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INEQUALITIES INVOLVING $\|f\|_p$ AND $\|f^{(n)}\|_q$ FOR f WITH n ZEROS

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Let $\|\ \|_p$ denote the L^p-norm. This paper determines the smallest possible constants C which satisfy

$$\|f\|_p \leqq C\cdot(b-a)^s\|f^{(n)}\|_q$$

for certain classes of n-times continuously differentiable functions having n zeros on some interval $[a, b]$. Particular interest is placed on functions having α zeros at a and $n-\alpha$ zeros at b. It is shown that smallest possible constants exist for all positive integers n, for all extended real numbers p and q not less than one, and for $\alpha = 0, \ldots, n$ providing the exponent s is chosen properly. Moreover, these constants can be used to determine best possible constants when the n zeros are restricted only by the condition that α are at a and $\beta < n-\alpha$ are at b.

Inequalities of the type studied in this paper have been investigated by a number of authors including [15] and [16] who were primarily concerned with special cases of p and q or were concerned with small n. For example, the inequality $\|f\|_2 \leqq 2(b-a)\|f'\|_2/\pi$ when $f(a)=0$, $f \in C^1[a, b]$ is included in [1] and [7] and $\|f\|_2 \leqq (b-a)^2\|f''\|_2/\pi^2$ when $f(a)=f(b)=0$, $f \in C^2[a, b]$ is found in [7]. Boyd [4] gives a method for finding best constants C for inequalities of the form

$$\int_a^b |f|^p|f^{(n)}|^r w(t)dt \leqq C\left\{\int_a^b |f^{(n)}|^q m(t)dt\right\}^{(p+r)/q}$$

$f(a) = f'(a) = \cdots = f^{(n-1)}(a) = 0$, $f^{(n-1)}$ absolutely continuous, $p > 0$, $q > 1$, and $0 \leqq r < q$. Not only is this one of the problems we are studying when $r = 0$ but his method works equally well on problems were the zeros are divided among the endpoints at least when $p \in [1, \infty)$ and $q \in (1, \infty)$. The cases $p = \infty$ and $q = 1$ or ∞ are studied separately in this paper. Hukuhara [10] indicates that Tumura [17] has found inequalities of the form

$$\|f\|_\infty \leqq (n-1)^{n-1}(b-a)^n\|f^{(n)}\|_\infty/n!\, n^n$$

and

$$\|f^{(k)}\|_\infty \leqq k(b-a)^{n-k}\|f^{(n)}\|_\infty/n(n-k)!\ , \quad k = 1, \cdots, n-1\ ,$$

for $f \in C^n[a, b]$ having n zeros in $[a, b]$ including zeros at a and b. This result is also mentioned in [3] and [13]. Coles [6] extends

Wirtinger's inequalities to include certain cases of the type which we are studying. Levin [12], [13], Hukuhara [10], and more recently Hartman [9] have established a related result in which f and its derivatives have a certain pattern of zeros. Levin's proof motivated the method used to establish Theorem 4.1 which allows us to concentrate our attention on functions having zeros only at the ends of the interval under consideration.

2. NOTATION. There are a few conventions needed throughout this paper. We will count zeros according to their multiplicity. Thus, the statement $y(a_j) = 0, j = 1, \cdots, n$ must be interpreted to mean $y^{(i)}(a_m) = 0, i = 0, \cdots, k-1$, if a_m appears k times among the a_j's. The term "f has k zeros" is used interchangeably with "f has at least k zeros" unless context demands otherwise. $PC^n[a, b]$ denotes the set of all functions with n piecewise continuous derivatives. The notation $[1, \infty]$ will denote the extended real numbers greater than or equal to one, and we will define $1/\infty \equiv 0$ whenever needed. As usual we will use

$$\|f\|_p = \left[\int_a^b |f(t)|^p dt\right]^{1/p}$$

as the norm on $L^p[a, b]$ for any $p \in [1, \infty)$ and

$$\|f\|_\infty = \sup\{|f(t)|: t \in [a, b]\}$$

for $L^\infty[a, b]$.

The basic set used throughout this paper is the set of all $f \in C^n[a, b]$ having at least n zeros on $[a, b]$ and it will be denoted $C^n\langle[a, b]\rangle$. We will need subsets of $C^n\langle[a, b]\rangle$ consisting of all functions $f \in C^n\langle[a, b]\rangle$ which satisfy one or more of the following properties:

$$(q) \qquad \|f^{(n)}\|_q = 1 .$$

(α, β) f has α zeros at a and β zeros at b.
(α) f has α zeros at a and $\beta = n - \alpha$ zeros at b.
These subsets will be denoted by including the appropriate symbol(s) inside the $\langle\ \rangle$. Particular values of q, α, or β will be included. For example, $C^n\langle[a, b], q = 2, \alpha, \beta = n - \alpha\rangle = C^n\langle[a, b], q = 2, \alpha\rangle$ denotes the subset of $C^n\langle[a, b]\rangle$ containing those functions f satisfying $\|f^{(n)}\|_2 = 1$ and having α zeros at a and $\beta = n - \alpha$ zeros at b.

In §3 we will be able to show that we need only consider functions with nonnegative nth derivative when searching for smallest constants $C = C(n, p, q, \alpha, \beta)$ satisfying

$$\|f\|_p \leqq C\cdot(b - a)^s \|f^{(n)}\|_q , \quad f \in C^n\langle[a, b], \alpha, \beta\rangle , \tag{2.1}$$

and in §4 we will show that it is not necessary to consider functions with interior zeros. In the following section, we find that if $s = n + 1/p - 1/q$ then

$$C = \max_{\alpha \leqq \alpha' \leqq n-\beta} \sup \{\|f\|_p : f \in C^n\langle[0, 1], q, \alpha'\rangle\} .$$

Combining this result with that in §4 motivates our study of

$$K(n, p, q, \alpha) = \sup \{\|f\|_p : f \in C^n\langle[0, 1], q, \alpha\rangle\}$$

in §§5–10. In addition to finding the value of the K's, the maximizing functions will also be determined whenever they exist. In §11, we consider inequality (2.1) for $\alpha = \beta = 1$ and $\alpha = \beta = 0$.

3. **Preliminaries.** In Theorem 3.4 we will show that we need only consider functions whose nth derivative is nonnegative. Some of the lemmas needed to establish this result will also be useful later in this paper.

LEMMA 3.1. *Suppose that $f \in C^n\langle[a, b]\rangle$, $f^{(n)} \geqq 0$, and f has exactly n zeros. Then the sign of $f(t)$ is determined for each t by the zeros of f alone.*

Proof. Let $t_0, \cdots, t_{n-1}$ be the largest zeros of $f, \cdots, f^{(n-1)}$, respectively. By use of the mean value theorem, one can see that $t_{n-1} < \cdots < t_\beta < b$ and $t_{\beta-1} = \cdots = t_0 = b$ where β is the exact number of zeros at b. As

$$f^{(i)}(t) = \int_{t_i}^{t} f^{(i+1)}(s)ds , \qquad i = 0, \cdots, n-1 ,$$

one can see that $f^{(n-1)}(t), \cdots, f^{(\beta)}(t)$ are all positive on the interval (t_β, b). However, on the same interval, $f^{(\beta-1)}(t), \cdots, f(t)$ must alternate in sign with $f^{(\beta-1)}(t) < 0$. Thus, the sign of $f(t)$ is $(-1)^\beta$ on (t_β, b). The sign of $f(t)$ can be determined between each of the other distinct zeros by tracing back from zero to zero. In particular, if $f \in C^n\langle[a, b], \alpha\rangle$ so that $\beta = n - \alpha$, then the sign of $f(t)$ is $(-1)^\beta$ if $f^{(n)} \geqq 0$.

It is useful to show that we need only consider functions whose nth derivative does not change signs, that is, functions whose nth derivative is either nonnegative or nonpositive.

This result requires a few preliminary lemmas generalizing the mean value theorem.

