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A DECOMPOSITION THEOREM FOR BIADDITIVE PROCESSES

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This paper treats a class of stochastic processes called biadditive processes and gives a proof of a decomposition of their sample functions. Informally, a biadditive proces $X(s, t)$ is a process indexed by two time parameters whose "increments" over disjoint rectangles are independent. The increments of such a process are the second differences

$$X(s_2, t_2) - X(s_1, t_2) - X(s_2, t_1) + X(s_1, t_1)$$

where $s_1 < s_2$ and $t_1 < t_2$. The decomposition theorem states that every centered biadditive process is the sum of four independent biadditive processes: one with jumps in both variables, two with jumps in one variable and continuous in probability in the other, and a fourth process which is jointly continuous in probability.

This decomposition is similar to one for processes with independent increments and in the proofs of both results a major role is played by the theory of centralized sums of independent random variables.

More formally, let $P_1 = \{s_1, s_2, \cdots, s_n\}$ and $P_2 = \{t_1, t_2, \cdots, t_m\}$ be two partitions of $[0, s_n]$ and $[0, t_m]$ respectively. Define $P_1 \times P_2$ to be the corresponding partition of $[0, s_n] \times [0, t_m]$ into rectangles whose vertices are the (s_i, t_j)'s. Let Δ_{ij} denote the increment

$$\Delta_{ij} = X(s_{i+1}, t_{j+1}) - X(s_i, t_{j+1}) - X(s_{i+1}, t_j) + X(s_i, t_j)$$

over the rectangle with vertices (s_{i+1}, t_{j+1}), (s_i, t_{j+1}), (s_{i+1}, t_j) and (s_i, t_j). Then if the increments

$$\{\Delta_{ij}: i = 0, 1, \cdots, n-1, j = 0, 1, \cdots, m-1\}$$

corresponding to any partition $P_1 \times P_2$ are independent and if $X(s, 0) = 0 = X(0, t)$ for all s and t not less than zero, $X(s, t)$ is called biadditive.

It is easy to construct some examples of biadditive processes. For instance, if $\{Y_{ij}\}_{i,j=0}^{\infty}$ is a doubly infinite sequence of independent random variables, then it is easy to see that the process

$$X(s, t) = \sum_{i<s} \sum_{j<t} Y_{ij}$$

is biadditive. A nontrivial example of a biadditive process is obtained when the space C_2 of continuous functions of two variables on $[0, \infty) \times [0, \infty)$ is given the Wiener-Yeh measure and the process $X(s, t)$ is the coordinate process (see [3]). In [1] it was shown that the only biadditive processes with versions having continuous sample surfaces are Gaussian with continuous mean and variance functions, a result analogous to the one parameter case.

In order to facilitate the reading of this note, a short summary without proofs of some results of the theory of centralized sums is given in §2. A very nice account with proofs is given in the lecture notes by K. Itô (see [2]).

2. Summary of the theory of centralized sums.

DEFINITION (J. L. Doob). If X is a random variable with probability distribution μ, the central value $\gamma(X)$ of X is defined to be the unique real number γ such that

$$\int_{-\infty}^{\infty} \operatorname{arc\,tan}(x-\gamma)\mu(dx) = 0 .$$

The dispersion $\delta(X)$ of X is defined to be

$$\delta(X) = -\log \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{-|x-y|\}\mu(dx)\mu(dy) .$$

Basic Properties.

(2.1) If β is any number, $\gamma(\pm X+\beta)=\pm\gamma(X)+\beta$ and $\delta(\pm X+\beta)=\delta(X)$.

(2.2) If c is any number and $X=c$ a.s., then $\gamma(X)=c$ and $\delta(X)=0$.

(2.3) A sequence of random variables $\{X_n\}$ converges in probability to a random variable X if and only if $\gamma(X_n)\rightarrow\gamma(X)$ and $\delta(X_n-X)\rightarrow 0$.

(2.4) If X and Y are independent random variables, then $\delta(X+Y)\geqq\delta(X)$. Furthermore, $\delta(X+Y)=\delta(X)$ if and only if Y is constant a.s.

