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CONTINUITY OF SAMPLE FUNCTIONS OF BIADDITIVE PROCESSES

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Let $\{X(s,t): 0 \leqq s, t \leqq 1\}$ be a stochastic process which has independent increments (second differences). Necessary and sufficient conditions are established to ensure the existence of a version with the property that almost every sample function is continuous. A corollary to these results is the existence of a class of measures on Wiener-Yeh space. The conditions are analogous to the usual case of additive processes $Z(t)$ indexed by one time parameter.

$X(s, t)$ will be said to have independent "increments" (second differences) if whenever $0 \leqq s_0 < s_1 < \cdots < s_m \leqq 1$ and $0 \leqq t_0 < t_1 < \cdots < t_n \leqq 1$ the random variables $X(s_i, t_j) - X(s_{i-1}, t_j) - X(s_i, t_{j-1}) + X(s_{i-1}, t_{j-1})$ $i = 1, \cdots, m$, $j = 1, \cdots, n$ are independent. If $X(s, t)$ has independent increments and $X(0, t) = X(s, 0) = 0$, then $X(s, t)$ will be called biadditive. Let $m(s, t) = E[X(s, t)]$ and $v(s, t) = \text{var}\,[X(s, t)]$. The following result is proved below:

There is a version of a biadditive process $X(s, t)$ with the property that almost every sample function is continuous if and only if $X(s, t)$ is Gaussian, $m(s, t)$ and $v(s, t)$ are continuous, and $v(s, t)$ is the distribution function of a Lebesgue-Stieltjes measure on $[0, 1] \times [0, 1]$.

A special case of this result occurs when $m(s, t) = 0$ and $v(s, t) = st$. This process is realized when the space C_2 of continuous functions of two variables on $[0, 1] \times [0, 1]$ is assigned the Wiener-Yeh measure and $X(s, t)$ is defined by $X(s, t)(f) = f(s, t)$ where $f \in C_2$. Theorem 2 will imply the existence of a class of Wiener-Yeh measures on C_2 corresponding to the choices of a pair of continuous functions $m(s, t)$ and $v(s, t)$.

The conditions on $m(s, t)$ and $v(s, t)$ are analogous to the well-known conditions for the usual case of a stochastic process indexed by one time parameter. The case for a process indexed by n-time parameters is similar. The proof here is probabilistic in nature, unlike the analytic proof given by Yeh in [2] for the special case above.

2. Statement of main results.

THEOREM 1. *Let $X(s, t)$ be a biadditive process having the property that almost every sample function is continuous. Then $X(s, t)$ is Gaussian and the increments of $X(s, t)$ are Gaussian. Furthermore the functions $m(s, t) = EX(s, t)$ and $v(s, t) = \text{var}\,(X(s, t))$ are continuous* *and determine the distribution of the process.*

The following corollary is easy and its proof will be omitted.

COROLLARY. *Let $X(s, t)$ be as in Theorem 1. If the increments of $X(s, t)$ are stationary, that is, if the distribution of $X(s+h_1, t+h_2) - X(s, t+h_2) - X(s+h_1, t) + X(s, t)$ depends only on h_1 and h_2, then there are constants c_1 and c_2 such that*

$$m(s, t) = EX(s, t) = c_1 st$$

$$v(x, t) = \operatorname{var}(X(s, t)) = c_2 st \,.$$

THEOREM 2. *Let $m(s, t)$ and $v(s, t)$ be continuous functions on $[0, 1] \times [0, 1]$ such that $m(s, 0) = 0 = m(0, t)$ and $v(s, 0) = 0 = v(0, t)$ for $0 \leqq s, t \leqq 1$. Suppose that $v(s, t)$ satisfies the condition*

$$\text{(A)} \qquad v(s'', t'') - v(s'', t') - v(s', t'') + v(s', t') \geqq 0$$

whenever

$$0 \leqq s' \leqq s'' \leqq 1 \quad \text{and} \quad 0 \leqq t' < t'' \leqq 1 \,.$$

Then there is a biadditive Gaussian process $X(s, t)$, $0 \leqq s, t \leqq 1$, such that

(i) *$EX(s, t) = m(s, t)$ and $\operatorname{var}(X(s, t)) = v(s, t)$ and*

(ii) *almost every sample function of $X(s, t)$ is continuous on $[0, 1] \times [0, 1]$.*

The distribution of $X(s, t)$ is determined by $m(s, t)$ and $v(s, t)$.

3. *Proof of Theorem* 1. We prove first that $X(s, t)$ is Gaussian.

LEMMA 3.1. *If almost every sample function of $X(s, t)$ is continuous on $[0, 1] \times [0, 1]$, then $X(s, t)$ and its increments are normally distributed.*

Proof. We show that the version of the central limit theorem in reference [1] (Theorem 2, p. 197) applies. Let (s, t) be a fixed point in $[0, 1] \times [0, 1]$ and define $s_i = s(i/n)$, $t_i = t(i/n)$, and

$$\Delta_{ij}(n) = X(s_i, t_j) - X(s_i, t_{j-1}) - X(s_{i-1}, t_j) + X(s_{i-1}, t_{j-1}) \,.$$

Let $\varepsilon > 0$ be given and let $A_n = [\max_{i,j=1,2,\cdots,n} |\Delta_{ij}(n)| \geqq \varepsilon]$. Then almost every sample function of $X(s, t)$ is uniformly continuous on $[0, 1] \times [0, 1]$, and consequently

$$P\{\limsup_{n\to\infty} A_n\} = 0 \,.$$

Hence $\limsup_{n\to\infty} P(A_n) = 0$.

Now $X(s, t)$ is the sum of independent random variables, that is,

$$X(s, t) = \sum_{i=1}^{n} \sum_{j=1}^{n} \Delta_{ij}(n) .$$

The $\Delta_{ij}(n)$ form an infinitesimal system because

$$\max_{i,j=1,2,\cdots,n} P[|\Delta_{ij}(n)| \geqq \varepsilon] \leqq P[\max_{i,j=1,2,\cdots,n} |\Delta_{ij}(n)| \geqq \varepsilon]$$

and since

$$\limsup_{n\to\infty} P(A_n) = 0 ,$$

$$\lim_{n\to\infty} \max_{i,j=1,2,\cdots,n} P[|\Delta_{ij}(n)| \geqq \varepsilon] = 0 .$$

It follows that $X(s, t)$ is normally distributed.

