PACIFIC JOURNAL OF MATHEMATICS
Vol. 42, No. 2, 1972

INTEGRABILITY OF TRIGONOMETRIC SERIES II

MASAKO IZUMI AND SHIN-ICHI IZUMI

There are proved theorems of the following type. Let $0<r<1$ and $g(t) \sim \sum_{n=1}^{\infty} b_n$ sin nt and suppose $\int_0^{\pi} g(t)\,|t-u|^{-r}dt \leqq A$ for all $u > 0$. Then $\sum_{n=1}^{\infty} |b_n|\, n^{r-1} < \infty$.

1. Sz.-Nagy [7] (cf. [1]) has proved the following

THEOREM 1. *Let $0 < r < 1$ and $g(t) \sim \sum_{n=1}^{\infty} b_n \sin nt$. If $g \downarrow$ and is bounded below, then*

$$t^{-r}g(t) \in L(0, \pi) \Longleftrightarrow \sum_{n=1}^{\infty} |b_n|/n^{1-r} < \infty . \tag{1}$$

The same holds for even functions.

In this theorem, the assumption $g \downarrow$ cannot be replaced by $g \geqq 0$, that is,

THEOREM 2. ([1]), [4]) *Let $0 < r < 1$, then there is an odd function*

$$g \sim \sum_{n=1}^{\infty} b_n \sin nt$$

such that

(i) $g \geqq 0$ *on* $(0, \pi)$,
(ii) $t^{-r}g(t) \in L(0, \pi)$,

but

(iii) $\sum_{n=1}^{\infty} |b_n|/n^{1-r} = \infty$.

The same holds for even functions.

Sunouchi [6], Edmonds [3] and Boas [2] (cf. [1]) proved that

THEOREM 3. *Let $0 < r < 1$ and $b_n \downarrow 0$. Then the function g defined by $g(t) = \sum_{n=1}^{\infty} b_n \sin nt$, satisfies the relation* (1). *The same holds for even functions.*

The monotonicity of (b_n) in this theorem cannot be replaced by positivity, that is,

THEOREM 4. ([1], [4]) *Let $0 < r < 1$, then there is an odd function*

$$g(t) \sim \sum_{n=1}^{\infty} b_n \sin nt$$

such that

(i) $b_n \geqq 0$ (*or* $b_n \to 0$ *and* $\sum |\Delta b_n| < \infty$),

(ii) $\sum_{n=1}^{\infty} |b_n|/n^{1-r} < \infty$,

but

(iii) $t^{-r}g(t) \notin L(0, \pi)$.

The same holds for even functions.

We shall prove the following theorems.

THEOREM 5. *Let* $0 < r < 1$ *and* $g(t) \sim \sum_{n=1}^{\infty} b_n \sin nt$, *then*

$$(2) \qquad \int_0^{\pi} \frac{g(t)}{|t-u|^r} dt \leqq A \text{ for all } u > 0 \Longrightarrow \sum_{n=1}^{\infty} |b_n|/n^{1-r} < \infty .$$

The same holds for even functions.

THEOREM 6. *Let* $0 < r < 1$ *and* $g(t) \sim \sum_{n=1}^{\infty} b_n \sin nt$. *Then*

$$(3) \qquad \sum_{n=1}^{\infty} n^r |\Delta b_n| < \infty \Longrightarrow t^{-r}g(t) \in L(0, \pi) .$$

The same holds for even functions.

By Theorem 4, the converse implication of (2) does not hold. We have proved that [5], for $0 < r < 1$,

$$\sum_{n=1}^{\infty} |b_n|/n^{1-r} < \infty \Longrightarrow \exists \int_{+0}^{\pi} g(t) t^{-r} dt .$$

The left side of (2) is satisfied when

(i) $g \downarrow$ on $(0, \delta)$ and bounded on (δ, π) and $t^{-r}g(t) \in L(0, \pi)$, or

(ii) there is an odd function g_1 such that

$$|g(t)| \leqq A\, g_1(t) \text{ on } (0, \pi)$$

and g_1 satisfies the condition in (i), or

(iii) there is a finite set of points $(x_1, x_2, \cdots, x_n)$ on $(0, \pi)$ such that the odd function g becomes monotonically infinite on one side or both side of each x_i and is bounded outside of each neighbourhood of x_i and

$$|g(t)|/|t - x_i|^r \in L(0, \pi) \qquad (i = 1, 2, \cdots, n) .$$

(i) shows that Theorem 5 contains the $\int \to \sum$ part of Theorem 1 as a particular case. (ii) and (iii) are generalizations of (i).

