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INTERPOLATION SETS FOR ANALYTIC FUNCTIONS

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Let U be a bounded open subset of the complex plane C. Criteria are obtained for a subset E of $\bar{U}$ to be an interpolation set for the algebra of all bounded analytic functions on U extending continuously to E.

In the case where U is the open unit disc Δ, this problem was treated by Détraz [3]. She showed that if E is a subset of the unit circle T then every bounded continuous function on E is the restriction of a bounded analytic function on Δ, extending continuously to $E \cup (T/\bar{E})$, if and only if E has measure zero. We extend this result to any U with connected complement, replacing linear measure on T by harmonic measure (Theorem 1). For the general case the same method yields a criterion in terms of representing measures for $A(U)$ (Theorem 2). Finally in Theorem 3 we use a localization argument to sharpen Theorem 1 and also treat the case where E contains points of U as well as ∂U.

NOTATION. If S is a plane set then $\bar{S}$ denotes its closure and ∂S its boundary. $A(U)$ denotes the algebra of all continuous functions on $\bar{U}$, analytic on U; $H^\infty(U)$ denotes the algebra of all bounded analytic functions on U; $H_E^\infty(U)$ denotes the algebra of all bounded continuous functions on $U \cup E$ which are analytic on U. If $y \in \bar{U}$, a *representing measure* for y with respect to $A(U)$ is a positive borel measure μ on $\bar{U}$ such that $f(y) = \int f d\mu$ for all $f \in A(U)$. We denote by $\|f\|$ the supremum of the function f over its domain of definition. $\Delta(z, \delta)$ denotes the disc with center z and radius δ.

We say that a set $S \subseteqq U \cup E$ is an *interpolation set* for $H_E^\infty(U)$ if for any bounded continuous f on S we can find $g \in H_E^\infty(U)$ with $g|S = f$. We say S is a *peak interpolation set* for $H_E^\infty(U)$ if for any bounded continuous f on S, and open set $V \supseteqq S$, and any $\varepsilon > 0$, we can find $g \in H_E^\infty(U)$ with $g|S = f$, $\|g\| \leqq \|f\|$, and $|g| < \varepsilon$ on $U \backslash V$.

THEOREM 1. *Suppose $C \backslash U$ is connected. Let F be a subset of ∂U with zero harmonic measure for each point of U (with respect to U). Then F is a peak interpolation set for $H_F^\infty(U)$.*

The proof follows from the following lemma.

LEMMA. *With U and F as in the theorem, let X be a compact* *subset of* $\bar{U}$, *W a neighborhood of X, and* $\varepsilon > 0$. *Then we can find* $f \in H^\infty_F$ *with* $\|f\| \leqq 2$, $|1 - f| < \varepsilon$ *on* $F \cap X$, *and* $|f| < \varepsilon$ *on* $U \backslash W$.

Proof. We can find a positive harmonic function σ on U such that $\sigma(\zeta) \to \infty$ as $\zeta \to z$, $\zeta \in U$, for each $z \in F$. Let τ be a harmonic conjugate to σ on U, and let $\theta = \sigma + i\tau$, an analytic function on U. Put $h = \theta/(\theta + 1)$. Since θ has positive real part, $h \in H^\infty(U)$ with $\|h\| \leqq 1$. Moreover $h(\zeta) \to 1$ as $\zeta \to z$, $\zeta \in U$, for each $z \in F$; hence we can regard h as an element of $H^\infty_F(U)$, with $h = 1$ on F.

Now let φ be a continuously differentiable function which is 1 on a neighborhood of X and zero outside W, with $\|\varphi\| = 1$. Then the function

$$g_n(\zeta) = \varphi(\zeta)h^n(\zeta) + \frac{1}{\pi}\int_U \frac{h^n(z)}{z - \zeta}\frac{\partial\varphi}{\partial\bar{z}}dm(z)$$

is in $H^\infty_F(U)$. (See [4], p. 210.) Moreover

$$\|g_n - \varphi h^n\| \leqq \frac{1}{\pi}\left\|\frac{\partial\varphi}{\partial\bar{z}}\right\| \|h^n\|_{L^3(U)} \sup_\zeta \left\|\frac{1}{z - \zeta}\right\|_{L^{3\backslash 2}(U)} .$$

The last term is bounded by a constant depending only on U, and $\|h^n\|_{L^3} \to 0$ as $n \to \infty$ since $|h| < 1$ in U. Choose n so that $\|g_n - \varphi h^n\| < \varepsilon$ and put $f = g_n$. Then f satisfies the requirements of the lemma.