LEMMA 3.2. *Suppose that $a < b$, $g \not\equiv 0$, $g \in C^m[a, b]$, $g(a) = g(b) = 0$, and that g has at least $m + 1$ zeros on $[a, b]$. Then if $m \geqq 2$ there* *exists a nondegenerate subinterval* $[c, d] \subset [a, b]$ *on which* g' *satisfies all of the above conditions with* m *and* $[a, b]$ *replaced by* $m - 1$ *and* $[c, d]$, *respectively.*

Proof. Since $g' \in C[a, b]$, by the definition of multiple zeros and by the mean value theorem, g' must have m zeros on $[a, b]$. Let $c = \min\{t: g'(t) = 0, t \geqq a\}$ and $d = \max\{t: g'(t) = 0, t \leqq b\}$. Hence, $g'(t)$ has m zeros on $[c, d]$. Suppose $c = d$ or $g' \equiv 0$ on $[c, d]$. Since $g'(t)$ must have a change of sign on $[a, b]$, we have $a < c \leqq d < b$. Without loss of generality, we may assume that $g'(t) < 0$ on $[a, c)$ and $g'(t) > 0$ on $(d, b]$. This would say that a and b would be the only zeros of the function g and that both must be simple zeros. Thus $c \neq d$ and g' can not be the zero function on $[c, d]$ and the lemma is established.

COROLLARY 3.3. *Suppose* $f \in C^n[a, b]$, *that* $f(a) = f(b) = 0$ *and that* f *has* $n + 1$ *zeros on* $[a, b]$. *Then if* $f \not\equiv 0$, $f^{(n)}$ *has a sign change on* $[a, b]$.

Proof. For $n = 1$, this result is an analogue of Rolle's Theorem. If $f \not\equiv 0$, f has a maximum or minimum at some point in (a, b) and f' must change signs in some interval about that point.

For $n \geqq 2$, Lemma 3.2 states that there exists a subinterval $[c_1, d_1]$ on which $f' \in C^{n-1}[c_1, d_1]$, f' has n zeros on $[c_1, d_1]$ including zeros at c_1 and d_1, and on this subinterval $f' \not\equiv 0$. We can use the lemma repeatedly until we find $f^{(n-1)}$ has 2 zeros on subinterval $[c_{n-1}, d_{n-1}]$. Showing that $f^{(n)}$ has a sign change is identical to the case $n = 1$ so the proof is complete.

Comment. If $f \in C^n[a', b']$ and f has at least $n + 1$ zeros on $[a', b']$ including at least two distinct zeros then we may apply Corollary 3.3 with $a = \min\{t \geqq a': f(t) = 0\}$ and $b = \max\{t \leqq b': f(t) = 0\}$, respectively, to show that $f^{(n)}$ has a sign change on $[a, b]$ and hence on $[a', b']$.

The following theorem will allow us to consider only those functions whose nth derivative is nonnegative when finding maximums over $C^n\langle[a, b], q, \alpha, \beta\rangle$. g is said to *enclose* f if $f, g \in C^n\langle[a, b], \alpha, \beta\rangle$ and

$$-|g(t)| \leqq f(t) \leqq |g(t)| \,. \tag{3.1}$$

$t_1, \cdots, t_n$ is a *zero set* for f if it is a set of n zeros for f of which α are at a and β at b.

THEOREM 3.4. *If* $f \in C^n\langle[a, b], q, \alpha, \beta\rangle$ *and* $f^{(n)}$ *changes signs, then* *there exists a function $g \in C^n\langle[a, b], q, \alpha, \beta\rangle$ which encloses f and whose nth derivative does not change signs. Hence, $\|f\|_p \leqq \|g\|_p$, $p \in [1, \infty]$.*

Proof. We will consider two cases.

Case (i). Suppose that the only zero set for f satisfies $t_1 = \cdots = t_n$ $f^{(i)}(t_1) = 0$ for $i = 0, \cdots, n-1$. Then

$$|f(t)| = \left|\int_{t_1}^{t} \cdots \int_{t_1}^{s_{n-1}} f^{(n)}(s_n)ds_n \cdots ds_1\right|$$
$$\leqq \left|\int_{t_1}^{t} \cdots \int_{t_1}^{s_{n-1}} |f^{(n)}(s_n)|\, ds_n \cdots ds_1\right| .$$

The iterated integral in the last step is the desired enclosing function g.

Case (ii). Suppose that f has a zero set with at least two distinct zeros. We will show that the function g defined by the conditions

$$g^{(n)}(t) = |f^{(n)}(t)|,\ g(t_i) = 0\ , \qquad i = 1, \cdots, n\ , \tag{3.2}$$

encloses f. Clearly $\|g^{(n)}\|_q = \|f^{(n)}\|_q = 1$ and $g \in C^n\langle[a, b], q, \alpha, \beta\rangle$. Moreover, we can use Corollary 3.3 to show that g has at most n zeros since $g^{(n)}$ does not change signs. Thus, g can have no other zeros than those given.

In order to show that g encloses f let us define $h(t) = f(t) - g(t)$. Then h also satisfies zero conditions having the same form as (3.2). we find that

$$\begin{aligned} h^{(n)}(t) &= f^{(n)}(t) - g^{(n)}(t) \\ &= f^{(n)}(t) - |f^{(n)}(t)| \\ &\leqq 0 \end{aligned}$$

so h has at most n zeros. By Lemma 3.1 the sign of $h(\mathrm{t})$ is equal to the sign of $-g(t)$ for each t. Thus, f is below g whenever g is positive and above g whenever g is negative. Likewise for $-f$ and g. Thus, inequality (3.1) is valid and the proof is complete.

4. Simplification of the problem. In the last section we showed that it was only necessary to consider functions whose nth derivative does not change signs. In this section we will further simplify the problem by showing that we need only consider functions without interior zeros.

THEOREM 4.1. *Let*

$$A = \{f \in C^n\langle[a, b], q, \alpha, \beta\rangle\text{: } f \textit{ has no interior zeros}\ ,\quad f^{(n)} \geqq 0\}$$

and let $h \in C^n\langle[a, b], q, \alpha, \beta\rangle$ *satisfying* $h, -h \notin A$. *Then*

$$(4.1) \qquad \|h\|_p < \sup\{\|f\|_p : f \in C^n\langle[a, b], q, \alpha, \beta\rangle\} = \sup\{\|f\|_p : f \in A\}\,, \quad p \in [1, \infty]\,,$$

if the supremum exists.

The proof of this theorem is similar to a proof used by Levin [12]. In fact, a proof of the equality in (4.1) could be carried out in a more elegant fashion using the Krein-Milman theorem [11] if we did not wish to establish the inequality as well.

We begin by looking at a fixed nth derivative $g(t)$ where $g \geqq 0$. We define the set

$$(4.2) \qquad M(g) = \{f \in C^n\langle[a, b], \alpha, \beta\rangle : f^{(n)} = g\}\,.$$

Our method is to show that $M(g)$ is compact and then to show that the maximum of any convex functional, e.g. $\|\ \|_p$, can not occur on a function with interior zeros if $\alpha + \beta > 0$. Theorem 4.1 can then be established.

The following two lemmas are useful in showing $M(g)$ is compact.

LEMMA 4.2 [12; p. 398]. *Let* x_0 *be a fixed element of a Banach space* X, *let* E *be a closed finite dimensional subspace of* X *and let* H *be a finite dimensional hyperplane* $\{y + x_0 : y \in E\}$. *Suppose that* M *is a closed subset of* X *which is also closed under scalar multiplication. If* $E \cap M = \{0\}$, *then* $H \cap M$ *is compact.*

LEMMA 4.3. *The set* $C^n\langle[a, b], \alpha, \beta\rangle$ *is closed with respect to the norm*

$$(4.3) \qquad \|f\| = \max_{0 \leqq m \leqq n}\ \max_{a \leqq t \leqq b} |f^{(m)}(t)|\,.$$

These lemmas can be used to establish the following theorem. It is not necessary to assume $g \geqq 0$ at this point.

THEOREM 4.4. *$M(g)$ is a compact set with respect to the norm* (4.3).