Centralized Sums. Let $\{X_n\}$ be a sequence of independent random variables and let $S_n=\sum_1^n X_k$. Then the sequence of dispersions $\{\delta(S_n)\}$ is a nondecreasing set of real numbers. There are two cases

(a) If $\lim_n \delta(S_n)<\infty$, then $\{S_n-\gamma(S_n)\}$ converges a.s.

(b) If $\lim_n \delta(S_n)=\infty$, then for every choice of a sequence of constants $\{c_n\}$, $\{S_n-c_n\}$ diverges a.s.

Let $\{X_\alpha\}_{\alpha\in A}$ be a countable family of independent random variables. Let F be a finite subset of A and set $S_F=\sum_{\alpha\in F}X_\alpha$ and $S_F^{\bullet}=S_F-\gamma(S_F)$. S_F is called the partial sum over F and $S_F^{\bullet}$ is called the centralized partial sum over F. We write $S_F^{\bullet}=\sum^{\bullet}_{\alpha\in F}X_\alpha$. (Also we will use $X\dotplus Y$ for $X+Y-\gamma(X+Y)$ and $X\dot{-}Y$ for $X-Y-\gamma(X-Y)$. Let

$$\delta(A)=\sup_F\delta(S_F)$$

where F ranges over all finite subsets of A.

THEOREM 2.1. *Suppose that* $\delta(A) < \infty$ *and that* $\{F_n\}$ *is a non-decreasing sequence of finite sets such that* $F_1 \subset F_2 \subset \cdots \to A$. *Then* $S^{\bullet}_{F_n}$ *converges a.s. and the limit* $S^{\bullet}_A$ *is independent of the choice of the sequence* $\{F_n\}$ *of finite subsets. Furthermore*

$$\gamma(S^{\bullet}_A) = 0 \quad \text{and} \quad \delta(S^{\bullet}_A) = \delta(A) .$$

Centralized sums behave in a very nice way. More precisely,

THEOREM 2.2. *Let* $\{X_\alpha\}_{\alpha \in A}$ *be a countable family of independent random variables such that* $\delta(A) < \infty$.

(a) *If* $A = \cup A_n$ (disjoint), *then* $S^{\bullet}_A = \sum^{\bullet} S^{\bullet}_{A_n}$ *a.s.*

(b) *If* $A_n \uparrow A$, *then* $S^{\bullet}_{A_n} \to S^{\bullet}_A$ *a.s.*

(c) *If* $B \subset A$ *and* $B_k \downarrow B$, *where* $B_k \subset A$ *for all* k, *then* $S^{\bullet}_{B_k} \to S^{\bullet}_B$ *a.s.*

3. The decomposition theorem.

DEFINITION. A centralized biadditive process $X(s, t)$ is for each s the sum of independent jumps occurring before time t if there exists a countable family of independent random processes $\{Z_t(s)\}$ such that

$$X(s, t) = \sum_{y \leqq t}{}^{\bullet} Z_y(s)$$

$X(s, t)$ is said to be the sum of independent jumps occurring before time (s, t) if there exists a countable family of independent random variables $\{T(x, y)\}$ such that

$$X(s, t) = \sum_{X \leqq s}{}^{\bullet} \sum_{y \leqq t}{}^{\bullet} T(x, y)$$

THEOREM 3.1. *Let* $\{X(s, t): s, t \geqq 0\}$ *be a biadditive process. Then* $X(s, t)$ *can be written as the sum of a deterministic part* $f(s, t)$ *and four independent centralized biadditive processes* $X_1(s, t)$, $X_2(s, t)$, $X_3(s, t)$, *and* $X_4(s, t)$ *which have the following properties*:

(a) $X_1(s, t)$ *is the sum of independent jumps occurring before time* (s, t).

(b) $X_2(s, t)$ *is for each* $t \geqq 0$ *continuous in probability in* s *and for each* s *is the sum of independent jumps occurring before time* t.

(c) $X_3(s, t)$ *is for each* $s \geqq 0$ *continuous in probability in* t *and for each* t *is the sum of independent jumps occurring before time* s.

(d) $X_4(s, t)$ *is continuous in probability on* $[0, \infty) \times [0, \infty)$.

