alpha_1 - (\alpha_0 - 1)$  generators and  $\alpha_2$  relators, where  $\alpha_i (i = 0, 1, 2)$  is the number of  $i$ -dimensional simplexes. Since  $M^3 - L$  has a connected 2-dimensional polyhedron as its deformation retract, say  $K$ , there is a presentation of  $\pi(M^3 - L)$  with deficiency

<sup>1</sup> This theorem is due to the referee.

$$\begin{aligned}\alpha_1 - (\alpha_0 - 1) - \alpha_2 &= 1 - p_0(K) + p_1(K) - p_2(K) \\ &= p_1(M^3 - L) - p_2(M^3 - L) .\end{aligned}$$

COROLLARY. *Let  $L$  be a polyhedron in  $S^3$  that does not separate  $S^3$ . Then  $\pi(S^3 - L)$  has a presentation with deficiency  $1 - p_0(L) + p_1(L)$ .*

*Proof.* By the Alexander duality theorem we have

$$\begin{aligned}p_1(S^3 - L) - p_2(S^3 - L) &= p_1(L) - (p_0(L) - 1) \\ &= 1 - p_0(L) + p_1(L) .\end{aligned}$$

THEOREM 8. *Let  $L$  be a polyhedron in  $S^3$  that does not separate  $S^3$ . Then for each 1-cycle  $l$  on  $L$  we have*

$$E_d(l) = (0), \text{ if } d < 1 - p_0(L) + p_1(L) .$$

REMARK. A greater number than  $1 - p_0(L) + p_1(L)$  can be obtained, if one (or more) component of  $L$  is contractible in itself.

The following Theorem 9 and Theorem 10 are corollaries of Theorem 2 and Theorems 5 and 6, respectively.

THEOREM 9. *Let  $L$  be a polyhedron in 3-sphere  $S^3$  that does not separate  $S^3$ . Then for each 1-cycle  $l$  on  $L$  we have*

$$\begin{cases} (E_d(l))^\circ = (0), \text{ if } d < p_1(L), \text{ and} \\ (E_d(l))^\circ = (1), \text{ if } d \geq p_1(L) . \end{cases}$$

THEOREM 10. *Let  $L$  be a polyhedron in 3-sphere  $S^3$  that does not separate  $S^3$ . Assume that  $p_1(L) > 0$  and let  $\mu = p_0(L)$ . Then we have*

$$\begin{cases} E_0[l] = 0, \text{ and} \\ E_d[l] \subset (1 - t_1, \dots, t - t_\mu)^{p_1(L)-d}, \text{ if } 1 \leq d < p_1(L) . \end{cases}$$

*Especially we have*

$$\begin{cases} E_0(l) = 0, \text{ and} \\ E_d(l) \subset (1 - t)^{p_1(L)-d}, \text{ if } 1 \leq d < p_1(L) . \end{cases}$$

4. EXAMPLES. Let  $L_1$  be a  $\theta$ -curve, which is trivially imbedded in  $S^3$ . Then the fundamental group  $G_{L_1}$  of  $S^3 - L_1$  is a free group of rank 2. From this it follows that for each 1-cycle  $l$  on  $L_1$  we have

$$\begin{cases} E_d(l) = (0), \text{ if } d < 2 , \\ E_d(l) = (1), \text{ if } d \geq 2 . \end{cases}$$

Now let  $L_2$  be a  $\theta$ -curve in  $S^3$ , which is shown in Fig. 1.

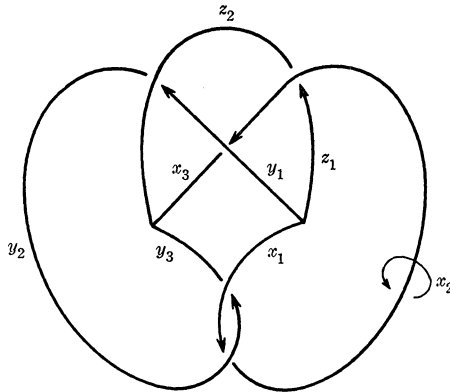


FIGURE 1

Then a presentation of the fundamental group  $G_{L_2}$  of  $S^3 - L_2$  is as follows:

$$(x_1, y_1, y_2, y_2x_1y_1y_2^{-1}x_1y_2y_1^{-1}y_2^{-1}x_1^{-1}y_2y_1^{-1}x_1^{-1}y_2^{-1}x_1y_2x_1^{-1} = 1).$$

Now let  $l$  be a 1-cycle on  $L_2$  and suppose that  $\text{link}(x_1, l) = c_1$  and  $\text{link}(y_1, l) = \text{link}(y_2, l) = c_2$ . Then we have

$$A(G_{L_2}, \psi) \approx (t^{c_1+c_2} + t^{c_2} + 1, 2, 0).$$

Hence, we have

$$\begin{cases} E_d(l) = (0), & \text{if } d < 2, \\ E_2(l) = (t^{c_1+c_2} + t^{c_2} + 1, 2), \\ E_d(l) = (1), & \text{if } d > 2. \end{cases}$$

This means that the position of  $L_1$  and that of  $L_2$  in  $S^3$  are topologically inequivalent. Note that any one of three simple closed curves on  $L_2$  is a trivial knot in  $S^3$ . (The example  $L_2$  was also discussed in [4], but there was a mistake in calculation, that was pointed out by R. H. Fox to the author of the paper. Of course the underlying theory in that paper is also different to that of this paper as noted in the introduction.) We may also note that generally  $E_2(l)$  of  $l$  on  $L_2$  in  $S^3$  is not a principal ideal, that generally  $E_2(l)$  does not satisfy the symmetricity property, that appears for knots and links in  $S^3$ . There are several of this kind of example in [6], too.