Proof. Any $f \in M(g)$ can be written in the form

$$(4.4) \qquad f(t) = p(t) + \int_a^t \cdots \int_a^{t_{n-1}} g(t_n)dt_n \cdots dt_1$$

where $p(t)$ is a polynomial of degree $n - 1$ or less. Hence, $M(g) = H \cap C^n\langle[a, b], \alpha, \beta\rangle$ where H is the set of all functions satisfying $f^{(n)} = g$ or, equivalently, equation (4.4). If we let E be the n-dimensional subspace of polynomials of degree $n - 1$ or less, then we see that H

is the n-dimensional hyperplane

$$\{p(t) + x_0(t): p(t) \in E\}$$

where x_0 is the iterated integral of equation (4.4). We take $M = C^n\langle[a, b], \alpha, \beta\rangle$ which is closed topologically as well as under scalar multiplication. Moreover $E \cap C^n\langle[a, b], \alpha, \beta\rangle$ is the set consisting of only the zero function since no other polynomial of degree $n - 1$ or less can have n zeros. This means all the conditions of Lemma 4.2 have been satisfied. Therefore, $M(g) = H \cap M$ is compact and the proof is complete.

The following lemma which has been verified by the author is stated without proof in [12, p. 397].

LEMMA 4.5. *Let $x(t)$ and $y(t) \in C^n[t_1, t_2]$ have at some point $c \in (t_1, t_2)$ zeros of order exactly r and $r - 1$, $1 \leqq r \leqq n$, respectively. Then there exists an $\varepsilon > 0$ such that each of the functions*

$$z_1(t) = x(t) + \varepsilon y(t) \quad \text{and} \quad z_2(t) = x(t) - \varepsilon y(t)$$

has not less than r zeros on $[t_1, t_2]$.

We are now ready to find $\sup\{\|f\|_p: f \in M(g)\}$. For $g \geqq 0$ we define $M_1 = M_1(g)$ to be the subset of $M(g)$ containing the functions without interior zeros and $M_2 = M_2(g)$ be the subset of $M(g)$ which contains functions f which have a zero of order greater than n at $t_0 \in (a, b)$ and which satisfy $f(t) \neq 0$ elsewhere on $[a, b]$. $M_2 = \varnothing$ unless $\alpha = \beta = 0$.

THEOREM 4.6. *If $g \geqq 0$ and $h \in M(g) - M_1$ then*

$$\|h\|_p < \sup\{\|f\|_p: f \in M(g)\} = \sup\{\|f\|_p: f \in M_1\},\ p \in [1, \infty] .$$

Proof. It was shown in Corollary 3.3 that because $g \geqq 0$, any function in $M(g)$ has exactly n zeros unless it has exactly one distinct zero and this zero is of order greater than n.

Since $M(g)$ is compact, there exists $f_0 \in M(g)$ such that

$$\|f_0\|_p = \sup\{\|f\|_p: f \in M(g)\} .$$

We begin by showing that f_0 must be in $M_1 \cup M_2$. Suppose otherwise. Let $t_1, \cdots, t_n$ be the set of zeros for f_0. Of these zeros α must at a and β at b. Since $f_0 \notin (M_1 \cup M_2)$, we may assume that there is an interior zero $t_1 = \cdots = t_r$ of order $r \leqq n - \alpha - \beta$.

Let

$$p(t) = (t - t_1)^{r-1}(t - t_{q+1}) \cdots (t - t_n)$$

so that $p(t)$ is a polynomial of degree $n-1$ with a zero of order $r-1$ at t_1. Hence by Lemma 4.5 there exists an $\varepsilon > 0$ such that both $z_1(t) = f_0(t) + \varepsilon p(t)$ and $z_2(t) = f_0(t) - \varepsilon p(t)$ have r zeros in some neighborhood of t_1 which may be chosen so small that it contains no other zeros of f_0. Moreover, z_1 and z_2 must be members of $M(g)$ as z_1 and z_2 have n zeros on $[a, b]$ and $z_1^{(n)} = z_2^{(n)} = g$. We have $f_0(t) = [z_1(t) + z_2(t)]/2$. Thus, by Minkowski's inequality

$$||f_0||_p \leqq [||z_1||_p + ||z_2||_p]/2 \tag{4.5}$$

so that either

$$||f_0||_p \leqq ||z_1||_p \quad \text{or} \quad ||f_0||_p \leqq ||z_2||_p\ . \tag{4.6}$$

In order to complete the contradiction to the definition of f_0 we must show that strict inequality holds for one of the possible inequalities in (4.6). We can do this as follows. For $p = 1$, equality holds in Minkowski's inequality and hence in (4.5) only if z_1 and z_2 always have the same sign. This is impossible for z_1 and z_2. For finite $p > 1$, equality is possible in Minkowski's inequality only if z_1 and z_2 satisfy $Dz_1(t) = Ez_2(t)$ for some $D, E \geqq 0$, $D^2 + E^2 > 0$. This is also impossible for our z_1 and z_2. The comments regarding equality in Minkowski's inequality can be deduced from an inspection of its proof. For $p = \infty$, we can show that either $||f_0||_\infty < ||z_1||_\infty$ or $||f_0||_\infty < ||z_2||_\infty$. We see that this is the case by noting that if $|f_0(t_0)| = ||f_0||_\infty$, then either

$$|z_1(t_0)| = |f_0(t_0) + \varepsilon p(t_0)| > |f_0(t_0)|$$

or

$$|z_2(t_0)| = |f_0(t_0) - \varepsilon p(t_0)| > |f_0(t_0)|\ .$$

The necessary contradiction to the definition of f_0 has now been obtained as strict inequality holds in at least one of inequalities (4.6).

We can complete the proof of the theorem by showing that if f_0 maximizes $||f||_p$ then $f_0 \notin M_2$. Suppose otherwise. Let $h \in C^n\langle[a, b]\rangle$ be a function which satisfies $|h(t)| = |f_0(t)|$ and whose sign will be determined later. Then $|h^{(n)}(t)| = |f_0^{(n)}(t)|$. Since f_0 and h must have a zero of order greater than n at some point c, we can write $h(t) = (t-c)^n h_n(t)$ where h_n is continuous and $h_n(t) = 0$ if and only if $t = c$. Consider the functions $h(t) \pm \varepsilon(t-c)^{n-1} = (t-c)^{n-1}[(t-c)h_n(t) \pm \varepsilon]$. As we will insist $h_n(t) > 0$ for $t \neq c$, we can find $\varepsilon_1, \varepsilon_2 > 0$ such that $z_1(t) = h(t) + \varepsilon_1(t-c)^{n-1}$ and $z_2(t) = h(t) - \varepsilon_2(t-c)^{n-1}$ have distinct zeros of orders 1 and $n-1$. As shown above, either z_1 or z_2 satisfies $||z_i||_p > ||h||_p = ||f_0||_p$. As shown in the proof of Theorem 3.4 we can find a function f such that $f^{(n)} = |z_i^{(n)}|$ and $||f||_p \geqq ||z_i||_p$. But $f^{(n)} = g$ so $f \in M(g)$ and we have the contradiction to the maximizing property of f_0. The theorem is valid.

The last theorem can now be used to establish Theorem 4.1.

Proof of Theorem 4.1. Recall that M_1 depends on g so that $A = \cup\{M_1(g): g \geqq 0\}$.

Let $f_0 \in C^n\langle[a, b], q, \alpha, \beta\rangle$. As shown in Theorem 3.4, there exists a function h in the same set which satisfies $h^{(n)} \geqq 0$ and $\|f_0\|_p \leqq \|h\|_p$. By Theorem 4.6 we have that if $h \notin M_1(h^{(n)})$, then

$$\|f_0\|_p \leqq \|h\|_p < \sup\{\|f\|_p: f \in M_1(h^{(n)})\} \leqq \sup\{\|f\|_p: f \in A\} .$$

The proof of the theorem is now complete.

