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ON THE DENSITY OF CERTAIN COHESIVE BASIC SEQUENCES

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It has been shown in previous investigations of the combinatorial properties of basic sequences that any cohesive basic sequence $\mathscr{B}$ which is contained in $\mathscr{M}$ (the set of all pairs of relatively prime positive integers) must be large in some sense. To be precise, it has been proved that if $\mathscr{B}$ is a cohesive basic sequence and $\mathscr{B} \subset \mathscr{M}$, then $C_{\mathscr{B}}(p)$ is infinite for every prime p, where $C_{\mathscr{B}}(p)$ is the set of prime companions of p in primitive pairs in $\mathscr{B}$. While this implies that $\mathscr{B}$ must contain a great many primitive pairs, no specific statement has been made about the density of $\mathscr{B}$. It is reasonable to ask, therefore, whether there are cohesive basic sequences $\mathscr{B}$, contained in $\mathscr{M}$, with density $\delta(\mathscr{B}) = 0$.

It is shown here that such basic sequences do exist, and a method is given for the construction of a large class of these sequences.

A proof that $C_{\mathscr{B}}(p)$ is infinite when $\mathscr{B}$ is cohesive and $\mathscr{B} \subset \mathscr{M}$ may be found in [2].

A *basic sequence* $\mathscr{B}$ is a set of pairs (a, b) of positive integers satisfying

(i) $(1, k) \in \mathscr{B}$ $(k = 1, 2, \cdots)$,

(ii) $(a, b) \in \mathscr{B}$ if and only if $(b, a) \in \mathscr{B}$,

(iii) $(a, bc) \in \mathscr{B}$ if and only if $(a, b) \in \mathscr{B}$ and $(a, c) \in \mathscr{B}$.

A pair (a, b) of positive integers is called a *primitive pair* if both a and b are primes. If $a \neq b$, the pair is a *type I* primitive pair; if $a = b$, the pair is a *type II* primitive pair. If Φ is a set of pairs (primitive or not) of positive integers, the basic sequence *generated* by Φ is defined to be

$$\Gamma[\Phi] = \bigcap \mathscr{D} ,$$

where the intersection is taken over all basic sequences $\mathscr{D}$ which contain Φ.

A basic sequence $\mathscr{B}$ is *cohesive* if for each positive integer k there is an integer $a > 1$ such that $(k, a) \in \mathscr{B}$.

Finally, we recall that the *density* of a basic sequence $\mathscr{B}$ is defined by

$$\delta(\mathscr{B}) = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \frac{{}^{\#}B_k}{d(k)} \tag{1.1}$$

if the limit exists, where $d(k)$ is the number of positive divisors of k, and ${}^{\#}B_k$ is the number of members (m, n) of $\mathscr{B}$ for which $mn = k$.

2. The main theorem. We will use the following notation.

$$P = \{p_1, p_2, \cdots\}$$

is the sequence of all primes, written in order of increasing magnitude;

$$Q = \{q_1, q_2, \cdots\}$$

is any sequence of primes, also written in order of increasing size; and

$$Q_i = \{q_i, q_{i+1}, q_{i+2}, \cdots\} \qquad (i = 1, 2, \cdots) .$$

We define $\mathscr{B}_Q$ to be the basic sequence generated by the primitive pairs

$$\{(p_1, q) \mid q \in Q_1\} \cup \{(p_2, q) \mid q \in Q_2\} \cup \cdots .$$

REMARK 1. $\mathscr{B}_Q$ is cohesive. For suppose $k > 1$, so that $k = p_{i_1}^{t_1} p_{i_2}^{t_2} \cdots p_{i_M}^{t_M}$ where $i_1 < i_2 < \cdots < i_M$. Then $(q_{i_M}, p_{i_j}) \in \mathscr{B}_Q$ for $j = 1, 2, \cdots, M$, so $(q_{i_M}, k) \in \mathscr{B}_Q$.

REMARK 2. $\mathscr{B}_Q \subset \mathscr{M}$ if $q_1 \geqq 3$. For if $q_1 \geqq 3\ (= p_2)$ then $q_i > p_i$ for every i, and $\mathscr{B}_Q$ will contain no type II primitive pairs.

THEOREM. *If* $\sum_{i=1}^{\infty} 1/q_i$ *converges, then* $\delta(\mathscr{B}_Q) = 0$.

Proof. Let L be a (large) fixed, but arbitrary positive integer which will be determined later. Decompose the set $\boldsymbol{Z}^+$ of positive integers as follows:

(a) $X' = \{k \mid {}^{\#}B_k = 2\}$,

(b) $X'' = \{k \mid k \notin X'$ and k has less than $4L$ different prime divisors$\}$,

(c) $Y = \{k \mid k \notin (X \cup X'')\}$.

In order to prove that $\delta(\mathscr{B}_Q) = 0$, let us consider

$$\frac{1}{N} \sum_{\substack{k=1 \\ k \in S}}^{N} \frac{{}^{\#}B_k}{d(k)} , \tag{2.1}$$

where $S = X'$, X'' and Y.

