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ON PERMANENTS OF CIRCULANTS

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A recurrence formula is obtained for permanents of circulants of the form $\alpha I_n + \beta P + \gamma P^2$ and explicit formulas are deduced from it. It is shown that for doubly stochastic circulants $\alpha I_n + \beta P + \gamma P^2$ the minimum permanent lies in the interval $(1/2^n, 1/2^{n-1}]$.

1. Introduction. The well-known unresolved conjecture of van der Waerden asserts that in Ω_n, the polyhedron of doubly stochastic $n \times n$ matrices, the permanent function takes its minimum value for the matrix J_n, all of whose entries are $1/n$, i.e.,

$$\min_{A \in \Omega_n} \operatorname{per}(A) = \operatorname{per}(J_n) . \tag{1}$$

By a theorem of Birkhoff, Ω_n is a convex polyhedron with the permutation matrices $P_1, \cdots, P_{n!}$ as vertices. Thus (1) can be written in the form

$$\min_{\theta} \operatorname{per}\left(\sum_{j=1}^{n!} \theta_j P_j\right) = \operatorname{per}\left(\sum_{j=1}^{n!} \frac{1}{n!} P_j\right) , \tag{2}$$

where the minimum is over all nonnegative $(n!)$-tuples $\theta = (\theta_1, \cdots, \theta_{n!})$ satisfying $\sum_{j=1}^{n!} \theta_j = 1$.

Since van der Waerden's conjecture is still unresolved, it is natural to ask whether

$$\min_{\omega} \operatorname{per}\left(\sum_{j=1}^{m} \omega_j P_j\right) = \operatorname{per}\left(\sum_{j=1}^{m} \frac{1}{m} P_j\right) , \tag{3}$$

for a fixed set of permutation matrices $\{P_1, \cdots, P_m\}$, where the minimum is over all nonnegative m-tuples $\omega = (\omega_1, \cdots, \omega_m)$ satisfying $\sum_{j=1}^{m} \omega_j = 1$.

In this paper we study circulants of the form $\alpha I_n + \beta P + \gamma P^2$, where I_n is the $n \times n$ identity matrix and P is the full-cycle permutation matrix with 1's in the positions $(1, 2), (2, 3), \cdots, (n-1, n), (n, 1)$. We obtain a recurrence formula and deduce explicit formulas for $\operatorname{per}(\alpha I_n + \beta P + \gamma P^2)$. We then specialize to doubly stochastic circulants of the form $\alpha I_n + \beta P + \gamma P^2$, obtain bounds for the minimum value of the permanent of such circulants, and show that (3) does not hold for the set $\{I_n, P, P^2\}$, $n \geqq 5$.

The author is indebted to Dr. David London for drawing his attention to the fact that $\operatorname{per}((1/2)I_n + (1/2)P) < \operatorname{per}((1/3)I_n + (1/3)P + (1/3)P^2)$, for sufficiently large n.

2. Results. We begin with two formulas for the permanent of a tridiagonal matrix of the form

$$\begin{bmatrix} \beta & \gamma & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 \\ \alpha & \beta & \gamma & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 \\ 0 & \alpha & \beta & \gamma & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & & & & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & & & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & & & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & & & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \beta & \gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \alpha & \beta & \gamma & 0 \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & \alpha & \beta & \gamma \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & \alpha & \beta \end{bmatrix} . \tag{4}$$

Let $F_n(\alpha, \beta, \gamma)$ denote the matrix (4) of order n and let the permanent of $F_n(\alpha, \beta, \gamma)$ be denoted by $f_n(\alpha, \beta, \gamma)$, or simply by f_n. Set $f_0 = 1$, $f_1 = \beta$, and $f_2 = \beta^2 + \alpha\gamma$.

LEMMA 1. *If $n \geqq 2$, then*

$$f_n = \beta f_{n-1} + \alpha\gamma f_{n-2} . \tag{5}$$

COROLLARY. *If $n \geqq 1$ and $\mu = \sqrt{\beta^2 + 4\alpha\gamma} \neq 0$, then*

$$f_n = \frac{1}{\mu} r_1^{n+1} - \frac{1}{\mu} r_2^{n+1} \tag{6}$$

where $r_1 = (\beta + \mu)/2$ and $r_2 = (\beta - \mu)/2$. If $\mu = 0$, then

$$f_n = (n + 1)(\beta/2)^n . \tag{6'}$$

(In other words, if the right side of (6) is considered as a polynomial expression in α, β, γ, then (6) holds even in the case $\mu = 0$.)

The lemma is proved easily by expanding the permanent of $F_n(\alpha, \beta, \gamma)$ by the first column. Formula (6) is obtained by solving the difference equation (5) subject to initial conditions.