Theorem 1 follows from the lemma in exactly the same way as Theorem 1 follows from Lemma 2 in [2]. (For an alternative approach see the proof of Theorem 4.3 of [3]).

We observe that if $A(U)$ is pointwise boundedly dense in $H^\infty(U)$ then using Theorem 2.1 of [5] we can modify the function f in the lemma so that it is in $H^\infty_{F\cup(\partial U\backslash\bar{F})}(U)$. Then we can prove that F is a peak interpolation set for $H^\infty_{F U(\partial u\backslash\bar{F})}(U)$.

In the general situation (where $C\backslash U$ need not be connected) the same method yields the following result. If $y \in U$ we denote by M_y the set of all (positive) representing measures for y with respect to $A(U)$ on $\bar{U}$. We assume U is connected.

THEOREM 2. *Let* $y \in U$ *and* $F \subseteqq \partial U$. *Suppose there is a decreasing sequence* $\{V_n\}$ *of open sets containing F, such that* $\mu(V_n) \to 0$ *uniformly for* $\mu \in M_y$.

Then F is a peak interpolation set for $H^\infty_F(U)$.

Proof. We may suppose $\mu(V_n) < 2^{-n}$ for each $\mu \in M_y$. For each n let $\{g_{nk}\}$ be an increasing sequence of nonnegative continuous functions converging to the characteristic function of V_n. Then $\int g_{nk}d\mu < 2^{-n}$

for $\mu \in M_y$ and so by Theorem II 2.1 of [4], we can find $h_{nk} \in A(U)$ with Re $h_{nk} \geqq g_{nk}$ on $\bar{U}$, Re $h_{nk}(y) < 2^{-n}$, and we can also suppose $\operatorname{Im} h_{nk}(y) = 0$. Passing to a subsequence we have $h_{nk} \to h_n$ as $k \to \infty$, pointwise in U, where h_n is analytic in U with $\operatorname{Re} h_n \geqq 1$ on $V_n \cap U$ and $|h_n(y)| \leqq 2^{-n}$, $\operatorname{Re} h_n \geqq 0$ on U, $\operatorname{Im} h_n = 0$. By Harnack's inequalities the series $\sum_{n=1}^{\infty} h_n$ converges pointwise on U to an analytic function h such that $\operatorname{Re} h \geqq 0$ on U and $\operatorname{Re} h(\zeta) \to \infty$ as $\zeta \to z, \zeta \in U$, for each $\zeta \in F$.

The rest of the proof follows Theorem 1.

Again we observe that if $A(U)$ is pointwise boundedly dense in $H^\infty(U)$ then the interpolation can be achieved by functions in $H^\infty_{F \cup (\partial U \setminus \bar{F})}(U)$. Moreover under the same assumption the converse to Theorem 2 holds, for if f is as in the definition of peak interpolation set, with V chosen so that $y \notin V$, and $g = 1$, then we can choose a neighborhood W of F so that $|1 - f| < \varepsilon$ on $U \cap W$; by Theorem 5.1 of [1] we can approximate f to within ε on compact subsets of W by a sequence $\{f_n\}$ in $A(U)$ with $||f_n|| \leqq 1$, so that $\mu(W)$ is small for all $\mu \in M_y$.

The question naturally arises: suppose $\mu(F) = 0$ for all $\mu \in M_y$. Must there exist open sets $V_n \supseteqq F$ such that $\mu(V_n) \to 0$ uniformly for $\mu \in M_y$? This is easily verified if F is σ-compact (in this case the conclusion of Theorem 2 can be deduced from the fact that each compact subset of F is a peak interpolation set for $A(U)$). We have no information of the general case.

LEMMA 2. *Let F be a subset of ∂U such that for each $z \in F$ there exists $\delta > 0$ such that $F_z = F \cap \{w: |w - z| \leqq \delta/2\}$ is a peak interpolation set for $H^\infty_{F \cap \Delta(z,\delta)}(U \cap \Delta(z, \delta))$, then F is a peak interpolation set for $H^\infty_F(U)$.*

Proof. First we show that F_z is a peak interpolation set for $H^\infty_F(U)$. Let g be a bounded continuous function on F_z, let $\varepsilon > 0$, and let V be an open neighborhood of F_z. Choose $f \in H^H_{F \cap \Delta(z,\delta)}(U \cap \Delta(z, \delta))$ such that $f = g$ on F_z, $||f|| = ||g||$, and $|f| < \varepsilon$ outside $V \cap \{w: |w - z| < 3\delta/4\}$.

