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ON BANACH SPACE VALUED EXTENSIONS FROM SPLIT FACES

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The object of this note is the following theorem: Suppose a is a continuous affine map from a closed split face F of a compact convex set K with values in a Banach space B enjoying the approximation property. Suppose also that p is a strictly positive lower semi-continuous concave function on K such that $\|a(k)\| \leqq p(k)$ for all k in F. Then a admits a continuous affine extension $\tilde{a}$ to K into B such that $\|\tilde{a}(k)\| \leqq p(k)$ for all k in K.

We shall use the methods of tensor products of compact convex sets as developed by Semadeni [12], Lazar [9], Namioka and Phelps [10] and Behrends and Wittstock [6] to reduce the problem to the case $B = \boldsymbol{R}$, and in this case the result follows from the work of Alfsen and Hirsberg [3] and the present author [4].

We shall be concerned with compact convex sets K_1 and K_2 in locally convex spaces E_1 and E_2 respectively. By $A(K_i)$ we shall denote the continuous real affine functions on K_i for $i = 1, 2$. We let $BA(K_1 \times K_2)$ be the Banach space of continuous biaffine functions on $K_1 \times K_2$. We observe that $1 \in BA(K_1 \times K_2)$ and that $BA(K_1 \times K_2)$ separates points of $K_1 \times K_2$. As usual we define the projective tensor product of K_1 and K_2, $K_1 \otimes K_2$, to be the state space of $BA(K_1 \times K_2)$ equipped with the w^*-topology. Then $K_1 \otimes K_2$ is a compact convex set, and we have a homeomorphic embedding $\omega_{K_1 \times K_2}$ (called ω, when no confusion can arise) from $K_1 \times K_2$ into $K_1 \otimes K_2$ defined by the following rule: For all a in $BA(K_1 \times K_2)$ and all (x_1, x_2) in $K_1 \times K_2$

$$\omega(x_1, x_2)(a) = a(x_1, x_2) .$$

We notice that ω is a biaffine map. It was proved in [10; Prop. 1.3, Th. 2.3] and [6; Satz 1.1.3] that $\partial_e(K_1 \otimes K_2) = \omega(\partial_e K_1 \times \partial_e K_2)$, where in general we denote the extreme points of a convex set K by $\partial_e K$.

For a in $A(K_1)$ and b in $A(K_2)$ we define the continuous biaffine function $a \otimes b$ by

$$a \otimes b(x_1, x_2) = a(x_1)b(x_2), \text{ all } (x_1, x_2) \in K_1 \times K_2 .$$

We let $A(K_1) \otimes A(K_2)$ be the real vector space

$$A(K_1) \otimes A(K_2) = \{\sum_{i=1}^{n} a_i \otimes b_i \mid a_i \in A(K_1), b_i \in A(K_2)\}$$

which is a copy of the algebraic tensor product of $A(K_1)$ and $A(K_2)$. We denote by $A(K_1)\otimes_\varepsilon A(K_2)$ the uniform closure of $A(K_1)\otimes A(K_2)$ in $BA(K_1\times K_2)$.

We recall that a Banach space B is said to have the approximation property if for each compact convex subset C of B and each $\varepsilon>0$ there is a continuous linear map $T\colon B\to B$ such that $T(B)$ is finite dimensional and such that $\|Tx-x\|<\varepsilon$ for all $x\in C$. It is proved in [10; Lem. 2.5] that if $A(K_1)$ (or $A(K_2)$) has the approximation property then $BA(K_1\times K_2)=A(K_1)\otimes_\varepsilon A(K_2)$.

Following Lazar [9] we define T_1 and T_2 as the natural embeddings of $A(K_1)$ and $A(K_2)$ into $BA(K_1\times K_2)$, i.e.

$$T_1a=a\otimes 1,\ \text{all}\ a\in A(K_1)$$
$$T_2b=1\otimes b,\ \text{all}\ b\in A(K_2)\ .$$

Let P_i be the adjoint map of T_i for $i=1,2$. Then P_i is an affine and continuous map of $K_1\otimes K_2$ onto K_i (= state space of $A(K_i)$), and

$$P_i\omega(k_1,k_2)=k_i,\ i=1,2\ .$$

The first part of the following proposition was proved by Lazar in the case where K_1 and K_2 are simplexes, but the proof holds in general. The last part was proved by Lazar in the simplex case by means of the Stone-Weierstrass Theorem for simplexes.