In the next lemma, formula (5) is used to obtain a relation between the permanent of the circulant $\alpha I_n + \beta P + \gamma P^2$ and permanents of tridiagonal matrices of the form (4).

LEMMA 2. *If $n \geqq 3$, then*

$$\text{per}\,(\alpha I_n + \beta P + \gamma P^2) = f_n + \alpha\gamma f_{n-2} + \alpha^n + \gamma^n \,. \tag{7}$$

Proof. A direct computation shows that the theorem holds for $n = 3$. Assume that $n \geqq 4$. Denote the matrix $\alpha I_n + \beta P + \gamma P^2$ by Q_n, and the submatrix of Q_n obtained by deleting rows i_1, i_2 and columns j_1, j_2 by $Q_n(i_1, i_2 | j_1, j_2)$. Expand the permanent of Q_n by the first two columns:

$$\begin{aligned}\text{per}\,(Q_n) &= \alpha^2\,\text{per}\,(Q_n(1, 2 | 1, 2)) + \beta\gamma\,\text{per}\,(Q_n(1, n-1 | 1, 2)) \\ &\quad + (\alpha\gamma + \beta^2)\,\text{per}\,(Q_n(1, n | 1, 2)) + \alpha\gamma\,\text{per}\,(Q_n(2, n-1 | 1, 2)) \\ &\quad + \alpha\beta\,\text{per}\,(Q_n(2, n | 1, 2)) + \gamma^2\,\text{per}\,(Q_n(n-1, n | 1, 2)) \\ &= \alpha^n + \alpha\beta\gamma f_{n-3} + (\alpha\gamma + \beta^2) f_{n-2} + \alpha^2\gamma^2 f_{n-4} + \alpha\beta\gamma f_{n-3} + \gamma^n \\ &= \beta f_{n-1} + \alpha\gamma f_{n-2} + \alpha\gamma(\beta f_{n-3} + \alpha\gamma f_{n-4}) + \alpha^n + \gamma^n \\ &= f_n + \alpha\gamma f_{n-2} + \alpha^n + \gamma^n \,.\end{aligned}$$

We now use the preceding result to obtain a recurrence formula for the permanent of $\alpha I_n + \beta P + \gamma P^2$, and then to deduce explicit formulas for these circulants.

THEOREM 1. *If* $Q_n = \alpha I_n + \beta P + \gamma P^2$ *and* $n \geqq 5$, *then*

$$\begin{aligned}\text{per}\,(Q_n) &= \beta\,\text{per}\,(Q_{n-1}) + \alpha\gamma\,\text{per}\,(Q_{n-2}) \\ &\quad + \alpha^{n-1}(\alpha - \beta - \gamma) + \gamma^{n-1}(\gamma - \alpha - \beta) \,.\end{aligned} \tag{8}$$

Proof. We use (7) and (5) to transform the right-hand side of (8) as follows:

$$\begin{aligned}&\beta\,\text{per}\,(Q_{n-1}) + \alpha\gamma\,\text{per}\,(Q_{n-2}) + \alpha^{n-1}(\alpha - \beta - \gamma) + \gamma^{n-1}(\gamma - \alpha - \beta) \\ &\quad = \beta f_{n-1} + \beta\alpha\gamma f_{n-3} + \beta\alpha^{n-1} + \beta\gamma^{n-1} + \alpha\gamma f_{n-2} + \alpha^2\gamma^2 f_{n-4} + \alpha^{n-1}\gamma \\ &\qquad + \alpha\gamma^{n-1} + \alpha^n - \alpha^{n-1}\beta - \alpha^{n-1}\gamma + \gamma^n - \alpha\gamma^{n-1} - \beta\gamma^{n-1} \\ &\quad = (\beta f_{n-1} + \alpha\gamma f_{n-2}) + \alpha\gamma(\beta f_{n-3} + \alpha\gamma f_{n-4}) + \alpha^n + \gamma^n \\ &\quad = f_n + \alpha\gamma f_{n-2} + \alpha^n + \gamma^n \\ &\quad = \text{per}\,(Q_n) \,.\end{aligned}$$

The difference equation (8) can now be solved subject to the conditions

$$\text{per}\,(Q_3) = \alpha^3 + \beta^3 + \gamma^3 + 3\alpha\beta\gamma$$

$$\text{per}\,(Q_4) = \alpha^4 + \beta^4 + \gamma^4 + 4\alpha\beta^2\gamma + 2\alpha^2\gamma^2$$

$$\text{per}\,(Q_5) = \alpha^5 + \beta^5 + \gamma^5 + 5\alpha\beta^3\gamma + 5\alpha^2\beta\gamma^2 \,, \quad \text{etc.,}$$

which are computed directly using a Laplace expansion. We obtain the following explicit formula.