5. The constants $K(n, p, q, \alpha)$. Let us return to the problem of finding best possible constants C which satisfy the inequality

$$\|f\|_p \leqq C\cdot(b-a)^s\|f^{(n)}\|_p,\ f \in C^n\langle[a, b], \alpha, \beta\rangle . \tag{5.1}$$

If we choose $s = n + 1/p - 1/q$ and transform the problem to the interval $[0, 1]$, the above inequality reduces to

$$\|f\|_p \leqq C\|f^{(n)}\|_q,\ f \in C^n\langle[0, 1], \alpha, \beta\rangle . \tag{5.2}$$

If in addition $f/\|f^{(n)}\|_q$ is replaced by f, we have that the last two inequalities are equivalent to

$$\|f\|_p \leqq C,\ f \in C^n\langle[0, 1], q, \alpha, \beta\rangle . \tag{5.3}$$

With this in mind we define

$$K(n, p, q, \alpha) = \sup\{\|f\|_p: f \in C^n\langle[0, 1], q, \alpha\rangle\} \tag{5.4}$$

for $n = 1, 2, \cdots; p, q \in [1, \infty]$; and $\alpha = 0, \cdots, n$. Then the smallest possible constant C satisfying inequalities (5.1, 2, 3) is

$$C = \max\{K(n, p, q, \alpha'): \alpha \leqq \alpha' \leqq n - \beta\} . \tag{5.5}$$

Methods for determining the constants $K(n, p, q, \alpha)$ will be studied in the following sections. But first we will look at some of their properties.

THEOREM 5.1. *For any $n = 1, 2, \cdots$ and any $\alpha = 0, \cdots, n$; the constants $K(n, p, q, \alpha)$ satisfy*

$$K(n, p, q', \alpha) \leqq K(n, p, q, \alpha) \quad \textit{for} \quad q \leqq q' , \tag{5.6}$$

$$K(n, p, q, \alpha) \leqq K(n, p', q, \alpha) \quad \textit{for} \quad p \leqq p' , \tag{5.7}$$

and

$$K(n, 1, \infty, \alpha) \leqq K(n, p, q, \alpha) \leqq K(n, \infty, 1, \alpha) \tag{5.8}$$

Proof. The first two of the inequalities can be proven using the following fact. For $h \in L^r[0, 1]$

$$\|h\|_p \leqq \|h\|_r \quad \text{if} \quad 1 \leqq p < r \tag{5.9}$$

where equality holds if and only if h is a constant function. This can be shown by use of Holder's inequality for finite p and r. The result is clearly valid for $r = \infty$.

Inequality (5.6) may be proven as follows. Suppose n and α are given and that $f \in C^n\langle[0, 1], \alpha\rangle$. By inequality (5.2) with

$$C = K(n, p, q, \alpha), \beta = n - \alpha$$

and by (5.9),

$$\|f\|_p \leqq K(n, p, q, \alpha)\|f^{(n)}\|_q \leqq K(n, p, q, \alpha)\|f^{(n)}\|_{q'}$$

providing $q \leqq q'$. But $K(n, p, q', \alpha)$ is the smallest number K such that $\|f\|_p \leqq K\|f^{(n)}\|_{q'}$ for all $f \in C^n\langle[0, 1], \alpha\rangle$. Inequality (5.6) follows directly.

For $p \leqq p'$, we have

$$\|f\|_p \leqq \|f\|_{p'} \leqq K(n, p', q, \alpha)\|f^{(n)}\|_q .$$

As before, $K(n, p, q, \alpha)$ is the smallest number K for which $\|f\|_p \leqq K\|f^{(n)}\|_q$ is valid for all $f \in C^n\langle[0, 1], \alpha\rangle$ and inequality (5.7) follows.

In order to show inequality (5.8) and complete the proof, we notice that

$$\begin{aligned} K(n, 1, \infty, \alpha) \leqq K(n, 1, q, \alpha) &\leqq K(n, p, q, \alpha) \\ &\leqq K(n, p, 1, \alpha) \leqq K(n, \infty, 1, \alpha) . \end{aligned}$$

Inequalities (5.6–8) would be valid in terms of the extended real numbers even if some of the constants were infinite. However, inequality (5.8) and the existence of the upper bound would be sufficient to show that $K(n, p, q, \alpha)$ exists and is finite for all p and q.

THEOREM 5.2. [8] *$\ln K(n, p, q, \alpha)^p$ is convex in p.*

Proof. Let $n, p_1, p_2, q,$ and α be given and consider the inequalities $\|f\|_{p_i} \leqq K(n, p_i, q, \alpha)$, $i = 1, 2$, which must hold for every

$$f \in C^n\langle[0, 1], q, \alpha\rangle .$$

Let $\theta_1, \theta_2 \geqq 0$ satisfy $\theta_1 + \theta_2 = 1$, raise the above inequalities to the $\theta_i p_i$ power and then multiply to obtain

$$\|f\|_{p_1}^{\theta_1 p_1}\|f\|_{p_2}^{\theta_2 p_1} \leqq [K(n, p_1, q, \alpha)]^{\theta_1 p_1}[K(n, p_2, q, \alpha)]^{\theta_2 p_2} .$$

But by the Riesz Convexity Theorem we have that the left hand side is at least as large as $\|f\|_r^r$ where $r = \theta_1 p_1 + \theta_2 p_2$. This means that for any $f \in C^n\langle[0, 1], q, \alpha\rangle$ we have

$$\|f\|_r \leqq [K(n, p_1, q, \alpha)]^{\theta_1 p_1/r}[K(n, p_2, q, \alpha)]^{\theta_2 p_2/r} .$$

But $K(n, r, q, \alpha)$ is the smallest number greater than $\|f\|_r$ for all f so that

$$K(n, r, q, \alpha) \leqq [K(n, p_1, q, \alpha)]^{\theta_1 p_1/r}[K(n, p_2, q, \alpha)]^{\theta_2 p_2/r} .$$

The proof is completed by raising the above inequality to the r power and then taking logs.

THEOREM 5.3. *For $n = 1, 2, \cdots; p, q \in [1, \infty]$; and $\alpha = 0, \cdots, n$;*

$$K(n, p, q, \alpha) = K(n, P, Q, \alpha)$$

where $1/p + 1/Q = 1$ and $1/q + 1/P = 1$.

The proof of this theorem will be delayed until we have studied methods for evaluating the K's.

We notice that by the symmetry in the boundary conditions

$$K(n, p, q, \alpha) = K(n, p, q, n - \alpha) , \tag{5.10}$$

6. $K(n, p, q, \alpha)$ for $q = \infty$. In this section we will show that the function g defined by the boundary value problem

$$g^{(n)}(t) = 1 , \tag{6.1'}$$

$$g^{(i)}(0) = 0 , \quad i = 0, \cdots, \alpha - 1 ; \tag{6.1''}$$

$$g^{(i)}(1) = 0 , \quad i = 0, \cdots, n - \alpha - 1 ; \tag{6.1'''}$$

maximizes $\|f\|_p$ over all functions f in the set $C^n\langle[0, 1], q = \infty, \alpha\rangle$ for every extended real number $p \geqq 1$. Using this fact, we can easily calculate $K(n, p, \infty, \alpha)$ as defined by equation (5.4).

THEOREM 6.1. *The solution of the boundary value problem* (6.1), *namely $g(t) = t^\alpha(t - 1)^{n-\alpha}/n!$, satisfies*

$$-|g(t)| \leqq f(t) \leqq |g(t)| , \quad t \in [0, 1] , \tag{6.2}$$

for every $f \in C^n\langle[0, 1], q = \infty, \alpha\rangle$ with equality for $t \in (0, 1)$ only if $f = \pm g$. Hence, the constants $K(n, p, \infty, \alpha)$ exist and

$$K(n, p, \infty, \alpha) = \|g\|_p , \quad p \in [1, \infty] , \tag{6.3}$$

$$K(n, p, \infty, \alpha)$$

$$= \begin{cases} \dfrac{1}{n!}\left[\dfrac{\Gamma(p\alpha+1)\Gamma(pn-p\alpha+1)}{\Gamma(pn+2)}\right]^{1/p}, & p \in [1, \infty) , \quad (6.4_p) \\ [(p\alpha)!\,(pn-p\alpha)!/(pn+1)!]^{1/p}/n! , & p = 1, 2, \cdots, \quad (6.4'_p) \\ \alpha^\alpha(n-\alpha)^{n-\alpha}/(n!\,n^n) , & p = \infty , \quad (6.4_\infty) \end{cases}$$

for $n = 1, 2, \cdots$ *and* $\alpha = 0, \cdots, n$.