PROPOSITION 1. *Let F_1 and F_2 be closed faces of compact convex sets K_1 and K_2 resp. Let $F=P_1^{-1}(F_1)\cap P_2^{-1}(F_2)$*

(i) *Then F is a closed face in $K_1\otimes K_2$ and $F=\overline{\mathrm{co}}(\omega(F_1\times F_2))$*

(ii) *If $A(F_1)$ or $A(F_2)$ has the approximation property then $F_1\otimes F_2$ is affinely homeomorphic to F.*

Proof. Since P_i is continuous and affine it is immediate that $P_i^{-1}(F_i)$ is a closed face of $K_1\otimes K_2$, and hence F is a closed face.

Now let $p=\omega(k_1,k_2)\in\omega(F_1\times F_2)$. Then $P_ip=k_i\in F_i$, and hence $p\in P_1^{-1}(F_1)\cap P_2^{-1}(F_2)=F$. By the Krein Milman Theorem: $\overline{\mathrm{co}}(\omega(F_1\times F_2))\subseteqq F$.

Conversely, let $p\in\partial_eF$. Since F is a closed face we get

$$p\in\partial_eF=F\cap\partial_e(K_1\otimes K_2)=F\cap\omega(\partial_eK_1\times\partial_eK_2)\ .$$

Hence $p=\omega(x_1,x_2)$, $x_i\in\partial_eK_i$. Then $P_ip=x_i$ belongs to F_i by the definition of F. Hence $p\in\omega(F_1\times F_2)$, and again by the Krein Milman Theorem $F\subseteqq\overline{\mathrm{co}}(\omega(F_1\times F_2))$, and (i) is proved.

Now we shall prove (ii) under the assumption that $A(F_1)$ has the approximation property. We shall define a continuous affine map

$T: F_1 \otimes F_2 \rightarrow K_1 \otimes K_2$ by

$$(T\varphi)(b) = \varphi(b|_{F_1 \times F_2}),\ \varphi \in F_1 \otimes F_2,\ b \in BA(K_1 \times K_2)\ .$$

Then $T(F_1 \otimes F_2)$ is compact and convex in $K_1 \otimes K_2$. If $\varphi \in \partial_e(F_1 \otimes F_2)$ then $\varphi = \omega_{F_1 \times F_2}(x_1, x_2)$, where $x_i \in \partial_e F_i$, $i = 1, 2$. But then

$$(T\varphi)(b) = b(x_1, x_2) = \omega_{K_1 \times K_2}(x_1, x_2)(b),\ \text{all}\ b \in BA(K_1 \times K_2)\ .$$

Hence $T\varphi = \omega_{K_1 \times K_2}(x_1, x_2) \in \overline{\mathrm{co}}(\omega_{K_1 \times K_2}(F_1 \times F_2)) = F$. By the Krein Milman Theorem we conclude that $T(F_1 \otimes F_2) \subseteq F$.

Conversely, if $\psi \in \partial_e F$ then as F is a closed face, we get by Milman's theorem

$$\psi \in \omega_{K_1 \times K_2}(F_1 \times F_2) \cap \omega_{K_1 \times K_2}(\partial_e K_1 \times \partial_e K_2) = \omega_{K_1 \times K_2}(\partial_e F_1 \times \partial_e F_2)\ .$$

If $\psi = \omega_{K_1 \times K_2}(x_1, x_2)$, $x_i \in \partial_e F_i$, then $\omega_{F_1 \times F_2}(x_1, x_2) \in \partial_e(F_1 \otimes F_2)$, and as above $\psi = T(\omega_{F_1 \times F_2}(x_1, x_2))$. By the Krein Milman Theorem we get $F \subseteq T(F_1 \otimes F_2)$, and so T is surjective.