4. Proof of the decomposition theorem. The first lemma follows immediately from the definition of biadditive processes.

LEMMA 4.1. *Let $\{X_\alpha(s): 0 \leqq s\}_\alpha$ be a finite set of independent additive processes such that $X_\alpha(0) = 0$ for all α. Then*

$$Y(s, t) = \sum_{0<\alpha<t} X_\alpha(s)$$

is biadditive.

DEFINITION. We write $s_n \downarrow s$ if $s_1 > s_2 > \cdots > s_n > \cdots$ and $\lim_n s_n = s$. Similarly $s_n \uparrow s$ means $s_1 < s_2 < \cdots < s$ and $\lim_n s_n = s$.

THEOREM 4.1. *Let $X(s, t)$ be a centralized biadditive process. Then if $s_n \uparrow s$ and $t_n \downarrow t$, $P - \lim_{n\to\infty} X(s_n, t_n)$ exists. Furthermore if $\{(s'_n, t'_n)\}$ is another sequence of points such that $s'_n \uparrow s$* and *$t'_n \downarrow t$, then $P - \lim_{n\to\infty} X(s'_n, t'_n)$ exists and is equal to $P - \lim_{n\to\infty} X(s_n, t_n)$.*

Proof. We show that in fact the almost everywhere limits, exist, the exceptional set depending on the particular sequence. Let $s_n \uparrow s$ and $t_n \downarrow t$. Then

$$X(s_n, t_n) = X(s_1, t_1) + \sum_{r=1}^{n-1} [X(s_r, t_{r+1}) - X(s_r, t_r)] + \sum_{r=1}^{n-1} [X(s_{r+1}, t_{r+1}) - X(s_r, t_{r+1})] .$$

Since each of the sums on the right are sums of independent random variables and the dispersions of their partial sums are dominated by $\delta[X(s, t_1)] < \infty$, each sum when centralized converges a.s. It follows that $X(s_n, t_n) + k_n$ converges a.s. for some sequence of constants $\{k_n\}$. Then

$$\gamma\Big(\lim_{n\to\infty} [X(s_n, t_n) + k_n]\Big) = \lim_{n\to\infty} \{\gamma(X(s_n, t_n)) + k_n\} = \lim_{n\to\infty} k_n$$

exists and hence $X(s_n, t_n) = (X(s_n, t_n) + k_n) - k_n$ converges a.s.

To show that $\lim_{n\to\infty} X(s'_n, t'_n) = \lim_{n\to\infty} X(s_n, t_n)$, form a new sequence $(\bar{s}_n, \bar{t}_n)$ converging monotonically to (s, t) by alternating points from $\{(s_n, t_n)\}$ and $\{(s'_n, t'_n)\}$.

From now on let $X(s, t)$ denote a centralized biadditive process. The last theorem and its obvious counterparts justify the notation

$$X(s+, t+) = P - \lim_{n\to\infty} X(s_n, t_n) \quad \text{if} \quad s_n \downarrow s \quad \text{and} \quad t_n \downarrow t$$
$$X(s-, t+) = P - \lim_{n\to\infty} X(s_n, t_n) \quad \text{if} \quad s_n \uparrow s \quad \text{and} \quad t_n \downarrow t$$
$$X(s+, t-) = P - \lim_{n\to\infty} X(s_n, t_n) \quad \text{if} \quad s_n \downarrow s \quad \text{and} \quad t_n \uparrow t$$

$$X(s-, t-) = P - \lim_{n\to\infty} X(s_n, t_n) \quad \text{if} \quad s_n \uparrow s \quad \text{and} \quad t_n \uparrow t$$
$$X(0-, t) = X(s, 0-) = 0 \text{ (convention).}$$

LEMMA 4.2. *Let* $0 \leqq s, t$. *If* $\delta\{X(s_0+, t_0) - X(s_0-, t_0)\} > 0$ *for some* t_0, *then* $\delta\{X(s_0+, t) - X(s_0-, t)\} > 0$ *for all* $t \geqq t_0$. *Similarly if* $\delta\{X(s_0, t_0+) - X(s_0, t_0-)\} > 0$ *for some* s_0, *then* $\delta\{X(s, t_0+) - X(s, t_0-)\} > 0$ *for all* $s \geqq s_0$.