Here $\Gamma(x)$ denotes the gamma function and $O^0 = 1$.

Proof. Clearly $g(t)$ satisfies the problem (6.1). Inequality (6.2) can be established by an enclosing argument as in the proof of Theorem 3.4, case (ii). Define $h(t) = f(t) - g(t)$. As f and g are both in $C^n\langle[0, 1], \alpha\rangle$, so is h. One sees that

$$h^{(n)}(t) = f^{(n)}(t) - g^{(n)}(t) = f^{(n)}(t) - 1 \leqq 0$$

since $\|f^{(n)}\|_\infty \leqq 1$.

For finite p, calculations may be simplified by expressing $\|g\|_p^p$ in terms of the beta function.

7. $K(n, p, q, \alpha)$ for $q = 1$.

In this section we will show that it is possible to express $K(n, p, 1, \alpha)$ in terms of a Green's function. Our main theorem is:

THEOREM 7.1. *For* $n = 1, 2, \cdots; p \in [1, \infty]; \alpha = 0, \cdots, n;$ *and* $f \in C^n\langle[0, 1], q = 1, \alpha\rangle$; *the constants* $K(n, p, 1, \alpha)$ *exist and*

$$\|f\|_p < K(n, p, 1, \alpha) = \max\{\|g(t, s)\|_p : 0 \leqq s \leqq 1\} \tag{7.1}$$

where $g(t, s)$ *is the Green's function for the boundary value problem*

$$y^{(n)}(t) = 0 ; \tag{7.2a$_n$}$$

$$y^{(i)}(0) = 0 , \quad i = 0, \cdots, \alpha - 1 ; \tag{7.2b$_{n\alpha}$}$$

$$y^{(i)}(1) = 0 , \quad i = 0, \cdots, n - \alpha - 1 . \tag{7.2c$_{n\alpha}$}$$

Since the only solution of this boundary value problem is the trivial one, $y \equiv 0$, the Green's function $g(t, s)$ exists and can be determined uniquely from the following properties [5, p. 192]:

(i) $(\partial^k/\partial t^k)g(t, s)$ is a continuous function for $k = 0, \cdots, n-2$, on the rectangle $0 \leqq t, s \leqq 1$ and is continuous for $k = n - 1$ and n on the triangles $0 \leqq t < s \leqq 1$ and $0 \leqq s < t \leqq 1$.

(ii) $(\partial^{n-1}/\partial t^{n-1})g(s+, s) - (\partial^{n-1}/\partial t^{n-1})g(s-, s) = 1$.

(iii) For each fixed $s \in [0, 1]$, $g(t, s)$ is a solution of the boundary value problem $(7.2_{n\alpha})$ except at $t = s$.

Of course

$$y(t) = \int_0^1 g(t, s)f(s)ds \tag{7.3}$$

is the solution of $y^{(n)}(t) = f(t)$ subject to the boundary conditions (7.2b, $c_{n\alpha}$).

The Green's functions $g(t, s)$ will be considered to be functions of the variable t indexed by the parameter s. This convention will allow us to use the notation $g^{(i)}(t, s)$ for the ith partial derivative of $g(t, s)$ with respect to t.

A major part of the proof of Theorem 7.1 is based on the fact that for $\alpha = 1, \cdots, n-1$ and $f \in C^n\langle[0, 1], q = 1\rangle$, there exists $s \in$ 0, 1) such that

$$-|g(t, s)| < f(t) < |g(t, s)| , \quad t \in (0, 1) ,$$

at least if $f^{(n)} \geqq 0$. This s satisfies the equation $f^{(n-1)}(0) = g^{(n-1)}(0, s)$.

As the first step in showing this, we note that $g(t, 0)$ has $n - \alpha$ zeros at $t = 1$ and by the continuity of the function and the first $n - 2$ partials, $g(t, 0)$ must have $\alpha \leqq n - 1$ zeros at $t = 0$. Thus on the interval $0 = s \leqq t \leqq 1$, $g(t, 0)$ is a polynomial of degree $n - 1$ at most, has n zeros, and thus, must be the zero function. Likewise for $s = 1$. Hence

$$g(t, 0) \equiv 0 \equiv g(t, 1) , \quad \alpha = 1, \cdots, n-1 .$$

Now that we have $g(t, 0)$ we can show the following:

LEMMA 7.2. *For $\alpha = 1, \cdots, n - 1$; $n = 2, 3, \cdots$; and $k \in (-1, 0)$, there exists $s \in (0, 1)$ such that $g^{(n-1)}(0, s) = k$.*

Proof. Despite the fact that $g(t, 0)$, appears to be an analytic function, in the limiting sense it must have a jump in the $n - 1$st derivative at $t = s = 0$. Since $g^{(n-1)}(t, 0) = 0$ for $t > 0$, we must have that $g^{(n-1)}(t, s) \to -1$ as $(t, s) \to (0, 0)$ with $0 \leqq t < s$. Clearly $g^{(n-1)}(0, 1) =$ 0. By the continuity of $g^{(n-1)}(t, s)$ on the triangle $0 \leqq t < s \leqq 1$, $g^{(n-1)}(0, s)$ must take on any given value of $k \in (-1, 0)$.

We need the corresponding results for $\alpha = 0$ and $\alpha = n$ but the reader can as easily verify the following:

THEOREM 7.3. *For any positive integer n, the Green's function for the boundary value problem $(7.2_{n\alpha})$ with $\alpha = 0$ is given by*

$$g(t, s) = \begin{cases} -(t-s)^{n-1}/(n-1)! , & \text{for } 0 \leqq t \leqq s \leqq 1 , \\ 0 , & \text{for } 0 \leqq s \leqq t \leqq 1 , \end{cases}$$

and for $\alpha = n$

$$g(t, s) = \begin{cases} 0\,, & \textit{for}\quad 0 \leqq t \leqq s \leqq 1\,, \\ (t-s)^{n-1}/(n-1)!\,, & \textit{for}\quad 0 \leqq s \leqq t \leqq 1\,. \end{cases}$$

Proof of Theorem 7.1. Suppose that we are given

$$f_0 \in C^n\langle[0, 1], q = 1, \alpha\rangle\ .$$

By Theorem 3.4 there exists a function f in the same set which satisfies $f^{(n)} \geqq 0$ and $\|f_0\|_p \leqq \|f\|_p$. By Theorem 4.1 we may assume f has no interior zeros. We will show that if $s = s_0$ is picked properly and satisfies

$$g^{(n-1)}(0, s_0) = f^{(n-1)}(0),\ g^{(n-1)}(1, s_0) = f^{(n-1)}(1) \tag{7.4}$$

then

$$-|g(t, s_0)| < f(t) < |g(t, s_0)|\ , \quad t \in (0, 1)\ . \tag{7.5}$$

But first we need to show that such an s_0 exists.

First for $\alpha = 1, \cdots, n-1$. Let $k = f^{(n-1)}(0)$. Since $f^{(n)}(t) \geqq 0$, we have

$$f^{(n-1)}(1) = k + \int_0^1 f^{(n)}(t)dt = k + \|f^{(n)}\|_1 = k + 1\ .$$

Since $f^{(n)} \geqq 0$, $f^{(n-1)}$ is increasing and because f has n zeros, $f^{(n-1)}$ changes sign so that $k \in (-1, 0)$. By Lemma 7.2 there is an s_0 such that equations (7.4) hold.

For $\alpha = 0$, we have that $f^{(n-1)}(1) = 0$ so that

$$f^{(n-1)}(0) = f^{(n-1)}(1) - 1 = -1\ .$$

By Theorem 7.3, equations (7.4) are satisfied in terms of limits by $s_0 = 1$. Likewise the equations are satisfied with $s_0 = 0$ for $\alpha = n$.