We proceed to show that T is injective. This is the case if $BA(K_1 \times K_2)|_{F_1 \times F_2}$ is dense in $BA(F_1 \times F_2)$. We show that $A(K_1) \otimes A(K_2)|_{F_1 \times F_2}$ is dense in $BA(F_1 \times F_2)$. Hence let $c \in BA(F_1 \times F_2)$ and $\varepsilon > 0$. Since $A(F_1)$ has the approximation property, we have that $A(F_1) \otimes_\varepsilon A(F_2) = BA(F_1 \otimes F_2)$, so there exist $a_1, \cdots, a_n \in A(F_1)$, $b_1, \cdots, b_n \in A(F_2)$ such that

$$\left\| c - \sum_{i=1}^n a_i \otimes b_i \right\|_{F_1 \times F_2} < \frac{\varepsilon}{2}\ .$$

Now $A(K_i)|_{F_i}$ is dense in $A(F_i)$, so we can choose $a_1' \in A(K_1)$, $b_i' \in A(K_2)$, $i = 1, \cdots n$, such that

$$\left\| \sum_{i=1}^n a_i \otimes b_i - \sum_{i=1}^n a_i' \otimes b_1' \right\|_{F_1 \times F_2} < \frac{\varepsilon}{2}\ .$$

Then $\| c - \sum_{i=1}^n a_i' \otimes b_i' \|_{F_1 \times F_2} < \varepsilon$, and the claim follows.

The next step is to prove that $\overline{\mathrm{co}}(\omega(F_1 \times F_2))$ is a closed split face of $K_1 \otimes K_2$ provided F_i is a closed split face of K_i for $i = 1, 2$, and f.ex. $A(F_1)$ has the approximation property.

We shall remind the reader of the following definitions and facts: If F is a closed face of a compact convex K, then the complementary σ-face F' is the union of all faces disjoint from F. It is always true that $K = \mathrm{co}(F \cup F')$. F is called a split face if F' is a face and each point in $K \backslash (F \cup F')$ can be decomposed uniquely as convex combination of a point in F and a point in F'. It follows from a slight modification of the proof of [2; Th. 3.5] that a closed face is a split face if and only if each nonnegative u.s.c. affine function of F admits an u.s.c.

affine extension to K, which is equal to 0 on F'. This characterization is sometimes inconvenient because of the "nonsymmetric" properties of the affine functions involved. Using the above characterization we shall give a new one involving the space $A_s(K)$ which is the smallest uniformly closed subspace of the bounded functions on K containing the bounded u.s.c. affine functions. This space has been used f.ex. by Krause [8] and Behrends and Wittstock [6] in simplex theory and by Combes [7] in C^*-algebra theory. We shall state some of the known properties of $A_s(K)$.

LEMMA 2.
(i) *If* $a \in A_s(K)$ *and* $a \geqq 0$ *on* $\partial_e K$ *then* $a \geqq 0$ *on* K.
(ii) *If* $a \in A_s(K)$ *then* $\|a\|_K = \|a\|_{\partial_e K}$.
(iii) *If* $a \in A_s(K)$ *then* a *satisfies the barycentric calculus.*

Sketch of proof. If s and t are u.s.c. affine functions on K and $s \leqq t$ on $\partial_e K$ it follows by [5; Lem. 1] that $s \leqq t$ on K. Hence (i) follows by a limit argument. Now (ii) follows by (i), since on $\partial_e K$: $-\|a\|_{\partial_e K} \leqq a \leqq \|a\|_{\partial_e K}$. Hence the same inequality holds on K, and so $\|a\|_K \leqq \|a\|_{\partial_e K}$. The converse inequality is trivial. Finally (iii) follows from Lebesgue's theorem on dominated convergence, since the barycentric calculus holds for (differences of) u.s.c bounded affine functions, cf. [1; Cor. I. 1.4].

PROPOSITION 3. *Let* F *be a closed face of a compact convex set* K. *Then* F *is a split face if and only if each* $a \in A_s(F)$ (*or* $A_s(F)^+$, $A(F)$, $A(F)^+$, $A(F; K)$, $A(F; K)^+$) *has an extension* $\tilde{a} \in A_s(K)$ *such that* $\tilde{a} = 0$ *on* F'. *If such an extension exists then it is unique.*

Proof. The uniqueness statement follows from Lemma 2 (ii), since $\partial_e K \subseteqq F \cup F'$.