Proof. Suppose that for some t_0, $\delta\{X(s_0+, t_0) - X(s_0-, t_0)\} > 0$. If $t \geqq t_0$,

$$X(s_0+, t) - X(s_0-, t) = X(s_0+, t_0) - X(s_0-, t_0) + \Delta$$

where

$$\Delta = X(s_0+, t) - X(s_0+, t_0) - X(s_0-, t) + X(s_0-, t_0)$$

is independent of $X(s_0+, t_0) - X(s_0-, t_0)$. Hence

$$0 < \delta\{X(s_0+, t_0) - X(s_0-, t_0)\} \leqq \delta\{X(s_0+, t) - X(s_0-, t)\} .$$

DEFINITION. The line $s = s_0$ is a *line of discontinuity* for the biadditive process $X(s, t)$ if for some $t \geqq 0$, $\delta\{X(s_0+, t) - X(s_0-, t)\} > 0$. Similarly $t = t_0$ is a *line of discontinuity* if for some $s \geqq 0$, $\delta\{X(s, t_0+) - X(s, t_0)\} > 0$. Let

$$D_1 = \{s \geqq 0 : \exists t \geqq 0 \text{ such that } \delta[X(s+, t) - X(s-, t)] > 0\}$$

and

$$D_2 = \{t \geqq 0 : \exists s \geqq 0 \text{ such that } \delta[X(s, t+) - X(s, t-)] > 0\} .$$

It is easy to see that D_1 and D_2 are countable sets. D_1 is the union over all positive integers n of the countable sets of fixed points of discontinuity of the additive process $Y_n(s) = X(s, n)$. (This follows from Lemma 4.2.)

From now on $X(s, t)$ will denote a centralized biadditive process. We define

$$\begin{aligned} X_1(s, t) &= \sum_{0 \leqq x < s}{}^{\cdot} \sum_{0 \leqq y < t}{}^{\cdot} \{X(x+, y+) - X(x-, y+) - X(x+, y-) + X(x-, y-)\} \\ &\quad \dotplus \sum_{0 \leqq y < t}{}^{\cdot} \{X(s, y+) - X(s-, y+) - X(s, y-) + X(s-, y-)\} \\ &\quad \dotplus \sum_{0 \leqq x < s}{}^{\cdot} \{X(x+, t) - X(x-, t) - X(x+, t-) + X(x-, t-)\} \\ &\quad \dotplus \{X(s, t) - X(s-, t) - X(s, t-) + X(s-, t-)\} . \end{aligned}$$

All sums above and from here on are really countable since for only

x's in D_1 and y's in D_2 are the random variables in the sums nonzero. Let

$$Y_1(s, t) = X(s, t) \dot{-} X_1(s, t) .$$

PROPOSITION 4.1. *$Y_1(s, t)$ and $X_1(s, t)$ as defined above are independent biadditive processes. Furthermore for all s and $t \geqq 0$,*

$$Y_1(s+, t+) \dot{-} Y_1(s-, t+) \dot{-} Y_1(s+, t-) \dot{+} Y_1(s-, t-) = 0 .$$

Proof. By approximating $X_1(s, t)$ with finite sums $X_1^{(n)}(s, t)$ and writing $Y_1^{(n)} = X - X_1^{(n)}$ so that $X_1^{(n)}$ and $Y_1^{(n)}$ are independent biadditive processes, we see that X_1 and Y_1 are the limits of independent biadditive processes. It follows that X_1 and Y_1 are independent biadditive processes.