THEOREM 2. *If* $n \geqq 3$, *then*

$$\operatorname{per}(\alpha I_n + \beta P + \gamma P^2) = r_1^n + r_2^n + \alpha^n + \gamma^n \tag{9}$$

where r_1 *and* r_2 *are the roots of* $x^2 - \beta x - \alpha\gamma = 0$.

Alternatively, formula (9) can be obtained from (7) and (6) if $\mu \neq 0$, or from (7) and (6′) in case $\mu = 0$. Thus if $\mu \neq 0$:

$$\begin{aligned}\operatorname{per}(\alpha I_n + \beta P + \gamma P^2) &= f_n + \alpha\gamma f_{n-2} + \alpha^n + \gamma^n \\ &= \frac{1}{\mu} r_1^{n+1} - \frac{1}{\mu} r_2^{n+1} + \frac{\alpha\gamma}{\mu} r_1^{n-1} - \frac{\alpha\gamma}{\mu} r_2^{n-1} \\ &\quad + \alpha^n + \gamma^n \\ &= \frac{1}{\mu}(r_1^{n+1} - r_2^{n+1} - r_1 r_2 (r_1^{n-1} - r_2^{n-1})) + \alpha^n + \gamma^n \\ &= \frac{1}{\mu}(r_1^n + r_2^n)(r_1 - r_2) + \alpha^n + \gamma^n \\ &= r_1^n + r_2^n + \alpha^n + \gamma^n,\end{aligned}$$

since $\alpha\gamma = -r_1 r_2$ and $\mu = r_1 - r_2$. The case $\mu = 0$ is proved similarly.

Formulas (8) and (9) have been obtained in [2] for the special case $\alpha = \beta = \gamma$.

THEOREM 3. *If* $n \geqq 3$, *then*

$$\operatorname{per}(\alpha I_n + \beta P + \gamma P^2) = \alpha^n + \beta^n + \gamma^n + \sum_{t=1}^{[n/2]} c_t^{(n)} \alpha^t \beta^{n-2t} \gamma^t \tag{10}$$

where $c_t^{(n)} = 2^{-(n-2t-1)} \sum_{k=t}^{[n/2]} \binom{n}{2k}\binom{k}{t}$.

Proof. Let $r_1 = (\beta + \mu)/2$ and $r_2 = (\beta - \mu)/2$, where $\mu = \sqrt{\beta^2 + 4\alpha\gamma}$. Then by formula (9),

$$\begin{aligned}&\operatorname{per}(\alpha I_n + \beta P + \gamma P^2) \\ &\quad = \alpha^n + \gamma^n + \left(\frac{\beta + \mu}{2}\right)^n + \left(\frac{\beta - \mu}{2}\right)^n \\ &\quad = \alpha^n + \gamma^n + 2^{-(n-1)} \sum_{k=0}^{[n/2]} \binom{n}{2k} \beta^{n-2k} (\beta^2 + 4\alpha\gamma)^k \\ &\quad = \alpha^n + \gamma^n + 2^{-(n-1)} \sum_{k=0}^{[n/2]} \binom{n}{2k} \sum_{t=0}^{k} \binom{k}{t} \beta^{n-2t} (4\alpha\gamma)^t \\ &\quad = \alpha^n + \beta^n + \gamma^n + \sum_{t=1}^{[n/2]} \left(\sum_{k=t}^{[n/2]} 2^{-(n-2t-1)} \binom{n}{2k}\binom{k}{t} \right) \alpha^t \beta^{n-2t} \gamma^t .\end{aligned}$$

The following alternative form of formula (10) can be proved by induction:

$$(11)\quad \begin{cases} \operatorname{per}(\alpha I_n + \beta P + \gamma P^2) = \alpha^n + \beta^n + \gamma^n + \sum_{t=1}^{[n/2]} c_t^{(n)} \alpha^t \beta^{n-2t} \gamma^t \,, \\ \text{where } c_1^{(n)} = n,\ c_{n/2}^{(n)} = 2 \text{ in case } n \text{ is even,} \\ \text{and } c_t^{(n)} = c_t^{(n-1)} + c_{t-1}^{(n-2)},\ 1 < t < n/2. \end{cases}$$