Assume F is a split face and let $a \in A_s(F)$. If a is u.s.c. affine and nonnegative a has as noted above an u.s.c. affine extension $\tilde{a}$ with $\tilde{a} = 0$ on F'. Hence the result follows if a is the difference of two nonnegative u.s.c. affine functions on K. In general there are b_n, c_n u.s.c. affine and nonnegative, $a_n = b_n - c_n$, such that $\|a_n - a\|_F \underset{n\to\infty}{\longrightarrow} 0$. We use Lemma 2 (ii) and the fact that $\partial_e K \subseteqq F \cup F'$ to conclude that

$$\|\tilde{a}_n - \tilde{a}_m\| = \|\tilde{a}_n - \tilde{a}_m\|_{\partial_e K} = \|a_n - a_m\|_{\partial_e F} = \|a_n - a_m\|_F \,.$$

Hence $\{\tilde{a}_n\}_1^\infty$ is Cauchy in $A_s(K)$. Then $\tilde{a} = \lim \tilde{a}_n \in A_s(K)$ will be an extension of a with $\tilde{a} = 0$ on F'.

Conversely, assume that each $a \in A(F; K)^+$ has an extension $\tilde{a} \in A_s(K)$ such that $\tilde{a} = 0$ on F'. Let $x \in K \backslash (F \cup F')$, $x = \lambda y + (1 - \lambda)z$, where $y \in F$, $z \in F'$ and $0 < \lambda < 1$. Then $\lambda = \tilde{1}(x)$, and since λ is uniquely determined, $\hat{\chi}_F$ is affine, and hence $F' = \hat{\chi}_F^{-1}(0)$ is a face, cf. [2; Prop. 1.1, Cor. 1.2]. Now the uniqueness of F, F' components is easy, since $A(F; K)^+$ separates points of F.

The following lemma can be derived from [6; Formula (1), p. 263, Satz 2.1.3]. For the readers convenience we shall give a proof.

LEMMA 4. *Let K_1 and K_2 be compact convex sets and $a \in A_s(K_1)$, $b \in A_s(K_2)$. Then there is a function $c \in A_s(K_1 \otimes K_2)$, denoted by $a \otimes b$, such that*

$$c(\omega(x_1, x_2)) = a(x_1)b(x_2), \text{ all } (x_1, x_2) \in K_1 \times K_2.$$

Proof. First we shall consider the case where a and b are nonnegative u.s.c. and affine. Then there exist nets $\{a_\alpha\} \subseteqq A(K_1)^+$, $\{b_\beta\} \subseteqq A(K_2)^+$ such that $a_\alpha \searrow a$, $b_\beta \searrow b$, pointwise. Then $\{a_\alpha \otimes b_\beta\}$ is a decreasing net in $BA(K_1 \times K_2)^+$, and therefore there is an u.s.c. affine function c on $K_1 \otimes K_2$ such that

$$c(\varphi) = \inf_{\alpha,\beta} \varphi(a_\alpha \otimes b_\beta), \text{ all } \varphi \in K_1 \otimes K_2 .$$

Especially, for all $(x_1, x_2) \in K_1 \times K_2$

$$c(\omega(x_1, x_2)) = \inf a_\alpha(x_1) b_\beta(x_2) = a(x_1)b(x_2) .$$

If

$$(*) \qquad a = a_1 - a_2, \; b = b_1 - b_2$$

where a_i is u.s.c. nonnegative and affine on K_1, b_i is u.s.c. nonnegative and affine on K_2, then $(x_1, x_2) \to a(x_1)b(x_2)$ is linear combination of four terms of the kind considered in the first part of the proof, and we can choose c as the corresponding linear combination of elements from $A_s(K_1 \otimes K_2)$.