To prove that

$$Y_1(s+, t+) \dot{-} Y_1(s-, t+) \dot{-} Y_1(s+, t-) \dot{+} Y_1(s-, t-) = 0$$

we note that if $s_n \downarrow s$ and $t_n \downarrow t$,

$$P - \lim_{n\to\infty} \sum_{0 \leqq y < t_n}^{\bullet} \{X(s_n, y+) - X(s_n-, y+) - X(s_n, y-) + X(s_n-, y-)\} = 0$$

$$P - \lim_{n\to\infty} \sum_{0 \leqq x < s_n}^{\bullet} \{X(x+, t_n) - X(x-, t_n) - X(x+, t_n-) + X(x-, t_n-)\} = 0$$

$$P - \lim_{n\to\infty} \{X(s_n, t_n) - X(s_n-, t_n) - X(s_n, t_n-) + X(s_n-, t_n-)\} = 0 .$$

The first equality is a consequence of (2.4). Since X is biadditive,

$$[X(s_n, t_1) - X(s+, t_1)] - \sum_{0 \leqq y < t_n}^{\bullet} \{X(s_n, y+) - X(s_n-, y+) - X(s_n, y-) + X(s_n-, y-)\}$$

and

$$\sum_{0 \leqq y < t_n}^{\bullet} \{X(s_n, y+) - X(s_n-, y+) - X(s_n, y-) + X(s_n-, y-)\}$$

are independent. Hence,

$$\delta\left\{ \sum_{0 \leqq y < t_n}^{\bullet} \{X(s_n, y+) - X(s_n-, y+) - X(s_n, y-) + X(s_n-, y-)\} \leqq \delta\{X(s_n, t_1) - X(s+, t_1)\}\right\} \to 0 \quad \text{as} \quad n \to \infty .$$

Since the sum is centralized, the first equality follows by (2.3). The other two equalities follow from similar arguments. We have from Theorem 2.2

$$X_1(s+, t+) = \sum_{0 \leqq y \leqq t}^{\bullet} \sum_{0 \leqq x \leqq s}^{\bullet} \{X(x+, y+) - X(x-, y+) - X(x+, y-) + X(x-, y-)\} .$$

Using the basic properties of centralized sums and dispersions in a similar manner, we obtain

$$X_1(s-, t+) = \sum_{0\leq x<s}^{\bullet} \sum_{0\leq y\leq t}^{\bullet} \{X(x+, y+) - X(x-, y+) - X(x+, y-) + X(x-, y-)\}$$

$$X_1(s+, t-) = \sum_{0\leq x\leq s}^{\bullet} \sum_{0\leq y<t}^{\bullet} \{X(x+, y+) - X(x-, y+) - X(x+, y-) + X(x-, y-)\}$$

$$X_1(s-, t-) = \sum_{0\leq x<s}^{\bullet} \sum_{0\leq y<t}^{\bullet} \{X(x+, y+) - X(x-, y+) - X(x+, y-) + X(x-, y-)\} .$$

We obtain from these equations,

$$X_1(s+, t+) \dot{-} X_1(s-, t+) \dot{-} X_1(s+, t-) \dot{-} X_1(s-, t-) = X(s+, t+) \dot{-} X(s-, t+) \dot{-} X(s+, t-) \dot{-} X(s-, t-) .$$

Since $Y_1 = X \dot{-} X_1$, the proposition is proved.

Now define

$$X_2(s, t) = \sum_{0\leq x<s}^{\bullet} \{Y_1(x+, t) - Y_1(x-, t)\} \dot{+} \{Y_1(s, t) - Y_1(s-, t)\}$$

and

$$Y_2(s, t) = Y_1(s, t) \dot{-} X_2(s, t) .$$

PROPOSITION 4.2. *$X_2(s, t)$ and $Y_2(s, t)$ are independent biadditive processes. Furthermore, for all s and t*

$$X_2(s, t+) = X_2(s, t-)$$

and

$$Y_2(s+, t+) \dot{-} Y_2(s+, t-) \dot{-} Y_2(s-, t+) \dot{+} Y_2(s-, t-) = 0 .$$

Proof. The fact that X_2 and Y_2 are independent biadditive processes is proved in the same way as the corresponding assertion in Proposition 4.1. Using the techniques of the theory of centralized sums, one may easily see that

$$X_2(s, t+) = \sum_{0\leq x<s}^{\bullet} \{Y_1(x+, t+) - Y_1(x-, t+)\} \dot{+} \{Y_1(s, t+) - Y_1(s-, t+)\}$$

and

$$X_2(s, t-) = \sum_{0\leq x<s}^{\bullet} \{Y_1(x+, t-) - Y_1(x-, t-)\} \dot{+} \{Y_1(s, t-) - Y_1(s-, t-)\} .$$