To show that the increments of $X(s, t)$ are normally distributed, let s_0 and t_0 be fixed and for $s \geqq s_0$, $t \geqq t_0$ consider the process

$$Y(s, t) = X(s, t) - X(s_0, t) - X(s, t_0) + X(s_0, t_0) .$$

It is biadditive and has continuous sample functions a.s. The above argument shows that $Y(s, t)$ is Gaussian and hence the increments of $X(s, t)$ are Gaussian.

To complete the proof of Theorem 1 we need to check that $m(s, t)$ and $v(s, t)$ are continuous and determine the distribution of the process. Since $X(s, t)$ is biadditive, we have for $s' < s''$ and $t' < t''$

$$\begin{aligned} \mathrm{var}\,(X(s'', t'')) = {} & \mathrm{var}\,(X(s'', t'') - X(s', t'') - X(s'', s') + X(s', t')) \\ & + \mathrm{var}\,(X(s', t'') - X(s', t')) + \mathrm{var}\,(X(s'', t') \\ & - X(s', t')) + \mathrm{var}\,(X(s', t')) \end{aligned}$$

$$\mathrm{var}\,(X(s', t'') - X(s', t')) + \mathrm{var}\,(X(s', t')) = \mathrm{var}\,(X(s', t''))$$

$$\mathrm{var}\,(X(s'', t') - X(s', t')) + \mathrm{var}\,(X(s', t')) = \mathrm{var}\,(X(s'', t')) .$$

From these equations using $v(s, t) = \mathrm{var}\,(X(s, t))$ we obtain

$$\begin{aligned} & \mathrm{var}\,(X(s'', t'') - X(s'', t') - X(s', t'') + X(s', t')) \\ & \quad = v(s'', t'') - v(s', t'') - v(s'', t') + v(s', t') . \end{aligned}$$

Since a similar relation holds for $m(s, t) = EX(s, t)$, the fact that the increments are Gaussian and $X(s, t)$ is biadditive implies that the distribution of $X(s, t)$ is determined by $m(s, t)$ and $v(s, t)$.

Since almost every sample function is continuous,

$$\lim_{h_1, h_2 \to 0} X(s + h_1, t + h_2) = X(s, t) .$$

Let $\varphi(h_1, h_2, u)$ denote the characteristic function of $X(s + h_1, t + h_2)$. Then

$$\varphi(h_1, h_2, u) = \exp\left\{ium(s + h_1, t + h_2) - \frac{u^2}{2}v(s + h_1, t + h_2)\right\}$$

and hence

$$\begin{aligned} v(s, t) &= -2\log|\varphi(0, 0, 1)| \\ &= -2\lim_{h_1, h_2\to 0}\log|\varphi(h_1, h_2, 1)| \\ &= \lim_{h_1, h_2\to 0} v(s + h_1, t + h_2) \end{aligned}$$

so $v(s, t)$ is continuous. To show $m(s, t)$ is continuous, we use Chebychef's inequality.

$$\lim_{h_1, h_2\to 0} P[|X(s + h_1, t + h_2) - X(s, t) - m(s + h_1, t + h_2) + m(s, t)| \geqq \varepsilon]$$

$$\leqq \lim_{h_1, h_2\to 0} \frac{v(s + h_1, t + h_2) - v(s, t)}{\varepsilon^2} = 0$$

so that

$$X(s + h_1, t + h_2) - X(s, t) - m(s + h_1, t + h_2) + m(s, t) \xrightarrow{P} 0 .$$

Since $X(s + h_1, t + h_2) \to X(s, t)$, it follows that $m(s, t)$ is continuous.

4. *Lemmas for Theorem* 2. In §3, we have shown that any biadditive stochastic process with almost all its sample functions continuous is Gaussian with continuous mean and variance functions. The next task is to show that given a pair of continuous functions $m(s, t)$ and $v(s, t)$ where $v(s, t)$ is a normalized distribution function for a Lebesgue-Stieltjes measure on $[0, 1] \times [0, 1]$, there is a biadditive process $X(s, t)$ such that $EX(s, t) = m(s, t)$ and $\operatorname{var}(X(s, t)) = v(s, t)$. For this proof a few preparatory results are needed. In the following Lemma, * denotes convolution.;

LEMMA 4.1. *Suppose there is a system of probability distributions* $\{\Phi(a_1, b_1, a_2, b_2) | 0 \leqq a_1 < a_2 \leqq 1, 0 \leqq b_1 < b_2 \leqq 1\}$ *such that for any* $\alpha > 0$ *and* $\beta > 0$

$$(1) \qquad \Phi(a_1, b_1, a_2 + \alpha, b_2) = \Phi(a_1, b_1, a_2, b_2) * \Phi(a_2, b_1, a_2 + \alpha, b_2)$$

$$(2) \qquad \Phi(a_1, b_1, a_2, b_2 + \beta) = \Phi(a_1, b_1, a_2, b_2) * \Phi(a_1, b_2, a_2, b_2 + \beta) .$$

Then there is a biadditive process $X(s, t)$ *such that the increment*

$$X(a_2, b_2) - X(a_1, b_2) - X(a_2, b_1) + X(a_1, b_1)$$

has the probability distribution $\Phi(a_1, b_1, a_2, b_2)$ *for* $0 \leqq a_1 < a_2 \leqq 1$ *and* $0 \leqq b < b_2 \leqq 1$.

Proof. The proof uses the Daniell-Kolmogorov extension theorem in the usual manner and is therefore omitted. Conditions (1) and (2) guarantee the consistency of the system.

LEMMA 4.2. (Ottaviani's Inequality). *Let* $\{X_1, X_2, \cdots, X_n\}$ *be independent random variables and let* $S_k \equiv \sum_{i=1}^{k} X_i$. *If for some* $\varepsilon > 0$,

$$P[|S_n - S_k| > \varepsilon] \leqq \frac{1}{2} \quad \textit{for} \quad k = 0, 1, 2, \cdots n \ ,$$

where $S_0 \equiv 0$, *then*

$$P[\max_{k=1,2\cdots n} |S_k| > 2\varepsilon] \leqq 2P[|S_n| > \varepsilon] \ .$$

Proof. The proof may be found in reference [3]. It is very similar to the following lemma which will be proved in full.