We can show that inequality (7.5) is valid with s_0 as chosen above. For any $n = 2, 3, \cdots$ and $\alpha = 1, \cdots, n-1$, we define

$$h(t) = f(t) - g(t, s_0) \in C^{n-2}[0, 1]\ .$$

Since both $f(t)$ and $g(t, s_0)$ satisfy the boundary conditions (7.2b, $c_{n\alpha}$), so does $h(t)$. Moreover, h can never be the zero function. We have

$$\begin{aligned} &h^{(n-1)}(t) = f^{(n-1)}(t) - g^{(n-1)}(t, s_0)\ , \\ &\begin{cases} h^{(n-1)}(t) \geqq 0 & \text{for}\quad 0 \leqq t < s_0\ , \\ h^{(n-1)}(t) \leqq 0 & \text{for}\quad s_0 < t \leqq 1\ . \end{cases} \end{aligned} \tag{7.6}$$

Clearly $h^{(n-1)}(t)$ has exactly one sign change on $[0, 1]$ and for each $t \neq s_0$, $h^{(n-1)}(t)$ and $g^{(n-1)}(t, s_0)$ have opposite signs. Suppose that $h(t)$ has an interior zero. In this case the continuous function $h^{(n-2)}(t)$

must have $(n+1)-(n-2)=3$ zeros, and since the function $h(t)$ has 3 distinct zeros, the 3 zeros of $h^{(n-2)}(t)$ are distinct. Two of these zeros must fall in either $[0, s_0]$ or $[s_0, 1]$. By the extension of the mean value theorem used in the proof of Corollary 3.3, h must change signs within that interval. Since this is impossible, $h(t)$ can have no interior zeros.

As in the other theorems established by this method, $h(t)$ and $g(t, s_0)$ have opposite signs. This can be established by noticing that the conclusion of Lemma 3.1 remains true if we only insist that $f^{(n-1)}$ is integrable, is nonpositive on $[a, t_{n-1}]$, and is nonnegative on $[t_{n-1}, b]$. Inequality (7.5) can then be established by use of an enclosing argument as in the proof of Theorem 3.4, case (ii). Hence $\|f\|_p < \|g(t, s_0)\|_p$. This can be shown for $\alpha = 0$ and n with s_0 chosen as indicated earlier, that is with $s_0 = 1$ and 0, respectively.

Now we can easily complete the proof of Theorem 7.1. Since $g(t, s)$ is a continuous function of two variables, $\|g(t, s)\|_p$ (norm over $t \in [0, 1]$) is a continuous function of $s \in [0, 1]$. Therefore, its maximum exists. We have

$$\|f_0\|_p \leqq \|f\|_p < \|g(t, s_0)\|_p \leqq \max\{\|g(t, s)\|_p : 0 \leqq s \leqq 1\} \tag{7.7}$$

for all $f_0 \in C^n\langle[0, 1], q = 1, \alpha\rangle$. Moreover, this is the least upper bound over $C^n\langle[0, 1], q = 1, \alpha\rangle$ because for any given $s \in [0, 1]$, the Green's function $g(t, s)$ can be approximated as closely as desired by functions in $C^n\langle[0, 1], q = 1, \alpha\rangle$ with respect to the norm $\|f\| = \max\{\|f^{(i)}\|_1 : i = 0, \cdots, n-1\}$. The theorem is now established.

It would seem desirable to have a formula for the Green's function for calculating numerical values of $K(n, p, 1, \alpha)$. The author derived one such formula involving a double summation and a comparable formula is stated in [14]. As these formulas are rather complicated, it appears easier to use a method suggested in Coddington and Levinson [5, p. 190–193]. It allows one to find the Green's function by solving a linear system of dimension $n - \alpha$.

In certain cases, the theory of the adjoint boundary value problem is useful as an aid in calculating $K(n, p, 1, \alpha)$.

THEOREM 7.4. *For any $n = 1, 2, \cdots$, and any $\alpha = 0, \cdots, n$, the adjoint of the boundary value problem* $(7.2_{n\alpha})$ *is the boundary value problem*

$$(-1)^n y^{(n)}(t) = 0 \tag{7.8a_n}$$

$$y^{(i)}(0) = 0 \quad \textit{for} \quad i = 0, \cdots, n-\alpha-1 \tag{7.8$\mathrm{b}_{n\alpha}$}$$

$$y^{(i)}(1) = 0 \quad \textit{for} \quad i = 0, \cdots, \alpha-1\,. \tag{7.8$\mathrm{c}_{n\alpha}$}$$

Moreover, the Green's function for the adjoint problem is

$$g^*(t, s) = g(s, t)$$

where $g(t, s)$ *is the Green's function for* $(7.2_{n\alpha})$.

This theorem can be established by using Coddington and Levinson [5, p. 284-297].

We can now derive a few formulas for the constants $K(n, p, 1, \alpha)$ *for certain special choices of* p *and* α. *Some hints for calculating the constants will be given for a few cases when special formulas are not available.*

The problem of finding $K(n, \infty, 1, \alpha)$ has been shown to be equivalent to finding some best possible bounds for the Green's function. Beesack [2] has obtained some results in this area and in particular gives $K(n, \infty, 1, 0) = K(n, \infty, 1, n)$. In fact, the cases $\alpha = 0$ and n are easy. By Theorem 7.3 one sees that $\|g(t, s)\|_p$ is maximized by $s = 1$ if $\alpha = 0$ and $s = 0$ if $\alpha = n$. Thus, for any positive integer n,

$$K(n, p, 1, 0) = K(n, p, 1, n) = [(n-1)!\,(pn - p + 1)^{1/p}]^{-1},\ p \in [1, \infty)\ ,$$
$$K(n, \infty, 1, 0) = K(n, \infty, 1, n) = [(n-1)!]^{-1}\ .$$

We have two results which are obtained by use of the adjoint boundary value problem. The first concerns $p = \infty$.

THEOREM 7.5. *For any* $n = 1, 2, \cdots$, *and any* $\alpha = 0, \cdots, n$, *the Green's function for the boundary value problem* $(7.2_{n\alpha})$ *obtains its maximum magnitude along the line* $s = 1 - t$.

Proof. Suppose that n and α are given. Let $g_\alpha(t, s)$ be the Green's function for problem $(7.2_{n\alpha})$, let $g_{n-\alpha}(t, s)$ be the Green's function for the symmetric problem $(7.2_{n,n-\alpha})$ and finally $g_\alpha^*(t, s)$ be the function for the adjoint problem $(7.8_{n\alpha})$. Suppose $|g_\alpha(t, s)|$ obtains its maximum at the order pair (t_α, s_α). Then by symmetry $|g_{n-\alpha}(t, s)|$ must obtain its maximum for some point with $t = 1 - t_\alpha$ and $|g_\alpha^*(t, s)|$ must obtain its maximum at some point with $t = s_\alpha$. But by consideration of the associated boundary value problems, we have $g_{n-\alpha}(t, s) = (-1)^n g_\alpha^*(t, s)$ so that $s_\alpha = 1 - t_\alpha$ and the proof is complete.

By use of the adjoint problem once again, we can show

$$K(n, 1, 1, \alpha) = K(n, \infty, \infty, n - \alpha) = K(n, \infty, \infty, \alpha) \tag{7.9}$$

or

$$K(n, 1, 1, \alpha) = \frac{\alpha^\alpha (n - \alpha)^{n-\alpha}}{n!\,n^n} \tag{7.10}$$

for any positive integer n and any $\alpha = 0, \cdots, n$. If n and α are given, we have

$$\begin{aligned} K(n, 1, 1, \alpha) &= \max\left\{\int_0^1 |g(t, s)|\,dt : \quad 0 \leqq s \leqq 1\right\} \\ &= \max\left\{\left|\int_0^1 g(s, t)ds\right| : \quad 0 \leqq t \leqq 1\right\} \\ &= \max\left\{\left|\int_0^1 g^*(t, s)ds\right| : \quad 0 \leqq t \leqq 1\right\} \end{aligned}$$

where $g^*(t, s)$ is the Green's function for the adjoint problem $(7.8_{n\alpha})$. The last integral is the solution of the nonhomogeneous equation $(-1)^n y^{(n)}(t) = 1$ satisfying the adjoint boundary conditions (7.8b, $c_{n\alpha}$). Therefore, $\left|\int_0^1 g^*(t, s)ds\right| = |(-1)^n t^{n-\alpha}(t-1)^\alpha/n!|$ which must be maximized over t to obtain $K(n, 1, 1, \alpha)$. For all practical purposes this was done when the function $g(t)$, as given in Theorem 6.1, was maximized in order to obtain $K(n, \infty, \infty, \alpha)$ so equation (7.9) and hence (7.10) are valid.