If $b_n \downarrow 0$, the right side of (1) is equivalent to the left side of (3), and then the $\sum \to \int$ part of Theorem 3 is a particular case of Theorem 6.

2. *Proof of Theorem* 5. We consider the function

$$h(t) = \sum_{n=2}^{\infty} (-1)^n \sin nt/n^{1-r}\,, \tag{3}$$

then

$$k(t) = (g*h)(t) = \frac{2}{\pi}\int_0^{\pi} g(u)h(t+u)du \sim \sum_{n=2}^{\infty} \frac{(-1)^n}{n^{1-r}} b_n \cos nt\,.$$

If $k \in \mathrm{Lip}\,1$, then its Fourier series converges absolutely by Bernstein's theorem, which gives the conclusion of the theorem. Therefore it is sufficient to prove that

$$|k(t+v) - k(t)| \leqq Av \text{ for small } v > 0 \text{ and } 0 < t < \pi\,. \tag{4}$$

Now,

$$\begin{aligned} k(t+v) - k(t) &= \frac{2}{\pi}\int_0^{\pi} g(u)(h(t+u+v) - h(t+u))du \\ &= \frac{2}{\pi}\int_0^{\pi} g(u-t)(h(u+v) - h(u))du\,, \end{aligned} \tag{5}$$

where, by (3),

$$\begin{aligned} h(u) &= \sum_{n=1}^{\infty}\left(\frac{\sin 2nu}{(2n)^{1-r}} - \frac{\sin(2n+1)u}{(2n+1)^{1-r}}\right) \\ &= \sum_{n=1}^{\infty}\left(\frac{1}{(2n)^{1-r}} - \frac{1}{(2n+1)^{1-r}}\right)\sin 2nu \\ &\quad + (1-\cos u)\sum_{n=1}^{\infty}\frac{\sin 2nu}{(2n+1)^{1-r}} - \sin u \sum_{n=1}^{\infty}\frac{\cos 2nu}{(2n+1)^{1-r}} \\ &= P(u) + Q(u) - R(u)\,. \end{aligned} \tag{6}$$

We write

$$p(u) = \frac{1}{(2u)^{1-r}} - \frac{1}{(2u+1)^{1-r}} \text{ for } u > 0$$

and

$$p(n) = p_n (n = 1, 2, \cdots)\,,$$

then

$$\begin{aligned} P(u) &= \sum_{n=1}^{\infty} p_n \sin 2nu \\ &= \int_{1/2}^{\infty} p(w)\sin 2uw\, dw + \int_{1/2}^{\infty} p(w)\sin 2uw\, dj(w) \end{aligned}$$

where

$$j(w) = -w + [w] + \frac{1}{2} \sim \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{\sin 2\pi m w}{m}$$

and then

$$\begin{aligned} P(u+v) - P(u) &= \left\{ \int_{1/2}^{\infty} p(w) \sin 2(u+v)w\, dw - \int_{1/2}^{\infty} p(w) \sin 2uw\, dw \right\} \\ &\quad + \left\{ \int_{1/2}^{\infty} p(w) \sin 2(u+v)w\, dj(w) - \int_{1/2}^{\infty} p(w) \sin 2uw\, dj(w) \right\} \\ &= S_1 + S_2 . \end{aligned}$$

We shall first show that $S_1 = 0(v/u^r)$.