Choose a continuously differentiable function φ such that $\varphi = 1$ in a neighborhood of $\{w: |w - z| \leqq 3\delta/4\}$ and supp $\varphi \subseteqq \{w: |w - z| < \delta\}$. Define

$$f_1(w) = f(w)\varphi(w) + \frac{1}{\pi}\int_{U \cap \Delta(z,\delta)} \frac{f(f)}{\zeta - w}\frac{f(\zeta)}{\zeta - w} dm(\zeta)$$

where $f(w)$ is defined to be zero outside $(F \cup U) \cap \Delta(z, \delta)$. Then $f_1 \in H_F^\infty(U)$ and given $t > 0$ we can choose $\varepsilon > 0$ so that $||f_1 - f|| < t$. Moreover $||f_1|| \leqq A||f||$, where A is an absolute constant. (See [4], p. 210.) Then we have $|f_1 - f| < \varepsilon$ on F_z and $|f_1| < \varepsilon$ on $U \backslash V$. It now follows by a standard argument (see e.g. [2], Theorem 1), that F_z is a peak interpolation set for $H_F^\infty(U)$.

Now let V be an open set containing F. Shrinking V if necessary we may suppose that V is contained in the union of the discs $\Delta(z, \delta)$, $z \in F$, constructed above. This implies that for any compact set $K \subseteqq V$, we have $K \cap F \subseteqq \bigcup_{i=1}^n F_{z_i}$ for some $z_1, \cdots, z_n \in F$, which easily implies that $K \cap F$ is a peak interpolation set for $H_F^\infty(U)$. The lemma now follows by the argument used to deduce Theorem 1 from Lemma 3 in [2].

We say that U is locally simple connected at a point $z \in \partial U$ if there exists $\delta > 0$ such that $C \backslash (U \cap \Delta(z, \delta))$ is connected. For example, if the diameters of the components of $C \backslash U$ are bounded away from zero then U is locally simply connected at each point of ∂U. (Note that $U \cap \Delta(z, \delta)$ is not required to be connected; we only require that each component be simply connected.)

THEOREM 3. *Let S be a subset of $\bar{U}$ such that U is locally simply connected at each point of $S \cap \partial U$. Then S is an interpolation set for $H_{S \cap \partial U}^\infty(U)$ if and only if*:

(i) *$U \cap S$ is an interpolating sequence for $H^\infty(U)$,*

(ii) *$S \cap \partial U$ has zero harmonic measure for each point of U, with respect to U.*

Proof. Assume first that S is an interpolation set for $H_{S \cap \partial U}^\infty(U)$. A simple normal family argument shows that (i) holds.

Now let $y \in U$ and choose $f \in H_{S \cap \partial U}^\infty(U)$ such that $||f|| \leqq 1$, $f = 0$ on $S \cap \partial U$, and $f(y) \neq 0$. Then $-\log |f|$ is a positive superharmonic function on U, tending to ∞ at each point of $S \cap \partial U$, and finite at y. Thus $S \cap \partial U$ has zero harmonic measure for y with respect to U which proves (ii).

Now assume (i) and (ii) hold, and let f be a bounded continuous function on S with $||f|| \leqq 1$. By Lemma 2 and Theorem 1, $\partial U \cap S$ is a peak interpolation set for $H_{\partial U \cap S}^\infty(U)$ so we can find $h \in H_{\partial U \cap S}^\infty(U)$ with $||h|| \leqq 1$ and $h = f$ on $\partial U \cap S$. Let $g_1 = f - h$ on S, then $g_1 = 0$ on $\partial U \cap S$ so that for any $\varepsilon > 0$ we can find $F \in H_{\partial U \cap S}^\infty(U)$ so that $F = 0$ on $\partial U \cap S$, $|1 - F| < \varepsilon$ on $\{z \in S: |g_1(z)| > \varepsilon\}$, $||F|| \leqq 2$. Then $|Fg_1 - g_1| \leqq 3\varepsilon$ on S. Choose $G \in H^\infty(U)$ so that $||G|| \leqq M||g_1|| \leqq 2M$ and $G = g_1$ on $S \cap U$, where M is the interpolation constant of $S \cap U$; then $FG \in H_{\partial U \cap S}^\infty$ and satisfies $|FG - g_1| \leqq 3\varepsilon$ on S. Let $\tilde{f} = FG + h \in H_{\partial U \cap S}^\infty(U)$, then $|\tilde{f} - f| \leqq 3\varepsilon$ on S and $||\tilde{f}|| \leqq 4M + 1$, so the theorem follows by choosing ε with $3\varepsilon < 1$.

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