8. $K(n, p, q, \alpha)$ for $p = \infty$.

The following theorem shows that Theorem 5.3 is valid for $p = \infty$ or $q = 1$.

THEOREM 8.1. *If* $n = 1, 2, \cdots; q \in [1, \infty]; \alpha = 0, \cdots, n;$ *and* $f \in C^n\langle[0, 1], q, \alpha\rangle$ *then*

$$\|f\|_\infty \leqq K(n, \infty, q, \alpha) = K(n, P, 1, \alpha)\ , \quad 1/P + 1/q = 1\ .$$

Equality holds if and only if (i) $q = \infty$ *and* $f^{(n)} = \pm 1$ *or* (ii) $q \in (1, \infty)$ *and* $|f^{(n)}(t)|^q$ *is a nonzero multiple of* $|g^*(t, s)|^p$, *the Green's function for the adjoint boundary value problem* $(7.8_{n\alpha})$, *where* s *is selected so that* $\|g^*(t, s)\|_p$ *is maximized.*

Proof. By Theorem 4.1 we need only consider functions for which neither f nor $f^{(n)}$ change signs. The Green's function $g(t, s)$ for the boundary value problem $(7.2_{n\alpha})$ does not change sign either. By (7.3), $f(t) = \int_0^1 g(t, s)f^{(n)}(s)ds$. Hence by Holder's inequality and by Theorem 7.4,

$$|f(t)| \leqq \|g(t, s)\|_P \|f^{(n)}\|_q = \|g^*(s, t)\|_P \|f^{(n)}\|_q\ , \quad t \in [0, 1]\ ,$$

where $\|\ \|_p$ indicates norms over $s \in [0, 1]$. Hence,

$$\|f\|_\infty \leqq \max\{\|g^*(s, t)\|_P : \quad t \in [0, 1]\}\cdot\|f^{(n)}\|_q$$

and $K(n, \infty, q, \alpha) \leqq K(n, P, 1, n-\alpha) = K(n, P, 1, \alpha)$. By considering the possibility of equality in Holder's inequality, we see that the statements regarding equality are valid for $q \in (1, \infty)$. For $q = \infty$ and 1, we use Theorems 6.1 and 7.1 and the proof is complete.

For hints regarding means of calculating

$$K(n, \infty, q, \alpha) = K(n, P, 1, \alpha) ,$$

we can refer to the last part of §7.

9. $K(n, p, q, \alpha)$ for $p = 1$. The following theorem shows that Theorem 5.3 is valid for $p = 1$ or $q = \infty$.

THEOREM 9.1. *If $n = 1, 2, \cdots; q \in [1, \infty]; \alpha = 0, \cdots, n;$ and $f \in C^n\langle[0, 1], q, \alpha\rangle$ then*

$$\|f\|_1 \leqq K(n, 1, q, \alpha) = K(n, P, \infty, \alpha) , \quad 1/P + 1/q = 1 .$$

Equality holds if and only if (i) $q = \infty$ *and* $f^{(n)}(t) = \pm 1$ *or* (ii) $q \in (1, \infty)$ *and* $|f^{(n)}(t)|^q$ *is nonzero multiple of* $|t^{n-\alpha}(t-1)^\alpha|^P$.

Proof. Let $f \in C^n\langle[0, 1], q, \alpha\rangle$, $\beta = n - \alpha$, and $g(t) = t^\beta(t-1)^\alpha/n!$. We notice that $\|g\|_P = K(n, P, \infty, \beta) = K(n, P, \infty, \alpha)$ and that $g^{(n)} = 1$. We may assume f does not change signs and that $f^{(n)} \geqq 0$. By integration by parts, we have

$$\begin{aligned}
\int_0^1 f(t)dt &= f(t)g^{(n-1)}(t)\Big|_0^1 - \int_0^1 f'(t)g^{(n-1)}(t)dt \\
&\;\;\vdots \\
&= \sum_{i=0}^{n-1} (-1)^i f^{(i)}(t)g^{(n-i-1)}(t)\Big|_0^1 \\
&\quad + (-1)^n\int_0^1 f^{(n)}(t)g(t)dt \\
&= (-1)^n\int_0^1 f^{(n)}(t)g(t)dt
\end{aligned}$$

due to the boundary conditions satisfied by f and g. Thus,

$$\|f\|_1 = \|f^{(n)}g\|_1 \leqq \|g\|_P\|f^{(n)}\|_q = K(n, P, \infty, \alpha)\cdot 1 .$$

For $q \in (1, \infty)$, equality holds if and only if $|f^{(n)}|^q$ is a nonzero multiple of $|g|^P$. Our other remarks concerning equality follow from Theorems 6.1 and 7.1.

Theorem 6.1 is useful for finding formulas for $K(n, 1, q, \alpha)$.

10. $K(n, p, q, \alpha)$ for other p and q. The following theorem is an application of a method suggested by Boyd [4]. Actually the technique can be used on a more general class of equalities than we are considering in this paper. The reader can verify that the result given here for $p = 1$ reduces to that found in the last section.

THEOREM 10.1. *If* $n = 1, 2, \cdots$; $p \in [1, \infty)$; $q \in (1, \infty)$; $\alpha = 0, \cdots, n$; *and* $f \in C^n\langle[0, 1], q, \alpha\rangle$ *then*

$$\|f\|_p \leqq K(n, p, q, \alpha) = \lambda^{1/p} \tag{10.1}$$

where λ *is the largest eigenvalue of the differential system*

$$y^{(n)} = (-1)^{n-\alpha}[(-1)^\alpha w/\lambda]^{1/(q-1)}\ , \tag{10.2a}$$

$$w^{(n)} = y^{p-1}\ , \tag{10.2b}$$

$$y^{(i)}(0) = w^{(i)}(1) = 0\ ,\quad i = 0, \cdots, \alpha - 1\ , \tag{10.2c}$$

$$y^{(i)}(1) = w^{(i)}(0) = 0\ ,\quad i = 0, \cdots, n - \alpha - 1\ , \tag{10.2d}$$

$$\|y^{(n)}\|_q = 1\ . \tag{10.2e}$$

Equality holds in inequality (10.1) *if and only if* $y = f(t)$ *is part of a solution of* (10.2).

Proof. If $g(t, s)$ is the Green's function for the boundary value problem (7.2), then by equation (14) in [4] we have

$$T[h(t)] = \int_0^1 g(t, s)h(s)ds$$

is a compact operator from $L^q[0, 1]$ to $L^p[0, 1]$. Thus, by Boyd's Lemma 1 there exists $f_0 \in C^n\langle[0, 1], q, \alpha\rangle$ such that $\|f_0\|_p = K(n, p, q, \alpha)$.

We can use Lemma 2 [4] to show that $y = f_0(t)$ is part of solution of system (10.2). We can show that the conditions of that lemma are satisfied as follows. Since $g^{(n-1)}(t, s)$ has only one sign change $g(t, s)$ can not change sign. $T[h(t)]$ as defined above is a bounded operator since it is compact. The boundary value problem (10.2) follows from equation (19) [4] and statements concerning λ also follow from the conclusion of the lemma so Theorem 10.1 is valid.

We can complete the proof of Theorem 5.3 at this time. The cases $p = \infty, 1$, or $q = \infty, 1$ where covered in the last two sections.