LEMMA 4.3. (An extended version of Ottaviani's Inequality). *Let* $s_0 < s_1 < \cdots s_m$ *and* $t_0 < t_1 < t_2 < \cdots < t_n$. *Define*

$$\Delta_{ij} \equiv X(s_i, t_j) - X(s_{i-1}, t_j) - X(s_i, t_{j-1}) + X(s_{i-1}, t_{j-1})$$

where $X(s, t)$ *is a biadditive process on* $D = [0,1] \times [0, 1]$. *Let* $R_l \equiv \sum_{i=1}^{m} \sum_{j=l+1}^{n} \Delta_{ij}$ *and* $Q_{kl} = \sum_{i=k+1}^{m} \Delta_{il}$. *If for all* $k = 1, 2, \cdots, m$ *and* $l = 0, 1, \cdots, n$

$$P\left[|R_l| > \frac{\varepsilon}{2}\right] \leqq 1 - \sqrt{\frac{1}{2}}$$

and

$$P\left[|Q_{kl}| > \frac{\varepsilon}{2}\right] \leqq 1 - \sqrt{\frac{1}{2}} \ ,$$

then

$$P\left[\max_{\substack{k=1,2,\cdots,m \\ l=1,2,\cdots,n}} |S_{kl}| > 2\varepsilon\right] \leqq 2P[|S_{mn}| > \varepsilon] \ .$$

Proof. Let A_{ij} be defined for $i = 1, 2, \cdots, m$, and $j = 1, 2, \cdots, n$ by

$$A_{ij} \equiv [|S_{kl}| \leqq 2\varepsilon \quad \text{for} \quad l < j \quad \text{and} \quad k \leqq m, |S_{kj}| \leqq 2\varepsilon \quad \text{for} \quad k < i, |S_{ij}| > 2\varepsilon]$$
$$A_{11} \equiv [|S_{11}| > 2\varepsilon] \ .$$

Let $T \equiv \{(i, j): 1 = 1, 2, \cdots, m \text{ and } j = 1, 2, \cdots, n\}$. It is clear that

$$\left[\max_{(i,j)\in T} |S_{ij}| > 2\varepsilon\right] = \bigcup_{i=1}^{m} \bigcup_{j=1}^{n} A_{ij}$$

and the A_{ij}'s are disjoint. Now let

$$B_{kl} \equiv \left[|R_l| < \frac{\varepsilon}{2}, |Q_{kl}| < \frac{\varepsilon}{2}\right].$$

Then,

$$A_{kl} \cap B_{kl} \subset [|S_{mn}| > \varepsilon]$$

and so,

$$\bigcup_{l=1}^{n} \bigcup_{k=1}^{m} (A_{kl} \cap B_{kl}) \subset [|S_{mn}| > \varepsilon] .$$

Since $X(s, t)$ is biadditive, A_{kl} and B_{kl} are independent events, and R_l and Q_{kl} are independent random variables. It follows that

$$P(B_{kl}) = P\left[|R_l| < \frac{\varepsilon}{2}\right] \cdot P\left[|Q_{kl}| < \frac{\varepsilon}{2}\right] \geqq \sqrt{\frac{1}{2}} \cdot \sqrt{\frac{1}{2}} = \frac{1}{2} .$$

Hence,

$$\begin{aligned} \frac{1}{2} P\left[\max |S_{ij}| > 2\varepsilon\right] &= \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{n} P(A_{ij}) \leqq \sum_{i=1}^{m} \sum_{j=1}^{n} P(A_{ij} \cap B_{ij}) \\ &= P\left(\bigcup_{i=1}^{m} \bigcup_{j=1}^{n} A_{ij} \cap B_{ij})\right) \\ &\leqq P[|S_{mn}| > \varepsilon] . \end{aligned}$$

LEMMA 4.4. *Let $X(s, t)$ be a biadditive process on a probability space $(\Omega, \mathfrak{B}, P)$ with $(s, t) \in D = [0, 1] \times [0, 1]$. Let $m(s, t) \equiv EX(s, t)$ and $v(s, t) \equiv \operatorname{var}(X(s, t))$ be continuous on D. Then for any point $(s_0, t_0) \in D$ and for any sequence of points $\{(s_n, t_n)\} \subset D$ such that*

$$\lim_{n\to\infty} (s_n, t_n) = (s_0, t_0)$$

$$P\left[\lim_{n\to\infty} X(s_n, t_n) = X(s_0, t_0)\right] = 1 .$$

Proof. Let $\varepsilon > 0$ be chosen arbitrarily except for the condition $\varepsilon < 1 - \sqrt{1/2} < 1/2$. Chebychef's Inequality and the uniform continuity of $m(s, t)$ and $v(s, t)$ imply that there is a $\delta > 0$ such that for (s', t') and $(s, t) \in [s_0 - \delta, s_0 + \delta] \times [t_0 - \delta, t_0 + \delta]$

$$(1) \qquad P\left[|X(s, t) - X(s', t')| \geqq \frac{\varepsilon}{2}\right] < \frac{\varepsilon}{4} .$$

Now let S be a countable dense set in D and let S_1, S_2, S_3, and S_4

denote the sets

$$S_1 \equiv S \cap ([s_0, s_0 + \delta] \times [t_0, t_0 + \delta])$$

$$S_2 \equiv S \cap ([s_0, s_0 + \delta] \times [t_0 - \delta, t_0])$$

$$S_3 \equiv S \cap ([s_0 - \delta, s_0] \times [t_0, t_0 + \delta])$$

$$S_4 \equiv S \cap ([s_0 - \delta, s_0] \times [t_0 - \delta, t_0]) .$$

The first part of the proof will show that

$$P\Big[\sup_{(s,t) \in S_1} |X(s, t) - X(s_0, t_0)| > 6\varepsilon\Big] \leqq 6\varepsilon . \tag{2}$$

The same kind of argument can be used to show that for $i = 2, 3,$ and 4

$$P\Big[\sup_{(s,t) \in S_i} |X(s, t) - X(s_0, t_0)| > 6\varepsilon\Big] \leqq 6\varepsilon \tag{3}$$

and so only the case for S_1 will be done here.