$$\begin{aligned} S_1 &= \frac{1}{2(u+v)} \int_{u+v}^{\infty} p\left(\frac{x}{2(u+v)}\right) \sin x\, dx - \frac{1}{2u} \int_{u}^{\infty} p\left(\frac{x}{2u}\right) \sin x\, dx \\ &= \frac{1}{2} \int_{u+v}^{\infty} \left(\frac{1}{u+v} p\left(\frac{x}{2(u+v)}\right) - \frac{1}{u} p\left(\frac{x}{2u}\right) \right) \sin x\, dx \\ &\qquad\qquad - \frac{1}{2u} \int_{u}^{u+v} p\left(\frac{x}{2u}\right) \sin x\, dx \\ &= \frac{1}{2} \int_{u+v}^{\infty} \left\{ \left(\frac{1}{(u+v)^r} - \frac{1}{u^r} \right) \frac{1}{x^{1-r}} \right. \\ &\qquad \left. - \left(\frac{1}{(u+v)^r (x+u+v)^{1-r}} - \frac{1}{u^r (x+u)^{1-r}} \right) \right\} \sin x\, dx \qquad (7) \\ &\quad - \frac{1}{2u^r} \int_{u}^{u+v} \left(\frac{1}{x^{1-r}} - \frac{1}{(x+u)^{1-r}} \right) \sin x\, dx \\ &= \frac{1}{2} \left(\frac{1}{(u+v)^r} - \frac{1}{u^r} \right) \int_{u+v}^{\infty} \left(\frac{1}{x^{1-r}} - \frac{1}{(x+u)^{1-r}} \right) \sin x\, dx \\ &\quad + \frac{1}{2(u+v)^r} \int_{u+v}^{\infty} \left(\frac{1}{(x+u)^{1-r}} - \frac{1}{(x+u+v)^{1-r}} \right) \sin x\, dx \\ &\quad - \frac{1}{2u^r} \int_{u}^{u+v} \left(\frac{1}{x^{1-r}} - \frac{1}{(x+u)^{1-r}} \right) \sin x\, dx \\ &= O(v/u^r) , \end{aligned}$$

since

$$\begin{aligned} &\int_{u+v}^{\infty} \left(\frac{1}{x^{1-r}} - \frac{1}{(x+u)^{1-r}} \right) \sin x\, dx \\ &= (1 - \cos u) \int_{2u+v}^{\infty} \frac{\sin x}{x^{1-r}} dx + \int_{2u+v}^{\infty} \frac{\sin u \cdot \cos x}{x^{1-r}} dx \\ &\qquad + \int_{u+v}^{2u+v} \frac{\sin x}{x^{1-r}} dx \end{aligned}$$

and similarly for the integral of the second term of the right side of (7). The estimation of S_2 is a little more complicated. By integration by parts,

$$S_2 = \int_{1/2}^{\infty} p(w) \sin 2(u+v)w \, dj(w) - \int_{1/2}^{\infty} p(w) \sin 2uw \, dj(w)$$

$$= - \int_{1/2}^{\infty} p'(w) \sin 2(u+v)w \, j(w) \, dw$$

$$- 2(u+v) \int_{1/2}^{\infty} p(w) \cos 2(u+v)w \, j(w) \, dw$$

$$+ \int_{1/2}^{\infty} p'(w) \sin 2uw \, j(w) \, dw$$

$$+ 2u \int_{1/2}^{\infty} p(w) \cos 2uw \, j(w) \, dw ,$$

and using the Fourier expansion of $j(w)$, we get

$$S_2 = - \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{1}{m} \Big\{ \int_{1/2}^{\infty} p'(w)(\sin 2(u+v)w \cdot \sin 2\pi m w$$

$$- \sin 2uw \cdot \sin 2\pi m w) dw$$

$$- 2 \int_{1/2}^{\infty} p(w)((u+v) \cos 2(u+v)w \cdot \sin 2\pi m w$$

$$- u \cos 2uw \sin 2\pi m w) dw \Big\}$$

$$= \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{1}{m} \{ 2(1-r) T_1 + 2T_2 \} .$$

In order to get $S_2 = O(v)$, it is sufficient to prove that T_1 and T_2 are of order $O(v/m)$. Now, for $m \geqq 3$, we have

$$T_1 = \int_{1/2}^{\infty} \left(\frac{1}{(2w)^{2-r}} - \frac{1}{(2w+1)^{2-r}} \right)$$

$$\cdot \{ (\cos 2(m\pi - u - v)w - \cos 2(m\pi - u)w)$$

$$- (\cos 2(m\pi + u + v)w - \cos (m\pi + u)w) \} dw$$

$$= \frac{1}{2} \Big\{ \int_{m\pi - u - v}^{\infty} \left(\left(\frac{m\pi - u - v}{x} \right)^{2-r} - \left(\frac{m\pi - u - v}{x + m\pi - u - v} \right)^{2-r} \right) \frac{\cos x}{m\pi - u - v} dx$$