In another paper the author of the paper will prove that for any integral polynomial  $f(t)$  with  $f(t) = \pm 1$  there exists a  $\theta$ -curve  $L$  in  $S^3$  and a 1-cycle  $l$  on  $L$  such that  $E_2(l) = (f(t))$ .

5. On knots and links. Let  $K$  be an oriented polyhedral ( $n -$

2)-sphere in  $S^n$  ( $n \geq 3$ ) and  $k$  an  $(n-2)$ -cycle on  $K$  such that  $k = ck'$ , where  $k'$  is the fundamental cycle of  $K$ . Consider the fundamental group  $G_K$  of  $S^n - K$  and its abelianization

$$\alpha: G_K \rightarrow H_K = (t': \quad).$$

We choose the generator  $t'$  in such a way that for each  $g \in G_K$

$$g^\alpha = (t')^{\text{link}(g, k')}.$$

On the other hand, we have

$$g^\psi = t^{\text{link}(g, k)} = t^{c(\text{link}(g, k'))}.$$

Now define a homomorphism  $\sigma$  of  $H_K$  into  $H$  by  $(t')^\sigma = t^c$ . Then we have  $\psi = \sigma\alpha$ . From this it follows that

$$E_d(k)(t) = E_d(k')(t^c)$$

for each  $d \geq 0$ . Further, since  $H_K$  is infinite cyclic, we have  $E_0(k) = (0)$  and  $(E_d(k))^\circ = (1)$  for  $d \geq 1$ . If  $n = 3$ , we have

$$E_1(k)(t) = (\Delta_K(t^c)),$$

where  $\Delta_K(t)$  is the Alexander polynomial of the oriented simple closed curve  $K$  in  $S^3$ .

Let  $L$  be an oriented polyhedral  $(n-2)$ -link with  $\mu$  components in  $S^n$  ( $n \geq 3$ ), i.e. an ordered collection of  $\mu$  number of mutually disjoint oriented polyhedral  $(n-2)$ -spheres in  $S^n$ . We assume that  $\mu \geq 2$ . Let  $L_i$  be the  $i$ th component of  $L$  for each  $i$  ( $i = 1, \dots, \mu$ ). Let  $l$  be an  $(n-2)$ -cycle on  $L$  and let  $l'_i$  be as follows:

$$\begin{cases} l'_i | L_i = \text{the fundamental cycle on } L_i, \text{ and} \\ l'_i | L_j = 0, \text{ if } i \neq j. \end{cases}$$

Hence, we have  $l = \sum_{i=1}^{\mu} c_i l'_i$ . Let  $l_i = c_i l'_i$  and  $l' = \sum_{i=1}^{\mu} l'_i$ . Consider the fundamental group  $G_L$  of  $S^n - L$  and its abelianization

$$\alpha: G_L \rightarrow H_L,$$

which is a free abelian group of rank  $\mu$ . We choose the generator  $t'_i$  ( $i = 1, \dots, \mu$ ) in such a way that for each  $g \in G_L$

$$g^\alpha = \prod_{i=1}^{\mu} (t'_i)^{\text{link}(g, l'_i)}.$$

On the other hand, we have

$$g^{\psi_0} = \prod_{i=1}^{\mu} t_i^{\text{link}(g, l_i)} = \prod_{i=1}^{\mu} (t_i)^{c_i \text{link}(g, l'_i)}.$$

Define a homomorphism  $\sigma$  of  $H_L$  into  $H_0$  by  $(t'_i)^\sigma = t_i^{c_i}$  for each  $i$  ( $i = 1, \dots, \mu$ ). Then we have  $\psi_0 = \sigma\alpha$ . From this it follows that



$$E_d[l](t_1, \dots, t_\mu) = E_d[l'](t_1^{c_1}, \dots, t_\mu^{c_\mu}) .$$

Further we have  $E_0(l) = 0$ ,  $(E_d(l))^\circ = (0)$  for  $d < \mu$ , and  $(E_d(l))^\circ = (1)$  for  $d \geq \mu$ . If  $n = 3$ , we have

$$E_1[l](t_1, \dots, t_\mu) = (1 - t_1^{c_1}, \dots, 1 - t_\mu^{c_\mu}) \Delta_L(t_1^{c_1}, \dots, t_\mu^{c_\mu}) ,$$

where  $\Delta_L(t_1, \dots, t_\mu)$  is the Alexander polynomial of the link  $L$  in  $S^3$  and, hence

$$E_1(l)(t) = (1 - t^c) \Delta_L(t^{c_1}, \dots, t^{c_\mu}) ,$$

where  $c = \text{g.c.d.}(c_1, \dots, c_\mu)$ .