Let the elements of S_1 be numbered in an arbitrary manner so that $S_1 = \{(s_i, t_i) : i = 1, 2, \cdots\}$. Then

$$\begin{aligned} &P\Big[\sup_{(s,t) \in S_1} |X(s, t) - X(s_0, t_0)| > 6\varepsilon\Big] \\ &\quad = \lim_{n\to\infty} P\Big[\max_{i=1,\cdots,n} |X(s_i, t_i) - X(s_0, t_0)| > 6\varepsilon\Big] . \end{aligned} \tag{4}$$

Thus it suffices to show that

$$P\Big[\max_{i=1,\cdots,n} |X(s_i, t_i) - X(s_0, t_0)| > 6\varepsilon\Big] \leqq 6\varepsilon \tag{5}$$

in order to prove (2). Now clearly

$$\begin{aligned} &P\Big[\max_{i=1,\cdots,n} |X(s_i, t_i) - X(s_0, t_0)| > 6\varepsilon\Big] \\ &\quad \leqq P\Big[\max_{i=1,\cdots,n} |X(s_i, t_i) - X(s_0, t_i) - X(s_i, t_0) + X(s_0, t_0)| > 2\varepsilon\Big] \\ &\qquad + P\Big[\max_{i=1,\cdots,n} |X(s_i, t_0) - X(s_0, t_0)| > 2\varepsilon\Big] \\ &\qquad + P\Big[\max_{i=1,\cdots,n} |X(s_0, t_i) - X(s_0, t_0)| > 2\varepsilon\Big] . \end{aligned} \tag{6}$$

Consider the first n points $(s_1, t_1), \cdots, (s_n, t_n)$ in S_1. Let $\sigma_1, \cdots, \sigma_n$ and $\tau_1, \cdots, \tau_n$ be rearrangements of $s_1, \cdots, s_n$ and $t_1, \cdots, t_n$ respectively so that $s_0 \leqq \sigma_1 \leqq \sigma_2 \leqq, \cdots, \leqq \sigma_n \leqq s_0 + \delta$ and $t_0 \leqq \tau_1 \leqq \tau_2 \leqq, \cdots, \leqq \tau_n \leqq t_0 + \delta$. Since $X(s, t)$ is biadditive,

$$X(\sigma_i, \tau_j) - X(\sigma_i, t_0) - X(s_0, \tau_j) + X(s_0, t_0)$$
$$= \sum_{m=1}^{i} \sum_{l=1}^{j} \{X(\sigma_m, \tau_l) - X(\sigma_{m-1}, \tau_l) - X(\sigma_m, \tau_{l-1}) + X(\sigma_{m-1}, \tau_{l-1})\}$$

$$X(\sigma_i, t_0) - X(s_0, t_0) = \sum_{m=1}^{i} \{X(\sigma_m, t_0) - X(\sigma_{m-1}, t_0)\}$$

$$X(s_0, t_j) - X(s_0, t_0) = \sum_{l=1}^{j} \{X(s_0, \tau_l) - X(s_0, \tau_{l-1})\}$$

are sums of independent random variables. Now if (s', t') and (s'', t'') are any two points in $[s_0 - \delta, s_0 + \delta] \times [t_0 + \delta, t_0 + \delta]$, then using (1) we may verify that the hypotheses of the Ottaviani inequalities, Lemmas 4.2 and 4.3, are satisfied. Thus

$$(7) \quad P\Big[\max_{i=1,\cdots,n} |X(\sigma_i, t_0) - X(s_0, t_0)| > 2\varepsilon\Big] \leqq 2P[|X(\sigma_n, t_0) - X(s_0, t_0)| > \varepsilon]$$

$$(8) \quad P\Big[\max_{j=1,\cdots,n} |X(s_0, \tau_j) - X(s_0, t_0)| > 2\varepsilon\Big] \leqq 2P[|X(s_0, \tau_n) - X(s_0, t_0)| > \varepsilon]$$

and

$$(9) \quad \begin{aligned} &P\Big[\max_{\substack{i=1,\cdots,n \\ j=1,\cdots,n}} |X(\sigma_i, \tau_j) - X(s_0, \tau_j) - X(\sigma_i, t_0) + X(s_0, t_0)| > 2\varepsilon\Big] \\ &\leqq 2P[|X(\sigma_n, \tau_n) - X(s_0, \tau_n) - X(\sigma_n, t_0) + X(s_0, t_0)| > \varepsilon] . \end{aligned}$$

From the choice of δ we see that the right sides of inequalities (7), (8), and (9) are each not greater than 2ε. Since the σ_i's are s_i's and τ_i's are t_i's, we have

$$(10) \quad P\Big[\max_{i=1,\cdots,n} |X(s_i, t_0) - X(s_0, t_0)| > 2\varepsilon\Big] \leqq 2\varepsilon$$

$$(11) \quad P\Big[\max_{j=1,\cdots,n} |X(s_0, t_j) - X(s_0, t_0)| > 2\varepsilon\Big] \leqq 2\varepsilon$$

and

$$(12) \quad P\Big[\max_{i=1,\cdots,n} |X(s_i, t_i) - X(s_0, t_i) - X(s_i, t_0) + X(s_0, t_0)| > 2\varepsilon\Big] \leqq 2\,\varepsilon .$$

Substituting (10), (11), and (12) into (6) we get (5), i.e.

$$P\Big[\max_{i=1,\cdots,n} |X(s_i, t_i) - X(s_0, t_0)| > 6\varepsilon\Big] \leqq 6\varepsilon .$$

Then

$$P\Big[\sup_{(s,t) \in S_1} |X(s, t) - X(s_0, t_0)| > 6\varepsilon\Big] \leqq 6\varepsilon .$$

Since the proof of (2) is similar, it is omitted.

Now let $V = S_1 \cup S_2 \cup S_3 \cup S_4$. Then

$$P\Big[\sup_{(s,t)\in V} |X(s, t) - X(s_0, t_0)| > 6\varepsilon\Big] \leqq \sum_{i=1}^{4} P\Big[\sup_{(s,t)\in S_i} |X(s, t) - X(s_0, t_0)| > 6\varepsilon\Big] \tag{13}$$

and hence

$$P\Big[\sup_{(s,t)\in V} |X(s, t) - X(s_0, t_0)| > 6\varepsilon\Big] \leqq 24\varepsilon\ . \tag{14}$$

Taking limits as $\delta \downarrow 0$, we obtain

$$P\Big[\limsup_{\delta\downarrow 0}\ \sup_{V} |X(s, t) - X(s_0, t_0)| > 6\varepsilon\Big] \leqq 24\varepsilon\ . \tag{15}$$

Now let $\varepsilon \downarrow 0$ and take complements to get

$$P\Big[\lim_{\delta\downarrow 0}\ \sup_{V} |X(s, t) - X(s_0, t_0)| = 0\Big] = 1\ . \tag{16}$$