$$- \int_{m\pi - u}^{\infty} \left(\left(\frac{m\pi - u}{x} \right)^{2-r} - \left(\frac{m\pi - u}{x + m\pi - u} \right)^{2-r} \right) \frac{\cos x}{m\pi - u} dx \Big\}$$

$$- \frac{1}{2} \Big\{ \int_{m\pi + u + v}^{\infty} \left(\left(\frac{m\pi + u + v}{x} \right)^{2-r} - \left(\frac{m\pi + u + v}{x + m\pi + u + v} \right)^{2-r} \right) \frac{\cos x}{m\pi + u + v} dx$$

$$- \int_{m\pi + u}^{\infty} \left(\left(\frac{m\pi + u}{x} \right)^{2-r} - \left(\frac{m\pi + u}{x + m\pi + u} \right)^{2-r} \right) \frac{\cos x}{m\pi + u} dx \Big\}$$

$$= \frac{1}{2} \Big\{ \int_{m\pi - u}^{\infty} \left(\frac{(m\pi - u - v)^{1-r}}{x^{2-r}} - \frac{(m\pi - u)^{1-r}}{x^{2-r}} \right.$$

$$\left. - \frac{(m\pi - u - v)^{1-r}}{(x + m\pi - u - v)^{2-r}} + \frac{(m\pi - u)^{1-r}}{(x + m\pi - u)^{2-r}} \right) \cos x \, dx + O(v/m) \Big\}$$

$$- \frac{1}{2} \{ \text{similar terms} \}$$

$$= O(v/m)$$

$$T_2 = \int_{1/2}^{\infty} p(w)\{(u+v)(\sin 2(m\pi+u+v)w + \sin 2(m\pi-u-v)w)$$
$$- u(\sin 2(m\pi+u)w + \sin 2(m\pi-u)w)\}\,dw$$
$$= \frac{1}{2}\left\{(u+v)\int_{m\pi+u+v}^{\infty}\left(\left(\frac{m\pi+u+v}{x}\right)\right)^{1-r}\right.$$
$$-\left(\frac{m\pi+u+v}{x+m\pi+u+v}\right)^{1-r}\Bigg)\frac{\sin x}{m\pi+u+v}dx$$
$$\left.- u\int_{m\pi+u}^{\infty}\left(\left(\frac{m\pi+u}{x}\right)^{1-r} - \left(\frac{m\pi+u}{x+m\pi+u}\right)^{1/r}\right)\frac{\sin x}{m\pi+u}dx\right\}$$
$$+ \frac{1}{2}\{\text{similar terms}\}$$
$$= O(v/m).$$

When $m = 1$ or 2, we get easily

$$|T_1| \leqq A\int_{1/2}^{\infty}\frac{v}{w^{2-r}}dw \leqq Av,\ |T_2| \leqq Av\,.$$

Thus we have proved $T_1 + T_2 = O(v/m)$ for all m, and then $S_2 = O(v)$. Collecting the above estimates, we get

$$(8)\qquad \int_0^{\pi}|g(u-t)|\,|P(u+v)-P(u)|\,du \leqq Av\int_0^{\pi}|g(u-t)|\,u^{-r}du \leqq Av\,.$$

Now, by (6),

$$Q(u+v) - Q(u) = (1-\cos(u+v))\sum_{n=1}^{\infty}\frac{\sin 2n(u+v)}{(2n+1)^{1-r}}$$
$$- (1-\cos u)\sum_{n=1}^{\infty}\frac{\sin 2nu}{(2n+1)^{1-r}}$$
$$= (\cos u - \cos(u+v))\sum_{n=1}^{\infty}\frac{\sin 2n(u+v)}{(2n+1)^{1-r}}$$
$$+ (1-\cos u)\left(\sum_{n=1}^{\infty}\frac{\sin 2n(u+v)}{(2n+1)^{1-r}} - \sum_{n=1}^{\infty}\frac{\sin 2nu}{(2n+1)^{1-r}}\right)$$
$$= U + (1-\cos u)V$$

where

$$|U| \leqq v(u+v)\left|\sum_{n=1}^{\infty}\Delta\left(\frac{1}{(2n+1)^{1-r}}\right)\tilde{D}_n(2u+v))\right| \leqq Av\,,$$

