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A DUALITY FOR QUOTIENT DIVISIBLE ABELIAN GROUPS OF FINITE RANK

DAVID M. ARNOLD

The usual duality for finite dimensional vector spaces induces a duality F on the category of torsion free quotient divisible abelian groups of finite rank with quasi-homomorphisms as morphisms. This duality preserves rank, is exact, hence preserves quasi-direct sums, sends free groups to divisible groups and conversely, and has the property that for all primes p, p-rank $FA =$ rank $A - p$-rank A.

A torsion free abelian group is *quotient divisible* if A has a free subgroup B such that A/B is the direct sum of a torsion divisible group and a group of bounded order. Let $\mathscr{C}$ be the category of quotient divisible abelian groups of finite rank (*rank* A is the cardinality of a maximal independent subset of A) with morphism sets $Q \otimes_Z \mathrm{Hom}(A, B)$, where Q is the field of rational numbers. Morphisms in $\mathscr{C}$ are quasi-homomorphisms of groups.

THEOREM A: There is a contravariant exact functor $F: \mathscr{C} \to \mathscr{C}$ such that F^2 is naturally equivalent to the identity functor on $\mathscr{C}$, rank $A =$ rank FA and A is free iff FA is divisible.

Let $R_p = \{m/n \in Q \mid (p, n) = 1\}$ be the localization of Z at a prime p and $\mathscr{C}_p = \{A_p = R_p \otimes_Z A \mid A \in \mathscr{C}\}$ be a category with morphism sets $Q \otimes_{R_p} \mathrm{Hom}(A_p, B_p)$. The duality F induces a duality on $\mathscr{C}_p$ which coincides with the duality given in [1].

For $A \in \mathscr{C}$, *p-rank* A is the Z/pZ dimension of A/pA.

COROLLARY B: *For all primes p, p-rank $FA =$ rank $A - p$-rank A.*

Notation is established in 1 and the relevant results of Beaumont-Pierce [2] are summarized in a series of lemmas. The proofs of Theorem A and Corollary B are contained in 2. Section 3 includes some easy consequences of the properties of the duality F.

1. Preliminaries. The ring of p-adic integers, p a prime, is denoted by R_p^* and Q_p^* is the quotient field of R_p^*, i.e., the p-adic completion of Q. There are subring inclusions $Z \subset R_p \subset Q \subset Q_p^*$ and $R_p \subset R_p^* \subset Q_p^*$ such that $R_p^* \cap Q = R_p$, $\cap \{R_p \mid p \text{ a prime}\} = Z$.

Each finite dimensional Q-vector space V may be regarded as a Q-subspace of $V_p^* = Q_p^* \otimes V$ by identifying v with $1 \otimes v$. If X is a subset of V and R a subring of Q_p^* then $RX = \{\sum r_i x_i \mid r_i \in R, x_i \in X\}$

is an R-submodule of V_p^*. Hence $ZX \subset R_pX \subset QX \subset V$ and $R_pX \subset R_p^*X \subset V_p^*$. Further, if A is a subgroup of V such that V/A is torsion then $R_p^*V = V_p^* = Q_p^*QA = Q_p^*A$ and rank $A = Q$-dimension of $V = Q_p^*$-dimension of $V_p^* = R_p^*$-rank of R_p^*A .

For the remainder of this note, V is a finite dimensional Q-vector space, X is a basis of V and δ_p is a Q_p^*-subspace of V_p^*. Define $(X, V, \delta) = V \cap (\cap \{R_p^*X + \delta_p \mid p \text{ is a prime}\})$.

LEMMA 1. *Let* $A = (X, V, \delta)$ *for some* X, V *and* δ.

(a) $R_pA = V \cap (R_p^*X + \delta_p)$;

(b) $R_p^*A = R_p^*X + \delta_p$ *and* $\delta_p = \cap \{p^i(R_p^*A) \mid i = 1, 2, \cdots\}$;

(c) $A \in \mathscr{C}$ *and* ZX *is a free subgroup of* A *with* A/ZX *torsion divisible*;

(d) *If* Y *is another basis of* V *and* $B = (Y, V, \delta)$ *then there are nonzero integers* m *and* n *with* $mA \subset B$ *and* $nB \subset A$.