If an arbitrary sequence (s_n, t_n) with $\lim_{n\to\infty} (s_n, t_n) = (s_0, t_0)$ is given, we extend the point set $\{s_n, t_n\}$ to a countable dense set S in D. Then

$$\Big[\lim_{n\to\infty} X(s_n, t_n) = X(s_0, t_0)\Big] \supset \Big[\lim_{\delta\downarrow 0}\ \sup_{V} |X(s, t) - X(s_0, t_0)| = 0\Big]$$

and by (16)

$$P\Big[\lim_{n\to\infty} X(s_n, t_n) = X(s_0, t_0)\Big] = 1\ .$$

LEMMA 4.5. *Let $X(s, t)$ be a biadditive process on a probability space $(\Omega, \mathfrak{B}, P)$ with $(s, t) \in D = [0, 1] \times [0, 1]$. Suppose that $v(s, t) \equiv$* var $(X(s, t))$ *is continuous over D. Furthermore, suppose that for any $\varepsilon > 0$,*

$$\lim_{n\to\infty} \sum_{j=1}^{n} \sum_{k=1}^{n} P\Big[\Big|X\Big(\frac{k}{n}, \frac{j}{n}\Big) - X\Big(\frac{k-1}{n}, \frac{j}{n}\Big) - X\Big(\frac{k}{n}, \frac{j-1}{n}\Big) + X\Big(\frac{k-1}{n}, \frac{j-1}{n}\Big)\Big| > \varepsilon\Big] = 0 \tag{1}$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} P\Big[\Big|X\Big(1, \frac{k}{n}\Big) - X\Big(1, \frac{k-1}{n}\Big)\Big| > \varepsilon\Big] = 0 \tag{2}$$

and

$$\lim_{n\to\infty} \sum_{j=1}^{n} P\Big[\Big|X\Big(\frac{j}{n}, 1\Big) - X\Big(\frac{j-1}{n}, 1\Big)\Big| > \varepsilon\Big] = 0\ . \tag{3}$$

Then there is a process $Y(s, t)$ equivalent to $X(s, t)$ such that almost *every sample function of* $Y(s, t)$ *is continuous on* D.

Proof. Let S be the set of all rational numbers in $[0, 1]$ and let $D' = S \times S$. Define Ω' by $\Omega' = \{\omega \in \Omega: X(s, t)$ is uniformly continuous on $D'\}$. In the first part of the proof, we show that $P(\Omega') = 1$.

Let Z_n be defined on $(\Omega, \mathfrak{B}, P)$ by

$$Z_n = \sup\left\{\left|X(s'', t'') - X(s', t')\right| : (s'', t'') \in D', (s', t') \in D' \text{ and } |s'' - s'| < \frac{1}{n}, |t'' - t'| < \frac{1}{n}\right\}.$$

Then $X(s, t)$ is uniformly continuous on D' if and only if $\lim_{n\to\infty} Z_n = 0$. Hence,

$$P(\Omega') = P\left[\lim_{n\to\infty} Z_n = 0\right]. \tag{4}$$

Let $S_j \equiv S \cap [(j-1)/n, j/n]$ $j = 1, \cdots, n$, and fix n. We number the elements of S_j in an arbitrary manner for each $j = 1, \cdots, n$. Let j and k be now fixed and let $s_1, \cdots, s_{m-1}$ and $t_1, \cdots, t_{m-1}$ denote the first $m - 1$ elements of S_j and S_k respectively. Let $\sigma_1, \cdots, \sigma_{m-1}$ and $\tau_1, \cdots, \tau_{m-1}$ be the arrangements of $\{s_1, \cdots, s_{m-1}\}$ and $\{t_1, \cdots, t_{m-1}\}$ respectively in ascending order so that $\sigma_1 < \sigma_2 < \cdots < \sigma_{m-1}$ and $\tau_1 < \tau_2 < \cdots < \tau_{m-1}$. Choose $\sigma_0 = (j-1)/n$, $\sigma_m = j/n$, $\tau_0 = (k-1)/n$, and $\tau_m = k/n$, and define $S_{jm} \equiv \{\tau_0, \tau_1, \cdots, \tau_m\}$. We will use the notation:

$$\Delta(s, t, s', t') \equiv X(s', t') - X(s, t') - X(s', t) + X(s, t).$$

Since $X(s, t)$ is biadditive, the three collections of random variables below are systems of independent random variables:

$$\{\Delta(\sigma_{\mu-1}, \tau_{\gamma-1}, \sigma_\mu, \tau_\gamma): \mu, \gamma = 1, \cdots, m\}$$

$$\left\{\Delta\left(\frac{j-1}{n}, \tau_{\gamma-1}, \frac{j}{n}, \tau_\gamma\right): \gamma = 1, \cdots, m \quad \text{and} \quad j = 1, \cdots, n\right\}$$

$$\left\{\Delta\left(\sigma_{\mu-1}, \frac{k-1}{n}, \sigma_\mu, \frac{k}{n}\right): \mu = 1, \cdots, m \quad \text{and} \quad k = 1, \cdots, n\right\}.$$

Let $\varepsilon > 0$ be chosen arbitrarily. Since $v(s, t)$ and $m(s, t)$ are continuous on D, they are uniformly continuous and if n is sufficiently large and if $0 < s'' - s' < 1/n$ or $0 < t'' - t' < 1/n$, then from Chebychef's inequality it follows that

$$P\left[|\Delta(s', t', s'', t'')| > \frac{\varepsilon}{2}\right] \leqq 1 - \sqrt{\frac{1}{2}}. \tag{5}$$

Let $Y_{n,j,k} \equiv \sup_{S_j \times S_k} |X(s, t) - X((j-1)/n, (k-1)/n))|$. Then from the triangle inequality we get

$$
\begin{aligned}
Y_n &\equiv \max_{j,k=1,\cdots,n} Y_{n,j,k} \\
&\leqq \max_{j,k,1,\cdots,n} \sup_{S_j\times S_k} \left| \Delta\left(\frac{j-1}{n}, \frac{k-1}{n}, s, t\right) \right| \\
&\quad + \max_{j,k=1,\cdots,n} \sup_{s\in S_j} \left| X\left(s, \frac{k-1}{n}\right) - X\left(\frac{j-1}{n}, \frac{k-1}{n}\right) \right| \\
&\quad + \max_{j,k=1,\cdots,n} \sup_{t\in S_k} \left| X\left(\frac{j-1}{n}, t\right) - X\left(\frac{j-1}{n}, \frac{k-1}{n}\right) \right| .
\end{aligned} \tag{6}
$$