Thus

$$X_2(s, t+) \dot{-} X_2(s, t-)$$
$$= \sum_{0 \leq x < s}^{\cdot} \{Y_1(x+, t+) \dot{-} Y_1(x-, t+) \dot{-} Y_1(x+, t-) \dot{+} Y_1(x-, t-)\}$$
$$\dot{+} \{Y_1(s, t+) \dot{+} Y_1(s-, t+) \dot{-} Y_1(s, t-) \dot{+} Y_1(s-, t-)\} = 0$$

by Proposition 4.1.

Since X_2 is centralized, $X_2(s, t+) = X_2(s, t-)$ follows. An almost identical argument shows that $X_2(s+, t+) = X_2(s+, t-)$ and

$$X_2(s-, t+) = X_2(s-, t-) .$$

The last equality follows immediately from these equations, Proposition 4.1, and the definition of Y_2.

We finally define

$$X_3(s, t) = \sum_{0 \leq y < t}^{\cdot} \{Y_2(s, y+) - Y_2(s, y-) \dot{+} \{Y_2(s, t) - Y_2(s, t-)\}$$

and

$$X_4(s, t) = Y_2(s, t) \dot{-} X_3(s, t) .$$

PROPOSITION 4.3. *X_3 and X_4 are independent biadditive processes. Also for all s and t*

$$X_3(s+, t) = X_3(s-, t) .$$

Furthermore, X_4 is continuous in probability since for all s and t

$$X_4(s+, t+) = X_4(s-, t-) .$$

Proof. The fact that X_3 and X_4 are independent follows just as similar previous assertions. Since

$$X_3(s+, t) = \sum_{0 \leq y < t}^{\cdot} \{Y_2(s+, y+) - Y_2(s+, y-)\} \dot{+} \{Y_2(s+, t) - Y_2(s+, t-)\}$$

and

$$X_3(s-, t) = \sum_{0 \leq y < t}^{\cdot} \{Y_2(s-, y+) - Y_2(s-, y-)\} \dot{+} \{Y_2(s-, t) - Y_2(s-, t-)\} ,$$

we have

$$X_3(s+, t) \dot{-} X_3(s-, t)$$
$$= \sum_{0 \leq y < t}^{\cdot} \{Y_2(s+, y+) \dot{-} Y_2(s+, y-) \dot{-} Y_2(s-, y+) \dot{+} Y_2(s-, y-)\}$$
$$\dot{+} \{Y_2(s+, t) \dot{-} Y_2(s+, t-) \dot{-} Y_2(s-, t) \dot{+} Y_2(s-, t-)\} = 0$$

by Proposition 4.2.

Since X_3 is centralized, $X_3(s+, t) = X_3(s-, t)$.

Similar computations yield

$$X_3(s+, t+) = \sum_{0 \leqq y \leqq t}^{\cdot} \{Y_2(s+, y+) - Y_2(s+, y-)\}$$

and

$$X_3(s-, t-) = \sum_{0 \leqq y < t}^{\cdot} \{Y_2(s-, y+) - X_2(s-, y-)\} .$$

Thus

$$\begin{aligned} &X_3(s+, t+) \dot{-} X_3(s-, t-) \\ &= \sum_{0 \leqq y < t}^{\cdot} \{Y_2(s+, y+) \dot{-} Y_2(s-, y+) - Y_2(s+, y-) \dot{+} Y_2(s-, y-)\} \\ &\quad \dot{+} \{Y_2(s+, t+) \dot{-} Y_2(s+, t-)\} \\ &= Y_2(s+, t+) \dot{-} Y_2(s+, t-) \end{aligned}$$

by Proposition 4.2. From the definition of X_4 it follows that

$$X_4(s+, t+) \dot{-} X_4(s-, t-) = 0 .$$

Since X_4 is centralized, the proposition is proved.

The decomposition theorem now follows immediately from Propositions 4.1, 4.2, and 4.3 and from the definitions of X_1, X_2, X_3 and X_4.

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