The cases $n = 3$ and 4 can be easily verified. If $Q_n = \alpha I_n + \beta P + \gamma P^2$, $n \geqq 5$, then by (8),

$$\begin{aligned} \operatorname{per}(Q_n) &= \beta \operatorname{per}(Q_{n-1}) + \alpha\gamma \operatorname{per}(Q_{n-2}) + \alpha^{n-1}(\alpha - \beta - \gamma) \\ &\quad + \gamma^{n-1}(\gamma - \alpha - \beta) \\ &= \alpha^{n-1}\beta + \beta^n + \beta\gamma^{n-1} + \sum_{t=1}^{[(n-1)/2]} c_t^{(n-1)} \alpha^t \beta^{n-2t} \gamma^t \\ &\quad + \alpha^{n-1}\gamma + \alpha\beta^{n-2}\gamma + \alpha\gamma^{n-1} + \sum_{s=1}^{[n/2]-1} c_s^{(n-2)} \alpha^{s+1} \beta^{n-2s+2} \gamma^{s+1} \\ &\quad + \alpha^{n-1}(\alpha - \beta - \gamma) + \gamma^{n-1}(\gamma - \alpha - \beta) \\ &= \alpha^n + \beta^n + \gamma^n + \alpha\beta^{n-2}\gamma + c_1^{(n-1)}\alpha\beta^{n-2}\gamma + \sum_{t=2}^{[(n-1)/2]} c_t^{(n-1)} \alpha^t \beta^{n-2t} \gamma^t \\ &\quad + \sum_{t=2}^{[n/2]} c_{t-1}^{(n-2)} \alpha^t \beta^{n-2t} \gamma^t \\ &= \begin{cases} \alpha^n + \beta^n + \gamma^n + (1 + c_1^{(n-1)})\alpha\beta^{n-2}\gamma + \sum_{t=2}^{[n/2]} (c_t^{(n-1)} + c_{t-1}^{(n-2)}) \alpha^t \beta^{n-2t} \gamma^t \,, \\ \hfill \text{if } n \text{ is odd,} \\ \alpha^n + \beta^n + \gamma^n + (1 + c_1^{(n-1)})\alpha\beta^{n-2}\gamma + \sum_{t=2}^{[n/2]-1} (c_t^{(n-1)} + c_{t-1}^{(n-2)}) \alpha^t \beta^{n-2t} \gamma^t \\ \quad + 2\alpha^{n/2}\gamma^{n/2} \,, \hfill \text{if } n \text{ is even,} \end{cases} \end{aligned}$$

and formula (11) follows easily.

Formula (11) allows us to construct a table of coefficients $c_t^{(n)}$ in the manner of Pascal's triangle.

n	$c_1^{(n)}$	$c_2^{(n)}$	$c_3^{(n)}$	$c_4^{(n)}$	$c_5^{(n)}$	$c_6^{(n)}$
3	3					
4	4	2				
5	5	5				
6	6	9	2			
7	7	14	7			
8	8	20	16	2		
9	9	27	30	9		
10	10	35	50	25	2	
11	11	44	77	55	11	
12	12	54	112	105	36	2

In the remainder of this paper we assume that $\alpha I_n + \beta P + \gamma P^2$ is doubly stochastic, i.e., that α, β, γ are nonnegative and $\alpha + \beta + \gamma = 1$.

THEOREM 4. *If α, β, γ are nonnegative then*

$$\frac{1}{2^n} < \min_{\alpha+\beta+\gamma=1} (\operatorname{per}(\alpha I_n + \beta P + \gamma P^2)) \leqq \frac{1}{2^{n-1}} . \tag{12}$$

Proof. The right inequality in (12) follows immediately from the fact that

$$\operatorname{per}\left(\frac{1}{2}I_n + \frac{1}{2}P\right) = \frac{1}{2^{n-1}} .$$

We prove the left inequality by showing that

$$\operatorname{per}(\alpha I_n + \beta P + \gamma P^2) > \frac{1}{2^n} \tag{13}$$

for any nonnegative α, β, γ satisfying $\alpha + \beta + \gamma = 1$. If any of α, β, γ exceeds 1/2 then (13) clearly holds, since by (10)

$$\operatorname{per}(\alpha I_n + \beta P + \gamma P^2) \geqq \alpha^n + \beta^n + \gamma^n .$$

Suppose that

$$0 \leqq \alpha \leqq \frac{1}{2},\ 0 \leqq \beta \leqq \frac{1}{2},\ 0 \leqq \gamma \leqq \frac{1}{2},\ \alpha + \beta + \gamma = 1 . \tag{14}$$