Consequently,

$$
\begin{aligned}
&P[Y_n > 6\varepsilon] \\
&\quad\leqq P\left[\max_{j,k=1,\cdots,n} \sup_{S_j\times S_k} \left| \Delta\left(\frac{j-1}{n}, \frac{k-1}{n}, s, t\right) \right| > 2\varepsilon\right] \\
&\qquad + P\left[\max_{j,k=1,\cdots,n} \sup_{s\in S_j} \left| X\left(s, \frac{k-1}{n}\right) - X\left(\frac{j-1}{n}, \frac{k-1}{n}\right) \right| > 2\varepsilon\right] \\
&\qquad + P\left[\max_{j,k=1,\cdots,n} \sup_{t\in S_k} \left| X\left(\frac{j-1}{n}, t\right) - X\left(\frac{j-1}{n}, \frac{k-1}{m}\right) \right| > 2\varepsilon\right] .
\end{aligned} \tag{7}
$$

For $(\sigma_\mu, \tau_\gamma) \in S_{jm} \times S_{km}$, we see that

$$
\Delta\left(\frac{j-1}{n}, \frac{k-1}{n}, \sigma_\mu, \tau_\gamma\right) = \sum_{q=1}^{\gamma} \sum_{p=1}^{\mu} \Delta(\sigma_{p-1}, \tau_{q-1}, \sigma_p, \tau_q)
$$

a sum of independent random variables. Now (5) implies that the hypotheses of the extended Ottaviani's Inequality (Lemma 4.3) are satisfied and consequently

$$
P\left[\max_{\mu,\gamma=1,\cdots,m} \left| \Delta\left(\frac{j-1}{n}, \frac{k-1}{n}, \sigma_\mu, \tau_\gamma\right) \right| > 2\varepsilon\right] \leqq 2P\left[\left| \Delta\left(\frac{j-1}{n}, \frac{k-1}{n}, \frac{j}{n}, \frac{k}{n}\right) \right| > \varepsilon\right] .
$$

Letting $m \to \infty$, it follows that

$$
P\left[\sup_{S_j\times S_k} \left| \Delta\left(\frac{j-1}{n}, \frac{k-1}{n}, s, t\right) \right| > 2\varepsilon\right] \leqq 2P\left[\left| \Delta\left(\frac{j-1}{n}, \frac{k-1}{n}, \frac{j}{n}, \frac{k}{n}\right) \right| > \varepsilon\right]
$$

and hence

$$
\begin{aligned}
&P\left[\max_{j,k=1,\cdots,n} \sup_{S_j\times S_k} \left| \Delta\left(\frac{j-1}{n}, \frac{k-1}{n}, s, t\right) \right| > 2\varepsilon\right] \\
&\quad \leqq 2\sum_{j=1}^{n} \sum_{k=1}^{n} P\left[\left| \Delta\left(\frac{j-1}{n}, \frac{k-1}{n}, \frac{j}{n}, \frac{k}{n}\right) \right| > \varepsilon\right] .
\end{aligned} \tag{9}
$$

Now if $\sigma_\mu \in S_{im}$, since $X(\sigma_\mu, 0) = X(0, (k-1)/n) = 0$, we have

$$
X\left(\sigma_\mu, \frac{k-1}{n}\right) - X\left(\frac{j-1}{n}, \frac{k-1}{n}\right) = \sum_{p=1}^{\mu} \sum_{q=1}^{k-1} \Delta\left(\sigma_{p-1}, \frac{q-1}{n}, \sigma_p, \frac{q}{n}\right) ,
$$

as before, a sum of independent random variables. Again, (5) allows us to use the extended Ottaviani's Inequality to obtain

$$P\Big[\max_{k=1,\cdots,n}\max_{t\in S_{jm}}\Big|X\Big(s,\frac{k-1}{n}\Big)-X\Big(\frac{j-1}{n},\frac{k-1}{n}\Big)\Big|>2\varepsilon\Big]$$
$$\leqq 2P\Big[\Big|X\Big(\frac{j}{n},1\Big)-X\Big(\frac{j-1}{n},1\Big)\Big|>\varepsilon\Big].$$

Letting $m\to\infty$, we get

$$P\Big[\max_{k=1,\cdots,n}\sup_{s\in S_j}\Big|X\Big(s,\frac{k-1}{n}\Big)-X\Big(\frac{j-1}{n},\frac{k-1}{n}\Big)\Big|>2\varepsilon\Big]$$
$$\leqq 2P\Big[\Big|X\Big(\frac{j}{n},1\Big)-X\Big(\frac{j-1}{n},1\Big)\Big|>\varepsilon\Big]$$

and

$$P\Big[\max_{j,k=1,\cdots,n}\sup_{s\in S_j}\Big|X\Big(s,\frac{k-1}{n}\Big)-X\Big(\frac{j-1}{n},\frac{k-1}{n}\Big)\Big|>2\varepsilon\Big]$$
$$(10)\qquad = P\Big\{\bigcup_{j=1}^{n}\Big[\max_{k=1,\cdots,n}\sup_{s\in S_j}\Big|X\Big(s,\frac{k-1}{n}\Big)-X\Big(\frac{j-1}{n},\frac{k-1}{n}\Big)\Big|>2\varepsilon\Big]\Big\}$$
$$\leqq 2\sum_{j=1}^{n}P\Big[\Big|X\Big(\frac{j}{n},1\Big)-X\Big(\frac{j-1}{n},1\Big)\Big|>\varepsilon\Big].$$

Similarly for $\tau_r\in S_{k,m}$

$$X\Big(\frac{j-1}{n},\tau_r\Big)-X\Big(\frac{j-1}{n},\ \frac{k-1}{n}\Big)=\sum_{p=1}^{j-1}\sum_{q=1}^{r}\Delta\Big(\frac{p-1}{n},\tau_{q-1},\frac{p}{n},\tau_q\Big),$$

a sum of independent random variables, and so by (5) we may again apply the extended Ottaviani's Inequality and take limits as $m\to\infty$. We get

$$(11)\qquad P\Big[\max_{j,k=1,\cdots,n}\sup_{t\in S_k}\Big|X\Big(\frac{j-1}{n},t\Big)-X\Big(\frac{j-1}{n},\frac{k-1}{n}\Big)\Big|>2\varepsilon\Big]$$
$$\leqq 2\sum_{k=1}^{n}P\Big[\Big|X\Big(1,\frac{k}{n}\Big)-X\Big(1,\frac{k-1}{n}\Big)\Big|>\varepsilon\Big].$$

