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ON THE RANGE SETS OF H^p FUNCTIONS

LEON BROWN AND LOWELL HANSEN

The object of this paper is to show that, even though every function in $H^p (0 < p < \infty)$ has a nontangential limit almost everywhere, "most" functions in H^p have surprisingly "wild" behavior at the boundary. The proof uses category arguments and a method of constructing non-normal functions in H^p.

Preliminaries. We denote the open unit disc by Δ, and the boundary of Δ by T. If f is a complex valued function on Δ, then the *cluster set* of f at the point $P \in T$, denoted by $C(f, P)$, is the set of points w for which there exists a sequence $\{z_n\} \subset \Delta$ with $z_n \to P$ and $f(z_n) \to w$. One easily sees that

$$C(f, P) = \bigcap_k \overline{f(D_k)} ,$$

where $D_k = \{z \in \Delta : |z - P| < 1/k\}$. The *range set* of f at P, $R(f, P)$, is the set of points w such that there exists a sequence $\{z_n\} \subset \Delta$ with $z_n \to P$ and $f(z_n) = w$. Thus,

$$R(f, P) = \bigcap_k f(D_k) .$$

f is said to possess the *angular limit* (nontangential limit) α at $P \in T$ if f converges to α when restricted to each Stolz angle

$$\{z \in \Delta : |\arg(P - z) - \arg P| < \delta\} , \qquad 0 < \delta < \pi/2 .$$

We say that α is an *asymptotic value* of f at $P \in T$ if there exists a Jordan curve $\varphi : \{0 \leqq t < 1\} \to \Delta$ such that $\lim_{t \to 1^-} \varphi(t) = P$ and $\lim_{t \to 1^-} f[\varphi(t)] = \alpha$.

A function defined on D is *normal* if the collection $\{f \circ S : S \in \Gamma\}$ is a normal family of functions, where Γ is the collection of conformal maps of Δ onto itself. Any holomorphic function which omits two complex values is a normal function. It follows from a theorem of Lehto and Virtanen that if a function f is meromorphic on Δ and has two different asymptotic values at $z = 1$, then f is not normal:

THEOREM 1. [9, Theorem 2, p. 53] *Let f be meromorphic and normal in the simply-connected region G, and let f have an asymptotic value α at a boundary point P along a Jordan curve lying in the closure of G. Then f possesses the angular limit α at the point P.*

The reader is referred to [7] for an excellent presentation of the theory of H^p spaces. We shall make strong use of the following decomposition theorem.

THEOREM 2. *Every function $f \not\equiv 0$ of class H^p $(0 < p \leqq \infty)$ has a unique factorization of the form*

$$f = BSF ,$$

where B is a Blaschke product, S is a singular inner function, and F is an outer function for the class H^p. That is,

$$B(z) = z^n \prod_n \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \bar{a}_n z} \tag{1}$$

where $\{a_n\}$ are the zeros of f,

$$S(z) = \exp\left\{ - \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(z) \right\} \tag{2}$$

where μ is a singular measure, and

$$F(z) = e^{i\gamma} \exp\left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log \psi(t)\, dt \right\} \tag{3}$$

where $\psi(t) = |f(e^{it})|$, $\psi(t) \geqq 0$, $\log \psi \in L^1$ and $\psi \in L^p$. Conversely, every such product BSF belongs to H^p.

Main result.

THEOREM 3. *For each function f in H^p $(0 < p < \infty)$ outside a set of first category, the range set $R(f, e^{i\theta})$ at each boundary point omits at most one complex number. Thus the cluster set $C(f, e^{i\theta})$ at each boundary point is the full Riemann sphere.*

Proof. The first step in our proof is to show that for "most" functions in H^p, $C \backslash R(f, 1)$ contains at most one point. Let

$$D_k = \left\{ z \in \Delta : |z - 1| < \frac{1}{k} \right\} .$$

Then we define $A_p(n, k) = A(n, k) = \{f \in H^p \mid \exists w_1, w_2 \notin f(D_k)$ with $|w_i| \leqq n$, $i = 1, 2$, and $|w_1 - w_2| \geqq 1/n\}$. Any function in the complement of $\bigcup_{n,k} A(n, k)$ has the required property. Thus it is sufficient to show that $A(n, k)$ is nowhere dense in H^p. We prove this by showing that $A(n, k)$ is closed and has a dense complement.

