PACIFIC JOURNAL OF MATHEMATICS
Vol. 42, No. 1, 1972

PRIMARY GROUPS WHOSE SUBGROUPS OF SMALLER CARDINALITY ARE DIRECT SUMS OF CYCLIC GROUPS

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Let G denote a primary abelian group. The conjecture (partially supported by a theorem of Nunke) that there exists, for each infinite cardinal m, a group G of cardinality m that is not a direct sum of cyclic groups but has the property that each subgroup of G having cardinality less than m is a direct sum of cyclic groups is shown to be false. More specifically, it is shown that if a primary group has cardinality $\aleph_\omega$ and each subgroup of smaller cardinality is a direct sum of cyclic groups, then so is the group.

Further, we show that if G is the union of a countable chain of pure subgroups, then G is a direct sum of cyclic groups if and only if the subgroups in the given chain are direct sums of cyclic groups.

All groups considered here are primary abelian groups. For simplicity of terminology, we say that the group G is an m-group if G has cardinality m and each subgroup of G having cardinality less than m is a direct sum of cyclic groups. The following problem, in one form of the other, has received considerable attention in recent years.

PROBLEM. For which cardinals m is evely m-group a direct sum of cyclic groups?

That every m-group is a direct sum of cyclic groups for a finite cardinal m follows immediately from the well-known structure of finite abelian groups. Prüfer [8] discovered early two interesting examples of an $\aleph_0$-group that is not a direct sum of cyclic groups: one of his examples was divisible and the other was reduced. Each of the two examples, but especially the reduced one, is often called the Prüfer group. Recently, Nunke [7] has shown, for any non-negative integer n, the existence of an m-group that is not a direct sum of cyclic groups with $m \geqq \aleph_n$. It has been informally conjectured that for every infinite cardinal m there exists an m-group that is not a direct sum of cyclic groups. However, the following theorem shows that this is not the case.

THEOREM 1. *Let G be an arbitrary primary abelian group. If*

G is an $\aleph_\omega$-group, then G is necessarily a direct sum of cyclic groups.

Proof. Let G be a group of cardinality $\aleph_\omega$ such that each subgroup of G having cardinality less than $\aleph_\omega$ is a direct sum of cyclic groups. Since any infinite subgroup can be imbedded in a pure subgroup of the same cardinality, it follows at once that G is the union of an ascending sequence of pure subgroups H_n of G such that $|H| = \aleph_n$ for $0 \leqq n < \omega$. For each $n < \omega$, let

$$H_n = \sum_{i \in I(n)} \{g_i\} .$$

For simplicity of notation, let μ denote the smallest ordinal having cardinality $\aleph_\omega$. We claim that there exist subgroups A_α of G, for $\alpha < \mu$, such that

(0) $A_0 = 0$.
(1) A_α is pure in G for each $\alpha < \mu$.
(2) $\{A_\alpha, H_n\}$ is pure in G for each $\alpha < \mu$ and each $n < \omega$.
(3) $A_{\alpha+1} \supseteqq A_\alpha$ for each α such that $\alpha + 1 < \mu$.
(4) $A_{\alpha+1}/A_\alpha$ is countable for each α such that $\alpha + 1 < \mu$.
(5) $A_\alpha \cap H_n = \sum_{i \in I(n,\alpha)} \{g_i\}$ for $\alpha < \mu$ and $n < \omega$, where $I(n, \alpha)$ is a subset of $I(n)$.
(6) $A_\beta = \bigcup_{\alpha<\beta} A_\alpha$ if β is a limit ordinal less than μ.
(7) $G = \bigcup_{\alpha<\mu} A_\alpha$.

The proof of the existence of subgroups A_α, $\alpha < \mu$, satisfying conditions (0) – (7) is, of course, by transfinite induction, and it employs the back-and-forth technique utilized in [1], [2], [3], [4], [5], [6] and a number of other recent papers. The reader who is well acquainted with these papers may need no further details concerning the existence of the subgroups A_α satisfying conditions (0) – (7); however, we shall include a brief outline of the proof.

Let $\gamma < \mu$, and suppose that we have chosen a subgroup A_α of G for each $\alpha < \gamma$ such that conditions (0) – (6) hold when μ is replaced by γ. We wish to select A_γ such that it is compatible with these conditions also. There are two cases.

Case 1. γ is a limit ordinal. In this case, we let $A_\gamma = \bigcup_{\alpha<\gamma} A_\alpha$. Observe that A_γ is pure in G since A_α is for each $\alpha < \gamma$. In the same way, $\{A_\gamma, H_n\}$ is pure G. Furthermore, we have honored condition (6) in the definition of A_γ. If we set $I(n, \gamma) = \bigcup_{\alpha<\gamma} I(n, \alpha)$ for each n, then it is easy to prove that $A_\gamma \cap H_n = \sum_{i \in I(n,\gamma)} \{g_i\}$. Hence all of the conditions (0) – (6) continue to hold if we pass from $\alpha < \gamma$ to $\alpha \leqq \gamma$.

Case 2. $\gamma - 1$ exists. We want to show that there exists a subgroup A_γ of G such that A_γ is a countable extension of $A_{\gamma-1}$ and such that

(i) A_γ is pure in G.

(ii) $\{A_\gamma, H_n\}$ is pure in G for each $n < \omega$.

(v) $A_\gamma \cap H_n = \sum_{i \in I(n,\gamma)}\{g_i\}$ for each n, where $I(n, \gamma)$ is a subset of $I(n)$.