If $a \in A_s(K_1)$, $b \in A_s(K_2)$ are arbitrary then we can find a'_n, b'_n of the type (*), such that $\|b - b'_n\|_{K_2} < 1/n$, $\|a - a'_n\|_{K_1} < 1/n$ and $c_n \in A_s(K \otimes K_2)$ such that

$$(**) \qquad c_n(\omega(x_1, x_2)) = a'_n(x_1)b'_n(x_2), \text{ all } (x_1, x_2) \in K_1 \times K_2 .$$

Then for all $(x_1, x_2) \in \partial_e K_2$

$$|a(x_1)b(x_2) - c_n(\omega(x_1, x_2))| < \frac{1}{n^2} + \frac{1}{n}(\|a\|_{K_1} + \|b\|_{K_2}) .$$

From this it follows that $\{c_n|_{\partial_e(K_1 \otimes K_2)}\}$ is Cauchy, and hence $\{c_n\}$ is Cauchy on $K_1 \otimes K_2$ by Lemma 2 (ii). Let $c = \lim c_n \in A_s(K_1 \otimes K_2)$. Then it is obvious from (**) that c satisfies the requirement.

THEOREM 5. *Let K_1 and K_2 be compact convex sets, and F_1 and F_2 closed faces of K_1 and K_2 respectively. Let F be the face $\overline{\text{co}}(\omega(F_1 \times F_2))$ in $K_1 \otimes K_2$. Then the following holds*

(i) *If F is a split face of $K_1 \otimes K_2$ then F_1 and F_2 are split faces of K_1 and K_2.*

(ii) *If either $A(F_1)$ or $A(F_2)$ has the the approximation property, and F_1 and F_2 are split faces of K_1 and K_2, then F is a split face of $K_1 \otimes K_2$.*

Proof. To prove (i) we assume that F is a split face. As noted before $\partial_e F = \omega(\partial_e F_1 \times \partial_e F_2)$. Let $a \in A(K_1)$ such that $a \geqq 0$ on F_1, i.e. $a|_{F_1} \in A(F_1; K_1)^+$. By Proposition 3 it will suffice to show that $(a\cdot\chi_{F_1})^\wedge$ is affine K_1. We know that $((a \otimes 1)\cdot\chi_F)^\wedge$ is u.s.c. and affine on $K_1 \otimes K_2$, since $a \otimes 1$ is nonnegative on $\omega(F_1 \times F_2)$ and hence on F. Now we fix $x_2 \in \partial_e F_2$. Then the function $g(x_2)\colon x \to ((a \otimes 1)\cdot\chi_F)^\wedge(\omega(x, x_2))$ is u.s.c. and affine on K_1. On F_1 $g(x_2)$ agrees with a, and since $\omega(\partial_e F_1' \times \partial_e F_2) \subseteqq F'$, we have that $g(x_2) = 0$ on $\partial_e F_1'$

Since $g(x_2)$ and $(a\cdot\chi_{F_1})^\wedge$ agree on $\partial_e K_1$, and $g(x_2)$ is u.s.c. affine, while $(a\cdot\chi_{F_1})^\wedge$ is u.s.c. concave it follows from Bauers principle [5; Lem. 1] that $g(x_2) \leqq (a\cdot\chi_{F_1})^\wedge$. Moreover $g(x_2) \geqq a\cdot\chi_{F_1}$, and since $(a\cdot\chi_{F_1})^\wedge$ is the smallest u.s.c. concave majorant of $a\cdot\chi_{F_1}$, we have $g(x_2) \geqq (a\cdot\chi_{F_1})^\wedge$, and (i) follows.

To prove (ii) we shall assume that F_1 and F_2 are split faces, and that $A(F_1)$ has the approximation property. By Proposition 3 we have to show that if $a \in A(F)^+$ then a admits an extension $\tilde{a} \in A_s(K_1 \otimes K_2)$ such that $\tilde{a} = 0$ on F'. Now $a \circ (\omega_{K_1 \times K_2}|_{F_1 \times F_2})$ belongs to $BA(F_1 \times F_2) = A(F_1) \otimes_\varepsilon A(F_2)$. If $\varepsilon > 0$ is arbitrary we can choose $a_1, \cdots, a_n \in A(F_1)$ and $b_1, \cdots, b_n \in A(F_2)$ such that

$$\left\| a \circ \omega_{K_1 \times K_2} - \sum_{i=1}^n a_i \otimes b_i \right\|_{F_1 \times F_2} < \varepsilon .$$

By Proposition 3 we can choose $\tilde{a}_i \in A_s(K_1)$, $\tilde{b}_i \in A_s(K_2)$ such that $\tilde{a}_i = a_i$ on F_1 and $\tilde{a}_i = 0$ on F_1', while $\tilde{b}_i = b_i$ on F_2 and $\tilde{b}_i = 0$ on F_2'.