(a) $A(n, k)$ is closed. If $f_j \in A(n, k)$ and $f_j \to f$ in H^p, we claim that $f \in A(n, k)$. We use the fact that H^p convergence implies uniform convergence on compact subsets of Δ [7, Lemma, p. 36]. If f is a constant function, f clearly is in $A(n, k)$. We assume that f is an open map. Let $w_j^1, w_j^2 \in \{z \mid |z| \leqq n\} \backslash f_j(D_k)$ and $|w_j^1 - w_j^2| \geqq 1/n$. By chooosing an appropriate subsequence we may assume that $w_j^1 \to w^1$ and $w_j^2 \to w^2$. It is clear that $|w^1 - w^2| \geqq 1/n$ and $|w^i| \leqq n$, $i = 1, 2$. If $w^1 \in f(D_k)$, choose $z^1 \in D_k$ such that $f(z^1) = w^1$. Then $f_j - w_j^1$ converges to $f - w^1$ uniformly on compact subsets of Δ. Let $N(z^1)$ be a neighborhood of z^1 such that $\overline{N(z^1)} \subset D_k$ and $w^1 \notin f(\partial N(z^1))$. An application of Rouche's theorem to the functions $f_j - w_j^1$ and $f - w^1$ implies that

$$w_j^1 \in f_j(N(z^1))$$

for j sufficiently large. This contradicts the fact that $w_j^1 \notin f_j(D_k)$. A similar argument shows that $w^2 \notin f(D_k)$. Thus we have shown that $A(n, k)$ is closed in H^p.

(b) The complement of $A(n, k)$ is dense. Let $f \in H^p$. By Theorem 1, we may write $f = BSF$. We shall approximate each of these factors as follows.

(i) If B is a finite product, put $B_n = B$. If B is an infinite product, put

$$B_n(z) = z^m \prod_{k=1}^{n} \frac{|a_k|}{a_k} \frac{a_k - z}{1 - \overline{a_k} z} .$$

Note that B_n converges uniformly on compact subsets to B, and on T, B_n is continuous and of modulus 1.

(ii) If $S \equiv 1$, we let $S_n \equiv 1$. Otherwise

$$S(z) = \exp\left\{ - \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t) \right\}$$

where μ is a singular positive measure. The set of measures with finite support is dense in the space of measures endowed with the weak* topology induced by the space of continuous functions. Thus, since H^p is separable, there exists a sequence of measures μ_n with finite support which converge to μ in the weak* topology. Let

$$S_n(z) = \exp\left\{ - \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu_n(t) \right\} .$$

We note that $\| S_n \|_\infty = 1$ and so $\{S_n\}$ forms a normal family. This fact, together with the pointwise convergence of S_n to S, implies that S_n conveges uniformly on compact subsets of Δ to S. Furthermore, on T, S_n is continuous and of modulus 1 except on the support of μ_n.

(iii) $$F(z) = e^{i\gamma} \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} \log \psi(t)dt\right\}$$

where $\psi(t) \geqq 0$, $\log \psi \in L^1$, $\psi \in L^p$, and $\psi(t) = |F(e^{it})| = |f(e^{it})|$ a.e. Let

$$\psi_n(t) = \begin{cases} t, \ 0 \leqq t \leqq \dfrac{1}{n} \\ |f(e^{it})|, \ \dfrac{1}{n} < t < 2\pi - \dfrac{1}{n} \\ (2\pi - t)^{-1/2p}, \ 2\pi - \dfrac{1}{n} \leqq t < 2\pi \end{cases}$$

and

$$F_n(z) = e^{i\gamma} \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} \log \psi_n(t)dt\right\} .$$

ψ_n has been defined so that $\psi_n \geqq 0$, $\log \psi_n \in L^1$, $\psi_n \in L^p$, $\log \psi_n \to \log \psi$ in L^1, and $\psi_n \to \psi$ in L^p. The L^1 convergence of $\log \psi_n$ to $\log \psi$ implies that F_n converges uniformly on compact subsets to F. In addition, since

$$\| F \| = \left(\frac{1}{2\pi}\int \psi(t)^p \, dt\right)^{1/p}$$

and $\psi_n \to \psi$ in L^p, we conclude that $\| F_n \| \to \| F \|$.