We assume, without loss of generality, that $\alpha \geqq \gamma$, and assert that under these conditions

$$r_1 \geqq \frac{1}{2} \quad \text{and} \quad |r_2| \leqq \alpha \tag{15}$$

where $r_1 = (1/2)(\beta + \sqrt{\beta^2 + 4\alpha\gamma})$ and $r_2 = (1/2)(\beta - \sqrt{\beta^2 + 4\alpha\gamma})$. We use the method of Lagrange's multipliers to determine the stationary points of the function $r_1 = r_1(\alpha, \beta, \gamma)$. Let

$$F(\alpha, \beta, \gamma) = \frac{1}{2}\left(\beta + \sqrt{\beta^2 + 4\alpha\gamma}\right) + \lambda(\alpha + \beta + \gamma - 1) .$$

The necessary conditions for a stationary point are

$$\frac{\partial F}{\partial \alpha} = \frac{\gamma}{\mu} + \lambda = 0 ,$$

$$\frac{\partial F}{\partial \beta} = \frac{1}{2}\left(1 + \frac{\beta}{\mu}\right) + \lambda = \frac{r_1}{\mu} + \lambda = 0 ,$$

$$\frac{\partial F}{\partial \gamma} = \frac{\alpha}{\mu} + \lambda = 0 .$$

Where $\mu = \sqrt{\beta^2 + 4\alpha\gamma}$, i.e., we must have $\alpha = \gamma = r_1$. But then

$$2\alpha = \beta + \sqrt{\beta^2 + 4\alpha^2} ,$$

i.e.,

$$4\alpha^2 - 4\alpha\beta + \beta^2 = \beta^2 + 4\alpha^2 ,$$

which implies that either $\beta = 0$ and $\alpha = \gamma = 1/2$, or $\alpha = \gamma = 0$ and $\beta = 1$. In any case the function $r_1(\alpha, \beta, \gamma)$ has no minimum in the interior of region (14). It is easy to verify that its minimum value on the boundary is 1/2.

We proceed to the second inequality in (15). Suppose that $|r_2| > \alpha$, i.e., that

$$\sqrt{\beta^2 + 4\alpha\gamma} - \beta > 2\alpha ,$$

or

$$\beta^2 + 4\alpha\gamma > \beta^2 + 4\alpha\beta + 4\alpha^2 . \tag{16}$$

Now α cannot be 0, since $\alpha \geqq \gamma$ and $\beta \leqq 1/2$. Hence (16) implies that

$$\gamma > \alpha + \beta ,$$

i.e.,

$$\gamma > \frac{1}{2} ,$$

which contradicts (14). Therefore the inequalities (15) hold. Thus for any α, β, γ satisfying (14) we have

$$\begin{aligned} \operatorname{per}(\alpha I_n + \beta P + \gamma P^2) &= r_1^n + r_2^n + \alpha^n + \gamma^n \\ &\geqq r_1^n + \gamma^n + (\alpha^n - |r_2|^n) \\ &> r_1^n \\ &\geqq \frac{1}{2^n} . \end{aligned}$$

THEOREM 5. *If α, β, γ are nonnegative numbers, $n \geqq 5$, then*

$$\min_{\alpha+\beta+\gamma=1} (\operatorname{per}(\alpha I_n + \beta P + \gamma P^2)) < \operatorname{per}\left(\frac{1}{3}I_n + \frac{1}{3}P + \frac{1}{3}P^2\right) . \tag{17}$$

In other words, the minimum of the permanent function on the convex hull of I_n, P, P^2, $n \geqq 5$, is not attained for $\alpha = \beta = \gamma = 1/3$.

Proof. By Theorem 4,

$$\min_{\alpha+\beta+\gamma=1} (\text{per}\,(\alpha I_n + \beta P + \gamma P^2) \leqq \frac{1}{2^{n-1}}\,.$$

From (9) we compute

$$\text{per}\left(\frac{1}{3}I_n + \frac{1}{3}P + \frac{1}{3}P^2\right) = \left(\frac{1+\sqrt{5}}{6}\right)^n + \left(\frac{1-\sqrt{5}}{6}\right)^n + \frac{1}{3^n} + \frac{1}{3^n}$$
$$> \left(\frac{1+\sqrt{5}}{6}\right)^n + \frac{1}{3^n}\,,$$

which is greater than $1/2^{n-1}$ for $n \geqq 10$. It can be checked by computation, that (17) holds for $5 \leqq n \leqq 9$ as well.

An explicit formula for $\min_{\alpha+\beta+\gamma=1}(\text{per}\,(\alpha I_n + \beta P + \gamma P^2))$, $\alpha, \beta, \gamma \geqq 0$, appears to be out of reach. The available numerical data for $n \leqq 18$ seem to indicate that the values of α, β, γ, at which the minimum is attained are the same for $n = 2k-1$ and $n = 2k$, for any k, but that otherwise they vary with n.

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