By Lemma 4 $\sum_{i=1}^n \tilde{a}_i \otimes \tilde{b}_i \in A_s(K_1 \otimes K_2)$ and on $\omega(F_1 \times F_2)$ it equals $\sum_{i=1}^n a_i \otimes b_i$, while $\sum_{i=1}^n \tilde{a}_i \otimes \tilde{b}_i = 0$ on $\partial_e(K_1 \otimes K_2)\backslash\partial_e F$.

As $A_s(K_1 \otimes K_2)$ is complete in $\| \quad \|_{\partial_e(K_1 \otimes K_2)}$ and the norm of $\sum_{i=1}^n \tilde{a}_i \otimes \tilde{b}_i$ is obtained at $\omega(F_1 \times F_2)$, this argument leads to the existence of $\tilde{a} \in A_s(K_1 \otimes K_2)$ such that $\tilde{a} = a$ on $\omega(F \times F_2)$, and $\tilde{a} = 0$ on $\partial_e F' = \partial_e(K_1 \otimes K_2)\backslash F$. It remains to show that $\tilde{a} = a$ on F and $\tilde{a} = 0$ on F'.

Now let $x \in F$ and represent x by a probability measure μ on $\omega(F_1 \times F_2)$. Since $\tilde{a}$ satisfies the barycentric calculus we get

$$\tilde{a}(x) = \int_{K_1 \otimes K_2} \tilde{a}\, d\mu = \int_{w(F_1 \times F_2)} \tilde{a}\, d\mu = \int_F a\, d\mu = a(x)$$

and so $\tilde{a} = a$ on F.

To show that $\tilde{a} = 0$ on F' we let $b \in A(K_1 \otimes K_2)$ with $b > 0$ on $K_1 \otimes K_2$ and $b > a$ on F. Then $b \geqq \tilde{a}$ on $\partial_e(K_1 \otimes K_2)$, and by Lemma 2 (*i*), $b \geqq \tilde{a}$ on $K_1 \otimes K_2$. For $\rho \in K_1 \otimes K_2$ we have

$$(a \cdot \chi_F)^\wedge(\rho) = \inf \{b(\rho) \mid b \in A(K_1 \otimes K_2),\, b > a \cdot \chi_F\} \geqq \tilde{a}(\rho) \geqq 0 .$$

Since $(a \cdot \chi_F)^\wedge = 0$ on F', we get $\tilde{a} = 0$ on F', and the proof is complete.

REMARK. It is easy to see from Lemma 4 that the embedding of the product of two parallel faces F_1 and F_2 in the sense of [11] gives rise to a parallel face F without the assumption of the presence of the approximation property in $A(F_1)$. In fact, $\hat{\chi}_F = \hat{\chi}_{F_1} \otimes \hat{\chi}_{F_2}$ is affine.

THEOREM 6. *Let F be a closed split face of a compact convex set K. Let B be a real Banach space having the approximation property. Let p be a concave l.s.c. strictly positive real function on K. Let $a: F \to B$ be an affine continuous map such that*

$$\|a(k)\| \leqq p(k), \text{ all } k \in F .$$

Then a has an extension to a continuous affine map $\tilde{a}: K \to B$ such that

$$\|\tilde{a}(k)\| \leqq p(k), \text{ all } k \in K .$$

Proof. Let C be the unit ball of B^* with w^*-topology. $B \times \boldsymbol{R}$ is normed by $\|(x, r)\| = \|x\| + |r|$. It was observed in [10] that $(x, r) \to (\cdot)(x) + r$ is an isometric isomorphism of $B \times \boldsymbol{R}$ onto $A(C)$. Hence if B has the approximation property then $A(C)$ has.