$\tilde{D}_n$ being the nth conjugate Dirichlet kernel, and

$$V = \left\{\int_{1/2}^{\infty}\frac{\sin 2(u+v)w}{(2w+1)^{1-r}}dw - \int_{1/2}^{\infty}\frac{\sin 2uw}{(2w+1)^{1-r}}dw\right\}$$
$$+\left\{\int_{1/2}^{\infty}\frac{\sin 2(u+v)w}{(2w+1)^{1-r}}dj(w) - \int_{1/2}^{\infty}\frac{\sin 2uw}{(2w+1)^{1-r}}dj(w)\right\}$$
$$= V_1 + V_2.$$

It is sufficient to prove that $(1 - \cos u)V = 0(v)$. Proceeding as for S_1 and S_2, we get

$$V_1 = \frac{1}{2(u+v)^r}\int_{u+v}^{\infty} \frac{\sin x}{(x+u+v)^{1-r}}dx - \frac{1}{2u^r}\int_u^{\infty}\frac{\sin x}{(x+u)^{1-r}}dx$$
$$= O(v/u)$$

and

$$V_2 = O(v) .$$

Therefore

$$Q(u+v) - Q(u) = O(v) ,$$

and

$$(9) \qquad \int_0^{\pi}|g(u-t)|\,|Q(u+v) - Q(u)|\,du \leqq Av\int_0^{\pi}|g(u)|\,du \leqq Av .$$

Finally we shall estimate $R(u+v) - R(u)$. By the definition (6),

$$R(u) = \sin u \sum_{n=1}^{\infty} \frac{\cos 2nu}{(2n+1)^{1-r}}$$
$$= \sin u \int_{1/2}^{\infty} \frac{\cos 2uw}{(2w+1)^{1-r}}dw + \sin u \int_{1/2}^{\infty}\frac{\cos 2uw}{(2w+1)^{1-r}}dj(w)$$

and

$$R(u+v) - R(u) = (\sin(u+v) - \sin u)\int_{1/2}^{\infty}\frac{\cos 2uw}{(2w+1)^{1-r}}dm$$
$$+ \sin(u+v)\int_{1/2}^{\infty}\frac{\cos 2(u+v)w - \cos 2uw}{(2w+1)^{1-r}}dw$$
$$+ \left\{\sin(u+v)\int_{1/2}^{\infty}\frac{\cos 2(u+v)w}{(2w+1)^{1-r}}dj(w)\right.$$
$$\left. - \sin u \int_{1/2}^{\infty}\frac{(2w+1)^{1-r}}{\cos 2uw}dj(w)\right\}$$
$$= W_1 + W_2 + W_3$$

where

$$|W_1| \leqq A\frac{v}{u^r}\left|\int_u^{\infty}\frac{\cos x}{(x+u)^{1-r}}dx\right| \leqq A\frac{v}{u^r} ,$$

$$|W_2| = \left|\frac{\sin(u+v)}{(u+v)^r}\int_{u+v}^{\infty}\frac{\cos x}{(x+u+v)^{1-r}}dx - \frac{\sin(u+v)}{u^r}\int_u^{\infty}\frac{\cos x}{(x+u)^{1-r}}dx\right|$$
$$= \left|\frac{\sin(u+v)}{(u+v)^r}\int_{u+v}^{\infty}\left(\frac{1}{(x+u+v)^{1-r}} - \frac{1}{(x+u)^{1-r}}\right)\cos x\,dx\right.$$
$$+ \left(\frac{1}{u^r} - \frac{1}{(u+v)^r}\right)\sin(u+v)\int_{u+v}^{\infty}\frac{\cos x}{(x+u)^{1-r}}dx$$