Proof. Beaumont-Pierce [2], §5.

LEMMA 2. *Every* $A \in \mathscr{C}$ *is an* (X, V, δ) *for some* X, V *and* δ.

Proof. Choose V such that $A \subset V$, V/A torsion; let X be a maximal Z-independent subset of A with A/ZX torsion divisible and let $\delta_p = \cap \{p^i(R_p^*A) \mid i = 1, 2, \cdots\}$. Then $R_p^*A = R_p^*X + \delta_p$ and $R_pA = R_p^*A \cap V$ for all primes p. Hence $A = \cap \{R_pA \mid p \text{ prime}\} = \{X, V, \delta\}$.

Note that if $A = (X, V, \delta)$ then p-rank A = rank A—(Q_p^*-dimension of δ_n).

Let A and B be torsion free abelian groups. Call $\phi: A \to B$ a *quasi-homomorphism* if there is $0 \neq n \in Z$ with $n\phi \in \text{Hom}(A, B)$. Observe that $\{\phi \mid \phi: A \to B$ is a quasi-homomorphism$\}$ may be identified with $Q \otimes_Z \text{Hom}(A, B)$. The groups A and B are *quasi-isomorphic* ($A \dot{\sim} B$) if there are monomorphisms $f: A \to B$, $g: B \to A$ such that $B/f(A)$ and $A/g(B)$ are bounded.

Assume that $A = (X, V, \delta)$ and $B = (Y, U, \sigma)$ are objects of $\mathscr{C}$ and that $\phi: A \to B$ is a quasi-homomorphism. Then ϕ induces a unique Q-linear transformation $\lambda: V \to U$ since V/A and U/B are torsion. Define $\phi_p = 1 \otimes \lambda: V_p^* \to U_p^*$, a Q_p^*-linear transformation extending λ, hence ϕ. There is an integer n such that $n\phi_p(R_p^*A) \subset R_p^*B$ so that $\phi_p(\delta_p) \subset \sigma_p$ for all primes p.

Conversely if $\theta: V \to U$ is a Q-linear transformation such that $\theta_p(\delta_p) \subset \sigma_p$ (where $\theta_p = 1 \otimes \theta: V_p^* \to V_p^*$) for all primes p, then $\theta: A \to B$ is a quasi-homomorphism. Observe that if W is a basis of U with $\theta(X) \subset W$ then $\theta(A) \subset D = (W, U, \sigma)$. By Lemma 1.d, there is $0 \neq n \in Z$ with $n\theta(A) \subset nD \subset B = (Y, U, \sigma)$.

Note that a quasi-homomorphism $\phi: A \to B$ is a quasi-isomorphism iff $\lambda: V \to U$ is an isomorphism and $\phi_p(\delta_p) = \sigma_p$ for all primes p, where λ is the unique extension of ϕ and $\phi_p = 1 \otimes \lambda$.

We summarize some of the categorical properties of $\mathscr{C}$, as given by Walker [4]. Assume that $\phi: A \to B$ is a quasi-homomorphism and that $f = n\phi \in \mathrm{Hom}(A, B)$: ϕ is epic in $\mathscr{C}$ iff $B/f(A)$ is bounded; ϕ is monic in $\mathscr{C}$ iff f is monic and $0 \to A \xrightarrow{\phi} B \xrightarrow{\theta} C \to 0$ is exact in $\mathscr{C}$ iff ϕ is monic, θ is epic and $(\mathrm{im}\, f + \ker g)/(\mathrm{im}\, f) \cap (\ker g)$ is bounded, where $g = m\theta \in \mathrm{Hom}(B, C)$. The direct sum in $\mathscr{C}$ is the *quasi-direct sum* of groups, $A \dot{\oplus} B$, where $M = A \dot{\oplus} B$ iff there are non-zero integers m and n with $mM \subset A \oplus B$ and $n(A \oplus B) \subset M$. A group $A \in \mathscr{C}$ is *strongly indecomposable* if A is indecomposable in $\mathscr{C}$, i.e., $A = B \dot{\oplus} C$ implies that $B = 0$ or $C = 0$.