We define a biaffine continuous function b on $F \times C$ by

$$b(x, x^*) = x^*(a(x)), \text{ all } x \in F,\, x^* \in C .$$

By Proposition 1 (ii) there is an affine homeomorphism between $F \otimes C$ and $\overline{\mathrm{co}}(\omega_{K \times C}(F \times C))$ defined by

$$T(\rho)(d) = \rho(d|_{F \times C}) \text{ for } d \in BA(K \times C) .$$

Since b is naturally a continuous affine function on $F \otimes C$ there is a continuous affine function b_1 on $\overline{\mathrm{co}}(\omega_{K \times C}(F \times C))$ such that

$$b_1(T\, \omega_{F \times C}(x, x^*)) = x^*(a(x)), \text{ all } (x, x^*) \in F \times C .$$

Moreover $\rho \to p(P_1(\rho))$ is concave, strictly positive and l.s.c. on $K \otimes C$. For $\rho \in \partial_e(\mathrm{co}(\omega_{K \times C}(F \times C))) = \omega_{K \times C}(\partial_e F \times \partial_e C)$ we have $\rho = \omega_{K \times C}(x, x^*)$ with $(x, x^*) \in \partial_e F \times \partial_e C$ and hence

$$|b_1(\rho)| = |x^*(a(x))| \leqq \|a(x)\| \leqq p(x) = p(P_1(\rho)) .$$

Since $\rho \to |b_1(\rho)|$ is convex and continuous and $\rho \to p(P_1(\rho))$ is concave and l.s.c., it follows from Bauers principle [5; Lem. 1] that $|b_1| \leqq p \circ P_1$ on $\overline{\text{co}}(\omega_{K\times C}(F \times C))$.

Now it follows from Theorem 5 that $\overline{\text{co}}(\omega_{K\times C}(F \times C))$ is a split face of $K \otimes C$. By [1; Th. II. 6. 12] and [3; Th. 2.2 and Th. 4.5] it follows that there is a function $c \in A(K \otimes C)$ such that c extends b_1 and

$$|c(\rho)| \leqq p(P_1(\rho)), \text{ all } \rho \in K \otimes C .$$

(Actually, it follows from [1; Cor. I. 5.2] that a concave l.s.c. function on a compact convex set is $A(K)$-superharmonic in the sense of [3]. Moreover it should be remarked that the theorems 2.2 and 4.5 of [3] are stated for complex spaces, but the proofs hold almost unchanged for the real case.)

Now we can define a continuous affine map $c_1: K \to A(C)$ by

$$c_1(k)(\cdot) = c(\omega(k, \cdot)) .$$

Then for $k \in K$

$$||c_1(k)|| = \sup_{x^* \in C} ||c(\omega(k, x^*))|| \leqq \sup p(P_1(k, x^*))) = p(k) .$$

By composing the isometry S between $A(C)$ and $B \times \boldsymbol{R}$ with the canonical projection Q from $B \times \boldsymbol{R}$ to B, which has norm 1. we get an affine continuous map $\tilde{a}(= Q \circ S \circ c_1)$ of K into B such that

$$||\tilde{a}(k)|| = ||(Q \circ S \circ c_1)(k)|| \leqq ||c_1(k)|| \leqq p(k)$$

for all $k \in K$. Moreover, for $k \in F$, $x^* \in C$

$$\begin{aligned} x^*(\tilde{a}(k)) &= x^*((Q \circ S \circ c_1)(k)) = c_1(k)(x^*) \\ &= c(\omega(k, x^*)) = b_1(\omega(k, x^*)) = x^*(a(k)) . \end{aligned}$$

Hence for $k \in F$: $\tilde{a}(k) = a(k)$.

Corollary. *Let F be a closed split face of a compact convex set K. Let B be a real Banach space having the approximation property. Let $a: F \to B$ be a continuous affine map. Then a admits an extension to a continuous affine function $\tilde{a}: K \to B$ such that* $\max_{k \in F} ||a(k)|| = \max_{k \in K} ||\tilde{a}(k)||$.

Remark. Conclusions similar to those of Theorem 6 and the Corollary hold with no assumptions on B, if instead we know that $A(F)$ has the approximation property. This is f.ex. the case, if K is a simplex.

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