Let B be any subgroup of G containing $A_{\gamma-1}$. If $B/A_{\gamma-1}$ is countable, there exists, according to Theorem 1 in [3], $C \supseteqq B$ with $|C/A_{r-1}| \leqq \aleph_0$ such that $C/A_{\gamma-1}$ is pure in $G/A_{\gamma-1}$ and such that

$$(C/A_{\gamma-1}, \{H_n, A_{\gamma-1}\}/A_{\gamma-1})/(\{H_n, A_{\gamma-1}\}/A_{\gamma-1})$$

is pure in $(G/A_{\gamma-1})/(\{H_n, A_{\gamma-1}\}/A_{\gamma-1})$ for each $n < \omega$. Due to the purity of $A_{\gamma-1}$ and $\{H_n, A_{\gamma-1}\}$, we conclude that $\{C, H_n\} = \{C, H_n, A_{\gamma-1}\}$ is a pure subgroup of G. Obviously, there is a countable extension $J(n)$ of the subset $I(n, \gamma-1)$ such that $C \cap H_n \subseteqq \sum_{i \in J(n)}(g_i)$. It follows from this if we replace B by C_i that there is an ascending sequence

$$C_0 \subseteqq C_1 \subseteqq C_2 \subseteqq \cdots \subseteqq C_k \subseteqq \cdots$$

of pure subgroups of G such that C_k is countable over $A_{\gamma-1}$ and such that $\{C_k, H_n\}$ is pure in G for all $k, n < \omega$. Letting

$$C_k \cap H_n \subseteqq \sum_{i \in j(n,k)} \{g_i\}$$

where $J(n, k)$ is a countable extension of the subset $I(n, \gamma-1)$ of $I(n)$, we choose $C_{k+1} \supseteqq \sum_{i \in J(k,n)}\{g_i\}$ for all n. Define $A_\gamma = \bigcup_{k<\omega} C_k$ and observe that if we set $I(n, \gamma) = \bigcup_{k<\omega} J(n, k)$, then

$$A_\gamma \cap H_n = \sum_{i \in I(n,\gamma)} \{g_i\} .$$

Thus conditions (0) – (6) remain valid for $\alpha < \gamma$. We have ignored condition (7) in the selection of A_γ, but there is no problem in choosing the A_α's such that they exhaust G. For example, if the index set $I(n)$ is chosen to be the set of ordinals less than $\aleph_n$, then we can easily carry along the condition that $\alpha \in I(n, \alpha)$ for all $n < \omega$ provided that $\alpha \in I(n)$.

In order to show that G is a direct sum of cyclic groups it remains only to show that A_α is a direct summand of $A_{\alpha+1}$ for each $\alpha < \mu$. Since A_α is pure and since $A_{\alpha+1}/A_\alpha$ is countable, it is enough to show that $A_{\alpha+1}/A_\alpha$ is without elements of infinite height, for it is well known (and quite easy to prove) that a countable primary group without elements of infinite height is pure-projective. Suppose that $x + A_\alpha \in p^\omega(A_{\alpha+1}/A_\alpha) \subseteqq p^\omega(G/A_\alpha)$. Then $x + A_\alpha \in p^\omega(\{A_\alpha, H_n\}/A_\alpha)$ for any n such that $x \in H_n$ because $\{A_\alpha, H_n\}$ is pure in G. However, $p^\omega(\{A_\alpha, H_n\}/A_\alpha) = 0$ since $\{A_\alpha, H_n\}/A_\alpha \cong H_n/(A_\alpha \cap H_n)$ is a direct sum of cyclic groups. Letting $A_{\alpha+1} = A_\alpha + C_\alpha$, we have that $G = \sum_{\alpha<\mu} C_\alpha$, and the theorem is proved.

The proof of the above theorem is obviously valid for the following more general result.

THEOREM 2. *Suppose that the ordinal* α *is the least upper bound of a* (countable) *sequence of smaller ordinals. Let the primary group* G *be a group of cardinality* $\aleph_\alpha$. *If each subgroup of* G *having cardinality less than* $\aleph_\alpha$ *is a direct sum of cyclic groups, then* G *is a direct sum of cyclic groups.*

A quick review of the proof of Theorem 1 also reveals the following result.

THEOREM 3. *If the primary group* G *is the union of a countable chain of pure subgroups, then* G *is a direct sum of cyclic groups if the subgroups are.*

The following corollary is well known in case the basic subgroups are countable but not (to our knowledge) in general.

COROLLARY 1. *The union of a countable chain of basic subgroups of a primary group* G *is again a basic subgroup of* G.

We can glean from the proof of Theorem 1 yet a little more; the results were actually discovered the way they are presented here.

THEOREM 4. *If the primary group* G *is the set-theoretic union of a countable number of pure subgroups* G_n, *then* G *is a direct sum of cyclic groups if* G_n *is for each* n.

In the proof of the above theorems we have used Prüfer's theorem that a countable primary group without elements of infinite height is a direct sum of cyclic groups, but essentially nothing beyond that is required. As an application of Theorem 4, we give a new and very simple proof of Kulikov's subgroup theorem.

COROLLARY 2 (Kulikov). *If the primary group* G *is a direct sum of cyclic groups, then any subgroup of* G *is a direct sum of cyclic groups.*

Proof. Suppose that G is a direct sum of cyclic groups and let H be a subgroup of G. Let G_n be an ascending sequence of pure subgroups of G whose union is G where G_n is bounded by p^n. Set $A_n = H \cap G_n$ for each n. Let $B_n \supseteq A_n$ be maximal in H with respect to $B_n \cap p^n H = 0$. It is a simple exercise to show that B_n is pure in H. Further, B_n is a direct sum of cyclic groups since it is bounded. Since $B_n \supseteqq A_n$, it follows immediately that H is the set theoretic union of its subgroups B_n. Applying Theorem 4, we have that H is a direct sum of cyclic groups.

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Received February 17, 1970 and in revised form December 14, 1971. This research was supported by NSF grant GP-12318.

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