Inserting (9), (10), and (11) into (7) and letting $n\to\infty$, we see from the hypotheses (1), (2), and (3) that

$$(12)\qquad \lim_{n\to\infty}P[Y_n>6\varepsilon]=0.$$

The inequality $Z_n\leqq 4Y_n$ can be checked by succesive applications of the triangle inequality. (If $|s'-s''|<1/n$ and $|t'-t''|<1/n$, $(s',t')\in[(j-1)/n,j/n]\times[(k-1)/n,k/n]$ implies that $(s'',t'')\in[(j-2)/n,(j+1)/n]\times[(k-2)/n,(k+1)n]$ and it suffices to check each possibility.) It follows that

$$P[Z_n > 24\varepsilon] \leqq P[Y_n > 6\varepsilon] .$$

Since $0 \leqq Z_n$ and $Z_{n+1} \leqq Z_n$ for all n,

$$\lim_{n\to\infty} P[Z_n > 24\varepsilon] = P\Big[\lim_{n\to\infty} Z_n > 24\varepsilon\Big] = 0$$

by (12). Letting $\varepsilon \downarrow 0$, we obtain

$$P\Big[\lim_{n\to\infty} Z_n > 0\Big] = 0 ,$$

and since $Z_n \geqq 0$, we get

$$P(\Omega') = P\Big[\lim_{n\to 0} Z_n = 0\Big] = 1 ,$$

which finishes the first part of the proof.

Now if $x(s, t)$ is any real-valued function uniformly continuous on a set D, it has a unique continuous extension to the closure of D. Let $Y(s, t, \omega)$ be defined for $\omega \in \Omega'$ by $Y(s, t, \omega) = X(s, t, \omega)$ if $(s, t) \in D'$.

If $(s, t) \notin D'$, choose a sequence of points (s_n, t_n) in D' such that $\lim_{n\to\infty} (s_n, t_n) = (s, t)$ and define $Y(s, t, \omega) \equiv \lim_{n\to\infty} Y(s_n, t_n, \omega)$ for $\omega \in \Omega'$. Since for $\omega \in \Omega'$ $Y(s, t, \omega)$ is uniformly continuous on D' which is dense in D, $Y(s, t, \omega)$ is well-defined for $\omega \in \Omega'$. If $\omega \notin \Omega'$, let $Y(s, t, \omega) \equiv 0$. Then for $(s, t) \in D'$,

$$P[Y(s, t) = X(s, t)] \geqq P(\Omega') = 1$$

and if $(s, t) \in D$ but $(s, t) \notin D'$,

$$P\Big[Y(s, t) = \lim_{n\to\infty} X(s_n, t_n)\Big] \geqq P(\Omega') = 1$$

for some sequence $\{(s_n, t_n)\}$ in D' such that $\lim_{n\to\infty} (s_n, t_n) = (s, t)$. But by Lemma 2.6,

$$P\Big[X(s, t) = \lim_{n\to\infty} X(s_n, t_n)\Big] = 1$$

and hence for any $(s, t) \in D$,

$$P[Y(s, t) = X(s, t)] = 1 .$$

That is, $Y(s, t)$ is a process which is equivalent to $X(s, t)$. It follows from the definition of $Y(s, t)$, that its sample functions are continuous on Ω', a set of probability one.

5. Proof of Theorem 2.

Proof. Let $\Phi(a, b, c, d)$ denote the normal probability distribution with mean zero and variance $v(c, d) - v(a, d) - v(c, b) + v(a, b)$ where $0 \leqq a < c \leqq 1$ and $0 \leqq b < d \leqq 1$. Then since the convolution of normal distributions is a normal distribution whose mean and variance are the respective sums of the means and variances of the original distributions, for any $\alpha > 0$ we have

$$\Phi(a, b, c + \alpha, d) = \Phi(a, b, c, d)*\Phi(c, b, c + \alpha, d)$$
$$\Phi(a, b, c, d + \alpha) = \Phi(a, b, c, d)*\Phi(a, d, c, d + \alpha)$$

where "$*$" denotes the operation of convolution.

By Lemma 4.1, there is a biadditive process $Y(s, t)$ such that for $s' < s''$ and $t' < t''$, $Y(s'', t'') - Y(s', t'') - Y(s'', t') + Y(s', t')$ is normally distributed with mean zero and variance $v(s'', t'') - v(s', t'') - v(s'', t') + v(s', t')$. If $Y(s, t)$ satisfies conditions (1), (2), and (3) of Lemma 4.5 there is a process $Y_0(s, t)$ equivalent to $Y(s, t)$ such that almost every sample function of $Y_0(s, t)$ is continuous over D. Define $X(s, t) = Y_0(s, t) + m(s, t)$. Then $X(s, t)$ satisfies (i) and (ii) and is biadditive since $Y_0(s, t)$ is. Furthermore almost every sample function of $X(s, t)$ is continuous over D.

Let Δ_{jk} denote the random variable

$$\Delta_{jk} \equiv Y\left(\frac{j}{n}, \frac{k}{n}\right) - Y\left(\frac{j-1}{n}, \frac{k}{n}\right) - Y\left(\frac{j}{n}, \frac{k-1}{n}\right) + Y\left(\frac{j-1}{n}, \frac{k-1}{n}\right)$$

where n is a positive integer. Conditions (1), (2), and (3) of Lemma 4.5 are

$$(1) \qquad \lim_{n\to\infty} \sum_{j=1}^{n} \sum_{k=1}^{n} P[|\Delta_{jk}| > \varepsilon] = 0$$

$$(2) \qquad \lim_{n\to\infty} \sum_{k=1}^{n} P\left[\left| Y\left(1, \frac{k}{n}\right) - Y\left(1, \frac{k-1}{n}\right)\right| > \varepsilon\right] = 0$$

$$(3) \qquad \lim_{n\to\infty} \sum_{j=1}^{n} P\left[\left| Y\left(\frac{j}{n}, 1\right) - Y\left(\frac{j-1}{n}, 1\right)\right| > \varepsilon\right] = 0$$

where $\varepsilon > 0$ is chosen in an arbitrary manner. We will use the following inequality which is valid for $\lambda > 0$.