LEMMA 3. *Suppose that* $A_i = (X_i, V_i, \delta_i) \in \mathscr{C}$, $i = 1, 2, 3$. *Then* $0 \to A_1 \xrightarrow{\phi_1} A_2 \xrightarrow{\phi_2} A_3 \to 0$ *is exact in* $\mathscr{C}$ *iff* $0 \to V_1 \xrightarrow{\lambda_1} V_2 \xrightarrow{\lambda_2} V_3 \to 0$ *is an exact sequence of Q-vector spaces where* λ_i *is the unique extension of* ϕ_i, $i = 1, 2$.

Proof. Observe that ϕ_1 monic iff λ_1 monic; ϕ_2 epic iff λ_2 epic and $(\ker f_2 + \mathrm{im}\, f_1)/(\ker f_2) \cap (\mathrm{im}\, f_1)$ is bounded iff $\ker \lambda_2 = \mathrm{im}\, \lambda_1$ where $f_i = n_i\phi_i \in \mathrm{Hom}(A_i, A_{i+1})$ for $0 \neq n_i \in Z$, $i = 1, 2$.

2. A Duality for $\mathscr{C}$.

Let $\mathscr{V}$ denote the category of finite dimensional Q-vector spaces with Q-linear transformations as morphisms. Define $G: \mathscr{V} \to \mathscr{V}$ by $G(V) = V' = \mathrm{Hom}_Q(V, Q)$; and for $f \in \mathrm{Hom}_Q(V, U)$, $G(f) = f'$ is an element of $\mathrm{Hom}_Q(U', V')$ defined by $f'(\alpha) = \alpha f$. It is well-known that G is a contravariant exact functor naturally equivalent to the identity functor on $\mathscr{V}$, i.e., $(fg)' = g'f'$; if $0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$ is an exact sequence of Q-vector spaces then $0 \to W' \xrightarrow{g'} V' \xrightarrow{f'} U' \to 0$ is exact; and for each $V \in \mathscr{V}$ there is a Q-isomorphism $h_V: V \to V''$ such that if $f \in \mathrm{Hom}_Q(V, U)$, $h_U f = f'' h_V$. If $\{x_1, \cdots, x_n\}$ is a basis for V then $\{x'_1, \cdots, x'_n\}$ is a basis for V' where x'_i is defined by $x'_i(x_j) = \delta_{ij}$, the Kronecker delta.

Proof of Theorem A.

(a) DEFINITION of F. If $A = (X, V, \delta) \in \mathscr{C}$ then there is a Q^*_p-exact sequence

$$0 \to \mathrm{Hom}(V^*_p/\delta_p, Q^*_p) \xrightarrow{j'_A} \mathrm{Hom}(V^*_p, Q^*_p) \xrightarrow{i'_A} \mathrm{Hom}(\delta_p, Q^*_p) \to 0$$

induced by the canonical Q^*_p-exact sequence

$$0 \to \delta_p \xrightarrow{i_A} V^*_p \xrightarrow{j_A} V^*_p/\delta_p \to 0 .$$

Define $F(A) = (X', V', \bar{\delta})$, where $V' = \operatorname{Hom}(V, Q)$, $X' = \{x' \mid x \in X\}$ and $\bar{\delta}_p = j'_A(\operatorname{Hom}(V_p^*/\delta_p, Q_p^*))$. Note that $\bar{\delta}_p$ may be regarded as a subspace of $(V')_p^*$ since $\operatorname{Hom}(V_p^*, Q_p^*)$ is naturally isomorphic to $Q_p^* \otimes V' = (V')_p^*$.

(b) *F is a contravariant functor.* Let $B = (Y, U, \sigma)$, $\theta: A \to B$ a quasi-homomorphism, $\lambda: V \to U$ the unique extension of θ and $\theta_p = 1 \otimes \lambda: V_p^* \to U_p^*$. Define $F(\theta) = \lambda' \in \operatorname{Hom}_Q(U', V')$. Then $F(\theta): F(B) \to F(A)$ is a quasi-homomorphism if for all primes p, $F(\theta)_p(\bar{\sigma}_p) \subset \bar{\delta}_p$, where $F(\theta)_p = 1 \otimes \lambda': (U')_p^* \to (V')_p^*$.