REMARK. Further, it is proved by Shinohara and Sumners [5] that if  $n \geq 4$ ,  $E_d(l) = (0)$  for  $d < \mu$ .

**6. On a Torres' formula for a link.** The following consideration may be interesting: Suppose that  $L$  is an oriented link with multiplicity  $\mu (> 1)$  in  $S^3$  and  $L_i$  the  $i$ th component of  $L$ . ( $i = 1, 2, \dots, \mu$ ). Let  $l$  be a 1-cycle on  $L$  and express  $l$  as  $\sum_{i=1}^{\mu} l_i$  as before. Denote  $\sum_{i=1}^{\mu-1} l_i$  by  $l^*$ . Now let  $L^\# = L - L_\mu$  and let  $l^\#$  be the 1-cycle on  $L^\#$  such that  $l^\# = l | L^\#$ . Hence we have  $l^\# = l^* | L^\#$ . Let  $H_0$  be a multiplicative free abelian group of rank  $\mu$ , i.e.  $H_0 = \prod_{i=1}^{\mu} H_i$ , where  $H_i = (t_i: ) (i = 1, 2, \dots, \mu)$ . Let  $H_0^\# = \prod_{i=1}^{\mu-1} H_i$ .

Now let  $\psi_0$  be the homomorphism of  $G_L$  into  $H_0$  such that for each  $g \in G_L$

$$g^{\psi_0} = \prod_{i=1}^{\mu} t_i^{\text{link}(g, l_i)} ,$$

$\psi_0^*$  the homomorphism of  $G_L$  into  $H_0$  such that for each  $g \in G_L$

$$g^{\psi_0^*} = \prod_{i=1}^{\mu-1} t_i^{\text{link}(g, l_i)}$$

and  $\psi_0^\#$  the homomorphism of  $G_L^\#$  into  $H_0^\#$  such that for each  $g \in G_L^\#$

$$g^{\psi_0^\#} = \prod_{i=1}^{\mu-1} t_i^{\text{link}(g, l_i)} .$$

Let  $c_i$  be the coefficient of  $l$  on  $L_i (i = 1, \dots, \mu)$ . Then we have the following theorem.

**THEOREM 11.** *We have*

$$E_1[l^*, \psi_0^*] = (t_1^{c_1 l_{1\mu}} \dots t_{\mu-1}^{c_{\mu-1} l_{\mu-1\mu}} - 1) E_1[l, \psi_0^\#] ,$$

where  $l_{ij} = \text{link}(L_i, L_j) (i, j = 1, 2, \dots, \mu)$ .

*Proof.* Let

$$E_i[l, \psi_0] = (t_1^{c_1} - 1, \dots, t_{\mu}^{c_{\mu}} - 1) \Delta_L(t_1^{c_1}, \dots, t_{\mu}^{c_{\mu}}).$$

Then we have

$$E_i[l^*, \psi_0^*] = (t_1^{c_1} - 1, \dots, t_{\mu-1}^{c_{\mu-1}} - 1) \Delta_L(t_1^{c_1}, \dots, t_{\mu-1}^{c_{\mu-1}}, 1).$$

On the other hand, we have

$$E_i[l^*, \psi^*] = (t_1^{c_1} - 1, \dots, t_{\mu-1}^{c_{\mu-1}} - 1) \Delta_{L^*}(t_1^{c_1}, \dots, t_{\mu-1}^{c_{\mu-1}}).$$

A Torres' formula of the Alexander polynomial of links ([7]) is as follows:

If  $\mu = 2$ , then

$$\Delta_L(t_1, 1) = \Delta_{L^*}(t_1)(t_1^{12} - 1)/(t_1 - 1).$$

If  $\mu > 2$ , then

$$\Delta_L(t_1, \dots, t_{\mu-1}, 1) = (t_1^{1\mu} \dots t_{\mu-1}^{1\mu} - 1) \Delta_{L^*}(t_1, \dots, t_{\mu-1}).$$

Hence if  $\mu > 2$ , the statement of the theorem follows immediately. If  $\mu = 2$  and  $c_1 \neq 0$ , we have

$$\begin{aligned} E_i[l^*, \psi_0^*] &= (t_1^{c_1} - 1) \Delta_L(t_1^{c_1}, 1) \\ &= (t_1^{c_1 t_{12}} - 1) \Delta_{L^*}(t_1^{c_1}) = (t_1^{c_1 t_{12}} - 1) E_i[l^*, \psi_0^*]. \end{aligned}$$

The statement of the theorem is trivial, if  $c_1 = 0$  and  $\mu = 2$ . Thus the proof of the theorem is complete.

REMARK. Theorem 11 can be proved directly and the Torres' theorem can be obtained as a corollary to this theorem.

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