$$\int_\lambda^\infty e^{-t^2/2} dt \leqq \frac{1}{\lambda}\int_\lambda^\infty te^{-t^2/2} dt = \frac{1}{\lambda} e^{-\lambda^2/2} .$$

For $\varepsilon > 0$ since Δ_{jk} is normally distributed,

$$P[|\Delta_{jk}| > \varepsilon] = \frac{2}{\sqrt{2\pi v_{jk}}}\int_\varepsilon^\infty e^{-(t^2/2v_{jk})} dt$$
$$= \frac{2}{\sqrt{2\pi}}\int_\lambda^\infty e^{-t^2/2} dt$$

or

$$P[|\varDelta_{jk}| > \varepsilon] \leqq \frac{2}{\lambda\sqrt{2\pi}} \exp\left\{-\frac{\lambda^2}{2}\right\} = \frac{2}{\varepsilon}\sqrt{\frac{v_{jk}}{2\pi}} \exp\left\{-\frac{\varepsilon^2}{2v_{jk}}\right\}$$

where

$$v_{jk} \equiv v\left(\frac{j}{n}, \frac{k}{n}\right) - v\left(\frac{j-1}{n}, \frac{k}{n}\right) - v\left(\frac{j}{n}, \frac{k-1}{n}\right) + v\left(\frac{j-1}{n}, \frac{k-1}{n}\right)$$

and $\lambda = \varepsilon(v_{jk})^{-(1/2)}$. Since $v(s, t)$ is uniformly continuous over D, we can choose N independently of j and k such that $n \geqq N$ implies $v_{jk}/\varepsilon^2 < 1/M_\delta^2$ where M_δ is determined as follows. Since $(1/x)\exp\{-(x^2/2)\} = o(x^{-2})$ as $x \to \infty$, we have for every positive integer δ, a number M_δ such that $x > M_\delta$ implies $x\exp\{-(x^2/2)\} < 1/\delta$, that is, for $x > M_\delta$,

$$\frac{1}{x}\exp\left\{-\frac{x^2}{2}\right\} < \frac{1}{\delta x^2} .$$

Now $v_{jk}/\varepsilon^2 < 1/M_\delta^2$ entails $\varepsilon/\sqrt{v_{jk}} > M_\delta$ and with $x = \varepsilon/\sqrt{v_{jk}}$ we get

$$\frac{\sqrt{v_{jk}}}{\varepsilon}\exp\left\{-\frac{\varepsilon^2}{2v_{jk}}\right\} \leqq \frac{1}{\delta}\frac{v_{jk}}{\varepsilon^2} .$$

Then for $n \geqq N$

$$P[|\varDelta_{jk}| > \varepsilon] \leqq \frac{2}{\sqrt{2\pi}} \cdot \frac{v_{jk}}{\varepsilon^2} .$$

But $v(1, 1) - v(1, 0) - v(0, 1) + v(0, 0) = v(1, 1) = \sum_{k=1}^{n}\sum_{j=1}^{n} v_{jk}$, and so

$$\sum_{j=1}^{n}\sum_{k=1}^{n} P[|\varDelta_{jk}| < \varepsilon] \leqq \frac{2}{\sqrt{2\pi}} \cdot \frac{1}{\delta\varepsilon^2} v(1, 1) .$$

Since we may take δ arbitrarily large, choosing N sufficiently large for each δ,

$$\lim_{n\to\infty}\sum_{j=1}^{n}\sum_{k=1}^{n} P[|\varDelta_{ik}| > \varepsilon] = 0$$

and (1) holds for $Y(s, t)$. A similar argument proves (2) and (3). Since $Y(s, 0) = Y(0, t) = 0$ for all (s, t) in D, $Y(1, k/n) - Y(1, (k-1)/n)$ is normally distributed with mean zero and variance $v(1, k/n) - v(1, (k-1)/n)$, and $Y(j/n, 1) - Y((j-1)/n, 1)$ is normally distributed with mean 0 and variance $v(j/n, 1) - v((j-1)/n, 1)$. Thus

$$P\left[\left| Y\left(1, \frac{k}{n}\right) - Y\left(1, \frac{k-1}{n}\right)\right| > \varepsilon\right] = \frac{2}{\sqrt{2\pi v_k}}\int_\varepsilon^\infty \exp\{-t^2/2v_k\}dt$$

$$\leqq \frac{2\sqrt{v_k}}{\sqrt{2\pi}\varepsilon}\exp\{-\varepsilon^2/2v_k\}$$

and

$$P\left[\left|Y\left(\frac{j}{n}, 1 - Y\left(\frac{j-1}{n}, 1\right)\right| > \varepsilon\right] = \frac{2}{\sqrt{2\pi v_j}}\int_{\varepsilon}^{\infty} \exp\{-t^2/2v\}dt$$
$$\leqq \frac{2\sqrt{v_j}}{\sqrt{2\pi}\varepsilon} \exp\{-\varepsilon^2/2v_j\}$$

where $v_j \equiv v(j/n, 1 - v((j-1)/n, 1)$ and $v_k \equiv v(1, k/n) - v(1, (k-1)/n)$. Again we may choose δ, M_δ, N', and N'' so that when $n \geqq N'$ or $n \geqq N''$, the respective inequalities

$$\frac{v_j}{\varepsilon^2} < \frac{1}{M_\delta^2} \quad \text{or} \quad \frac{v_k}{\varepsilon^2} < \frac{1}{M_\delta^2}$$

hold. Since $v(1, 1) = \sum_{j=1}^n v_j = \sum_{k=1}^n v_k$,

$$\sum_{k=1}^n P\left[\left|X\left(1, \frac{k}{n}\right) - X\left(1, \frac{k-1}{n}\right)\right| > \varepsilon\right] \leqq \frac{2}{\sqrt{2\pi}\delta\varepsilon^2} v(1, 1)$$

and

$$\sum_{j=1}^n P\left[\left|X\left(\frac{j}{n}, 1\right) - X\left(\frac{j-1}{n}, 1\right)\right| > \varepsilon\right] \leqq \frac{2}{\sqrt{2\pi}\delta\varepsilon^2} v(1, 1)$$

when $n > N''$ or $n > N'$ respectively. Thus there is a process $Y_0(s, t)$ equivalent to $Y(s, t)$ such that almost every sample function of Y_0 is continuous over D. Setting $X(s, t) = Y_0(s, t) + m(s, t)$ we obtain a biadditive process satisfying (i), (ii), and (iii).

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