Since $\theta_p(\delta_p) \subset \sigma_p$ there is a canonical homomorphism $\phi_p: V_p^*/\delta_p \to U_p^*/\sigma_p$ such that $\phi_p j_A = j_B \theta_p$. Thus $j'_A \phi'_p = \theta'_p j'_B$. It now follows that $F(\theta)_p(\bar{\sigma}_p) \subset \bar{\delta}_p$ since $\theta'_p = (1 \otimes \lambda)'$ is identified with $1 \otimes \lambda' = F(\theta)_p$ by the natural isomorphism of (a).

It is now clear that F is a contravariant functor in $\mathscr{C}$, since G is a contravariant functor in U.

(c) *F^2 is naturally equivalent to the identity.* For $A = (X, V, \delta) \in \mathscr{C}$, define $g_A: A \to F^2 A = (X'', V'', \bar{\bar{\delta}})$ to be the restriction of the Q-isomorphism $h_V: V \to V''$. It follows that g_A is a quasi-isomorphism since $(g_A)_p = 1 \otimes h_V: V_p^* \to (V'')_p^*$ has the property that $(g_A)_p(\delta_p) = \bar{\bar{\delta}}_p$.

Let $\theta: A \to B = (X, U, \sigma)$ be a quasi-homomorphism. Then $g_B \theta = F^2(\theta) g_A$ since $h_U \lambda = \lambda'' h_V$, where λ is the unique extension of θ, $\lambda: V \to U$. Therefore, F^2 is naturally equivalent to the identity functor on $\mathscr{C}$.

(d) *F is exact.* Assume $0 \to A_1 \xrightarrow{\phi_1} A_2 \xrightarrow{\phi_2} A_3 \to 0$ is an exact sequence in $\mathscr{C}$. By Lemma 3, $0 \to V_1 \xrightarrow{\lambda_1} V_2 \xrightarrow{\lambda_2} V_3 \to 0$ is exact hence $0 \to V'_3 \xrightarrow{\lambda'_2} V'_2 \xrightarrow{\lambda'_1} V'_1 \to 0$ is exact. Again by Lemma 3, $0 \to F(A_3) \xrightarrow{F(\phi_2)} F(A_2) \xrightarrow{F(\phi_1)} F(A_1) \to 0$ is exact. Consequently, F is an exact functor.

(e) *A is free iff FA is divisible.* Observe that $A = (X, V, \delta)$ is free iff $\delta_p = 0$ for all primes p and divisible iff $\delta_p = R_p^* A$ for all primes p.

Proof of Corollary B. A consequence of the definition of F and Lemma 3.

Note that A is strongly indecomposable iff FA is strongly indecomposable.

3. Examples and applications. If A is a rank 1 quotient divisible group with type (k_i), then $k_i = 0$ or ∞. It is easy to see that FA is a rank 1 quotient divisible group with type (l_i) where $l_i = 0$ if $k_i = \infty$ and $l_i = \infty$ if $k_i = 0$.

A torsion free abelian group A is *locally free* if R_pA is a free R_p-module for all primes p. The only locally free quotient divisible modules of finite rank are free, since if A is such a group FA is divisible (R_pFA is divisible for all primes p) hence A is free.

For $A \in \mathscr{C}$, let $E(A)$ be the quasi-endomorphism ring of A. Then F induces a ring anti-isomorphism from $E(A)$ to $E(FA)$ which is an isomorphism if $E(A)$ is commutative.

Beaumont-Pierce [3], Corollary 4.6, prove that a torsion free group A, of finite rank, is isomorphic to the additive group of a full subring of a semi-simple rational algebra (i.e., *has semi-simple algebra type*) iff A is quotient divisible and $A \dot{\sim} B_1 \oplus \cdots \oplus B_n$, B_i strongly indecomposable, and each $E(B_i)$ is an algebraic number field, whose dimension over Q is the rank of B_i. It follows that A has semi-simple algebra type iff FA does.

One can show, as in [1], that if rank $A = n + 1$ and p-rank $A = n$ for all primes p, $F(A) = \Lambda^n A$, the nth exterior power of A. A module theoretic characterization of F, in general, is unknown to the author.

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NEW MEXICO STATE UNIVERSITY