$$
\begin{aligned}
&- \frac{\sin(u+v)}{u^r}\int_u^{u+v}\frac{\cos x}{(x+u)^{1-r}}dx\Big| \\
&\leqq A(u+v)^{1-r}\int_{u+v}^{5\pi/2}\frac{v}{(x+u)^{2-r}}dx + A\frac{v(u+v)}{u^{1+r}}\left|\int_{u+v}^{\infty}\frac{\cos x}{(x+u)^{1-r}}dx\right| \\
&+ A\,\frac{u+v}{u}\int_u^{u+v}dx \\
&\leqq Av + Av/u^r + Av
\end{aligned}
$$

and $W_3 = O(v)$ by an estimation similar to that used for S_2. Therefore

$$
\int_0^{\pi}|g(u-t)|\;|R(u+v)-R(u)|\,du \leqq Av\,. \tag{10}
$$

Combining (8), (9) and (10), we get the required result (4). Thus we have completed the proof of Theorem 5.

3. *Proof of Theorem* 6. We write $s = [\pi/t]$ and

$$
\begin{aligned}
\int_0^{\pi}t^{-r}|g(t)|\,dt &\leqq \int_0^{\pi}t^{-r}\left|\sum_{n=1}^{s}b_n\sin nt\right|dt \\
&+ \int_0^{\pi}t^{-r}\left|\sum_{n=s+1}^{\infty}b_n\sin nt\right|dt = U + V
\end{aligned}
$$

where

$$
\begin{aligned}
U &\leqq \int_0^{\pi}t^{1-r}\left|\sum_{n=1}^{s}nb_n\right| = \sum_{m=1}^{\infty}\int_{\pi/(m+1)}^{\pi/m}t^{1-r}\left|\sum_{n=1}^{m}nb_n\right|dt \\
&\leqq A\sum_{m=1}^{\infty}\frac{1}{m^{3-r}}\sum_{n=1}^{m}n|B_n| \leqq A\sum_{n=1}^{\infty}|b_n|/n^{1-r} \\
&\leqq A\sum_{n=1}^{\infty}n^r|\varDelta b_n| < \infty
\end{aligned}
$$

and

$$
\begin{aligned}
V &= \int_0^{\pi}t^{-r}\Big| - b_{s+1}\tilde{D}_s(t) + \sum_{n=s+1}^{\infty}\varDelta b_n\tilde{D}_n(t)\Big|\,dt \\
&\leqq A\sum_{m=1}^{\infty}\frac{|b_m|}{m^{1-r}} + A\sum_{m=1}^{\infty}\frac{1}{m^{1-r}}\sum_{n=m+1}^{\infty}|\varDelta b_n| \\
&\leqq A\sum_{n=1}^{\infty}n^r|\varDelta b_n| < \infty\,.
\end{aligned}
$$

Thus we have proved $t^{-r}g(t) \in L$ under the assumption

$$
\sum_{n=1}^{\infty}n^r|\varDelta b_n| < \infty\,.
$$

References

1. R. P. Boas, Jr., *Integrability theorems for trigonometric transforms*, Ergebnisse der Math. u. ihrer Grenzgebiete, **38** (1967).
2. ———, *Integrability of trigonometric series II*, Quart. J. Math. Oxford Ser., (2) **3** (1952), 217-221.
3. S. M. Edmonds, *The Parseval formula for monotone functions, I, II, III, IV*, Proc. Cambridge Phil. Soc., **43** (1947), 289-306; **46** (1950), 231-248, 249-267; **49** (1953), 218-229.
4. M. and S. Izumi, *Integrability of trigonometric series*, J. of Math. Anal. and Appl. **32** (1970), 584-589.
5. S. Izumi and M. Satô, *Integrability of trigonometric series I*, Tôhoku Math. J., (2) **5** (1954), 258-263.
6. G. Sunouchi, *Integrability of trigonometric series*, J. Math., (Tokyo) **1** (1953), 99-103.
7. B. Sz-Nagy, *Séries et intégrales de Fourier des fonctions monotones non bornées*, Acta Sci. Math., (Szeged) **13** (1949), 118-135.

Received April 1, 1971.

MATHÉMATIQUE
FACULTÉ DES SCIENCES
UNIVERSITÉ DE SHERBROOKE
SHERBROOKE, CANADA
AND
DEPARTMENT OF MATHEMATICS,
INSTITUTE FOR ADVANCED STUDIES
THE AUSTRALIAN NATIONAL UNIVERSITY
CANBERRA, AUSTRALIA