By Lemma 3.2 in [1], we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\substack{k=1 \\ k \in X'}}^{N} \frac{{}^{\#}B_k}{d(k)} \leqq \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \frac{2}{d(k)} = 0 , \tag{2.2}$$

while by Theorem 11.8 in [3] we have

(2.3) $$\lim_{N\to\infty}\frac{1}{N}\sum_{\substack{k=1\\ k\in X''}}^{N}\frac{{}^{\#}B_k}{d(k)} \leqq \lim_{N\to\infty}\frac{1}{N}\sum_{\substack{k=1\\ k\in X''}}^{N} 1 = 0 .$$

It remains to estimate the sum in (2.1) when $S = Y$. Since

(2.4) $$\frac{1}{N}\sum_{\substack{k=1\\ k\in Y}}^{N}\frac{{}^{\#}B_k}{d(k)} \leqq \frac{1}{N}\sum_{\substack{k=1\\ k\in Y}}^{N} 1 ,$$

we will find an upper bound for the number of elements of Y which do not exceed N. Our estimate will depend on the following

LEMMA. *Every integer in Y is divisible by at least one of the primes q_i with $i \geqq L$.*

Proof of the Lemma. Let k be an element of Y. Then ${}^{\#}B_k > 2$, so there are integers u, v such that

$$k = uv,\ u > 1,\ v > 1,\ (u, v) \in \mathscr{B}_Q .$$

Suppose that u and v are expressed canonically as products of prime powers:

$$u = p_{i_1}^{a_1} p_{i_2}^{a_2} \cdots p_{i_r}^{a_r} , \qquad v = p_{j_1}^{b_1} p_{j_2}^{b_2} \cdots p_{j_s}^{b_s} ,$$

where $r \geqq 1$, $s \geqq 1$, $p_{i_1} < p_{i_2} < \cdots < p_{i_r}$, $p_{j_1} < p_{j_2} < \cdots < p_{j_s}$. Since k is divisible by at least $4L$ distinct primes, we have $r + s \geqq 4L$. At least one of the numbers r, s must be $\geqq 2L$, say

$$r \geqq 2L .$$

If $p_{j_1} \notin Q$, then every prime divisor of u is in Q since every primitive pair in $\mathscr{B}_Q$ contains at least one member from Q. Hence $p_{i_r} = q_i$ (for some q_i in Q) and $q_i \geqq q_r \geqq q_{2L}$.

Suppose, on the other hand, that p_{j_1} is in Q. Now separate the primes $p_{i_1}, \cdots, p_{i_r}$ into two classes, depending on whether or not they are in Q. Let $x_1, \cdots, x_\lambda$ be those not in Q, written in order of ascending size, and let $y_1, \cdots, y_\nu$ be those in Q, also given in ascending order. Thus

$$u = x_1^{c_1} \cdots x_\lambda^{c_\lambda} y_1^{d_1} \cdots y_\nu^{d_\nu} ,$$

with

(2.5) $$\lambda + \nu = r \geqq 2L .$$

It follows from (2.5) that either $\lambda \geqq L$ or $\nu \geqq L$.

If $\lambda \geqq L$, then $x_\lambda = p_m$ for some $m \geqq L$. Since $p_m \notin Q$, only the primes in Q_m appear as companions of p_m in primitive pairs of $\mathscr{B}_Q$. In particular, since $(p_m, p_{j_1}) \in \mathscr{B}_Q$, we have

$$p_{j_1} \in Q_m \subset Q_L .$$

Thus $p_{j_1} \in Q$, $p_{j_1} \geqq q_L$, and $p_{j_1} \mid k$.

If $\nu \geqq L$, then $y_\nu \in Q$, $y_\nu \geqq q_L$, and $y_\nu \mid k$.

That proves the Lemma.

We return to the estimation of the second sum in (2.4). As a consequence of the Lemma we have

$$\begin{aligned} \sum_{\substack{k=1 \\ k \in Y}}^{N} 1 &\leqq \sum_{\substack{k=1 \\ q_i \mid k \text{ for some } i \geqq L}}^{N} 1 \\ &\leqq \sum_{i=L}^{\infty} \left[\frac{N}{q_i} \right] \\ &\leqq N \sum_{i=L}^{\infty} \frac{1}{q_i} , \end{aligned}$$

and this together with (2.4) gives

$$\frac{1}{N} \sum_{\substack{k=1 \\ k \in Y}}^{N} \frac{\# B_k}{d(k)} \leqq \sum_{i=L}^{\infty} \frac{1}{q_i} . \tag{2.6}$$

Now let $\varepsilon > 0$ be given and choose L large enough so that

$$\sum_{i=L}^{\infty} \frac{1}{q_i} < \frac{\varepsilon}{3}$$

(L depends only on ε and Q). Then from (2.6) we have

$$\frac{1}{N} \sum_{\substack{k=1 \\ k \in Y}}^{N} \frac{\# B_k}{d(k)} < \frac{\varepsilon}{3} , \tag{2.7}$$

and it follows from (2.2), (2.3) and (2.7) that there is an integer $N_0(\varepsilon)$ such that

$$\frac{1}{N} \sum_{k=1}^{N} \frac{\# B_k}{d(k)} = \frac{1}{N} \Bigg(\sum_{\substack{k=1 \\ k \in X'}}^{N} + \sum_{\substack{k=1 \\ k \in X''}}^{N} + \sum_{\substack{k=1 \\ k \in Y}}^{N} \Bigg) \frac{\# B_k}{d(k)} < \varepsilon$$

when $N \geqq N_0(\varepsilon)$.

That proves $\delta(\mathscr{B}_Q) = 0$, and completes the proof of the Theorem.

By Remarks 1 and 2 and the Theorem, each sequence Q of distinct odd primes such that $\Sigma\, 1/q_j$ converges leads to a cohesive basic sequence $\mathscr{B}_Q$ in $\mathscr{M}$ such that $\delta(\mathscr{B}_Q) = 0$.

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