We assume that $p > 1$ and claim that $f_n = B_n S_n F_n$ converges to f in H^p. We recall that B_n, S_n, and F_n converge uniformly on compact subsets of Δ to B, S, and F respectively, and

$$\| B_n B_n F_n \| = \| F_n \| \longrightarrow \| F \| = \| f \| .$$

Hence f_n converges weakly to f in H^p and, since H^p $(1 < p < \infty)$ is a uniformly convex Banach space, $f_n \to f$ in H^p.

Let $n \geqq 1$ be fixed. Then since the support of μ_n is finite, there exists an arc $\{e^{i\theta}: |\theta| < \delta\}$ on which, except possibly at $\theta = 0$, $|S_n(e^{i\theta})| = 1$. Therefore, since $|B_n(e^{i\theta})| \equiv 1$ and $|F_n(e^{i\theta})| = \psi_n(\theta)$,

$$\lim_{\theta \to 0^+} f_n(e^{i\theta}) = 0 \quad \text{and} \quad \lim_{\theta \to 2\pi^-} f_n(e^{i\theta}) = \infty .$$

Let ϕ be a conformal mapping from $\overline{\Delta}$ onto $\overline{D_k}$ such that $\phi(1) = 1$. Thus $f_n \circ \phi$ has 0 and ∞ as asymptotic values at $z = 1$. It follows from Theorem 1 that $f_n \circ \phi$ is not normal and therefore $f_n \circ \phi(\Delta) = f_n(D_k)$ omits at most one value of the complex plane. Thus $f_n \notin A(n, k)$.

Suppose that $0 < p \leqq 1$. Then $H^2 \backslash A_2(n, k) \subset H^p \backslash A_p(n, k)$. We have already shown that $H^2 \backslash A_2(n, k)$ is dense in H^2, and so must be dense in the polynomials (in the H^2 norm). Since $\| \ \|_p \leqq \| \ \|_2$,

$H^2\backslash A_2(n, k)$ is also dense in the polynomials in the topology of H^p. Therefore $H^2\backslash A_2(n, k)$ is dense in H^p and so $H^p\backslash A_p(n, k)$ is dense in H^p. This completes the proof that $A(n, k)$ is nowhere dense.

To complete the proof, let $\{e^{i\theta_n}\}$ be a countable dense subset of T. We have just proved that for each fixed n, the set $W(n)$ of functions in H^p for which $C\backslash R(f, e^{i\theta_n})$ has at most one point is a residual set. Thus $\bigcap W(n)$ is a residual set. We claim that if $f \in \bigcap W(n)$, then $C\backslash R(f, e^{i\theta})$ is at most a singleton for every θ. If not, let $f \in \bigcap W(n)$ and $w_1, w_2 \in C\backslash R(f, e^{i\theta})$. Thus there exists $0<r<1$ such that $w_1, w_2 \notin f[\{z \in \Delta: |z - e^{i\theta}| < r\}]$. If $|e^{i\theta} - e^{i\theta_n}| < r$, then $w_1, w_2 \in C\backslash R(f, e^{i\theta_n})$, which is a contradiction. This completes the proof of the theorem.

We remark that $\{\theta \mid w \notin R(f, e^{i\theta})\}$ is an open subset of T. Hence if $C\backslash R(f, e^{i\theta})\}$ is at most a singleton for each θ, then $\bigcup C\backslash R(f, e^{i\theta})$ is at most countable.

We conclude with some historical remarks. A point ζ on the unit circle, T, is called an *ambiguous point* of f if f has two different asymptotic values at ζ. F. Bagemihl [1] proved that the set of ambiguous points of f is at most countable, even if f is an arbitrary function on Δ. F. Bagemihl and W. Seidel [2] proved that if E is any countable subset of T, then there exists a function, holomorphic and of bounded characteristic on Δ, for which every element of E is an ambiguous point. G. T. Cargo [5] has constructed such a function which is in H^p for all $p < \infty$. As we have shown in our theorem, the range set of any holomorphic function at an ambiguous point omits at most one complex value. Thus if we choose a dense countable subset of T, Cargo's construction yields a function with the "wild" behavior of Theorem 3.

Professor W. Seidel has kindly pointed out to us that the Picard-type behavior of a holomorphic function in a neighborhood of an ambiguous point is an old result of E. Lindelöf [10, p. 13].

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WAYNE STATE UNIVERSITY