Proof of Theorem 5.3 *for* $p, q \in (1, \infty)$. Let $P = q/(q - 1)$ and $Q = p/(p - 1)$. We can show that the system (10.2) can be rewritten in an equivalent manner that allows us to determine $K(n, P, Q, \alpha)$. Notice that

$$\|w^{(n)}\|_Q = \left[\int_0^1 |y|^p dt\right]^{1/Q} = [\|y\|_p^p]^{1/Q} = \lambda^{1/Q}\ .$$

We define $Y=(-1)^{\alpha}w/\lambda^{1/Q}$ and $W=(-1)^{n-\alpha}\lambda^{(P-1)/p}y$. After noting $q = P/(P-1)$, $q-1 = 1/(P-1)$, $p = Q/(Q-1)$, and $p-1 = 1/(Q-1)$, we see that system (10.2) becomes

$$\begin{aligned}
Y^{(n)} &= (-1)^{\alpha}[(-1)^{n-\alpha}W/\lambda^{P/p}]^{1/(Q-1)} ,\\
W^{(n)} &= Y^{P-1} ,\\
Y^{(i)}(0) &= W^{(i)}(1) = 0 , \quad i = 0, \cdots, n-\alpha-1 ,\\
Y^{(i)}(1) &= W^{(i)}(0) = 0 , \quad i = 0, \cdots, \alpha-1 ,\\
&\| Y^{(n)} \|_Q = 1 .
\end{aligned}$$

Comparing this system to (10.2) we see that it is the system for determining $K(n, P, Q, n-\alpha)$. Hence,

$$K(n, P, Q, \alpha) = K(n, P, Q, n-\alpha) = [\lambda^{P/p}]^{1/P} = \lambda^{1/p} = K(n, p, q, \alpha)$$

and the proof is complete.

Theorem 10.1 can be reduced to Theorem 9.1 if $p = 1$. The following corollary concerns another special case.

COROLLARY 10.2. *For $q = 2$, system* (10.2) *can be replaced by*

$$\begin{aligned}
y^{(2n)} &= (-1)^{n}y^{p-1}/\lambda ,\\
y^{(i)}(0) &= y^{(n+i)}(1) = 0 , \quad i = 0, \cdots, \alpha-1 ,\\
y^{(i)}(1) &= y^{(n+i)}(0) = 0 , \quad i = 0, \cdots, n-\alpha-1 ,\\
&\| y^{(n)} \|_2 = 1 .
\end{aligned}$$

EXAMPLE. $K(2, 2, 2, 1) = 1/\pi^2 = 0.101321\cdots$. This follows since $1/\pi^4$ is the largest eigenvalue of the above system if $n = p = 2$ and $\alpha = 1$. The maximizing function y is $f(t) = 2^{1/2}\sin(\pi t)/\pi^2$.

11. Maximums over $C^n\langle[a, b], \alpha, \beta\rangle$ for certain α and β. The cases of most general interest may well be $\alpha = \beta = 1$ and $\alpha = \beta = 0$. We define

$$L(n, p, q) = \sup\{\|f\|_p : f \in C^n\langle[0, 1], q, \alpha = 1, \beta = 1\rangle\},\ n = 2, 3, \cdots ,$$

and

$$M(n, p, q) = \sup\{\|f\|_p : f \in C^n\langle[0, 1], q\rangle\} , \quad n = 1, 2, \cdots .$$

By equations (5.1–5) we have that $L = L(n, p, q)$ and $M = M(n, p, q)$ are the smallest constants satisfying

$$\|f\|_p \leqq L\cdot(b-a)^s\|f^{(n)}\|_q , \quad f \in C^n\langle[a, b], \alpha = 1, \beta = 1\rangle ,$$

and

$$||f||_p \leqq M \cdot (b-a)^s ||f^{(n)}||_q , \quad f \in C^n\langle [a, b] \rangle ,$$

where $s = n + 1/p - 1/q$. By equation (5.5), we see that

$$L(n, p, q) = \max \{K(n, p, q, \alpha) : \quad 1 \leqq \alpha \leqq n-1\}$$

and

$$M(n, p, q) = \max \{K(n, p, q, \alpha) : \quad 0 \leqq \alpha \leqq n\} .$$

The last equations show the need to investigate $K(n, p, q, \alpha)$ as a function of α. On the basis of numerical data for $q = 1, \infty$ and Theorem 5.1, it is conjectured that $K(n, p, q, \alpha)$ is a convex function symmetric about $\alpha = n/2$ for fixed n, p, and q. If this is true,

$$L(n, p, q) = K(n, p, q, 1) = K(n, p, q, n-1)$$

and

$$M(n, p, q) = K(n, p, q, 0) = K(n, p, q, n) .$$

We can show that these equations are valid in case $q = \infty$ or $p = 1$ as follows.

For $q = \infty$, we know that $K(n, p, \infty, \alpha) = [I(\alpha)]^{1/p}/n!$ where

$$I(\alpha) = \int_0^1 |t^\alpha (t-1)^{n-\alpha}|^p dt .$$

It is easy to show that the integral $I(\alpha)$ is concave up as a function of α. We have

$$I'(\alpha) = p\int_0^1 \ln [t/(1-t)][t^\alpha (1-t)^{n-\alpha}]^p dt$$

and

$$I''(\alpha) = p^2\int_0^1 \{\ln [t/(1-t)]\}^2 \{t^\alpha (1-t)^{n-\alpha}\}^p dt$$

which is clearly positive. With this knowledge of the graph of the integral, we see that its maximum must occur at an endpoint of the interval in α. Thus, by symmetry, $K(n, p, \infty, \alpha)$ must have it maximum at $\alpha = 1$ and $n-1$ in the case of $L(n, p, \infty)$, and at $\alpha = 0$ and n in the case of $M(n, p, \infty)$. The case $p = 1$ follows from Theorem 5.3.

The following formulas can be obtained by using Theorem 6.1 and equations (7.9, 10). For $n = 2, 3, \cdots$,

$$L(n, p, \infty) = \frac{1}{n!}\left\{\frac{\Gamma(p+1)\Gamma(np-p+1)}{\Gamma(np+2)}\right\}^{1/p}, \; p \in [1, \infty) ,$$

$$L(n, 1, 1) = L(n, \infty, \infty) = (n-1)^{n-1}/(n!n^n) ,$$

and for $n = 1, 2, \cdots$,

$$M(n, p, \infty) = (np+1)^{-1/p}/n!, \; p \in [1, \infty) ,$$

$$M(n, 1, 1) = M(n, \infty, \infty) = 1/n! .$$

According to Theorem 5.3, we have that

$$L(n, p, q) = L(n, P, Q) \quad \text{and} \quad M(n, p, q) = M(n, P, Q)$$

where $1/p + 1/Q = 1$ and $1/q + 1/P = 1$.

Tables 11.1 and 11.2 contain some values of $L(n, p, q)$ and $M(n, p, q)$. According to the last paragraph, $L(n, 1, 2) = L(n, 2, \infty)$ and $L(n, \infty, 2) = L(n, 2, 1)$. Likewise for the M's. The tables can also be used for finding values of $K(n, p, q, \alpha)$ for $\alpha = 0, 1, n-1$, and n.

Table 11.1 $L(n, p, q)$

p	q	$n = 2$	$n = 3$	$n = 4$
1	1	$1/8 = .125$	$2/81 = .02469$	$9/2048 = .00439$
2	1	$1/4\sqrt{3} = .14434$	.02927	.00549
∞	1	$1/4 = .25$	.04508	.00902
2	2	$1/\pi^2 = .10132$		
1	∞	$1/12 = .08333$	$1/72 = .01389$	$1/480 = .00208$
2	∞	$1/2\sqrt{30} = .09129$	$1/6\sqrt{105} = .01626$	$1/48\sqrt{63} = .00262$
∞	∞	$1/8 = .125$	$2/81 = .02469$	$9/2048 = .00439$

Table 11.2 $M(n, p, q)$

p	q	$n = 1$	$n = 2$	$n = 3$
1	1	1	$1/2 = .5$	$1/6 = .16667$
2	1	1	$1/\sqrt{3} = .57735$	$1/2\sqrt{5} = .22361$
∞	1	1	1	$1/2 = .5$
2	2	$2/\pi = .63662$		
1	∞	$1/2 = .5$	$1/6 = .16667$	$1/24 = .04167$
2	∞	$1/\sqrt{3} = .57735$	$1/2\sqrt{5} = .22361$	$1/6\sqrt{7} = .06299$
∞	∞	1	$1/2 = .5$	$1/6 = .16667$

12. REMARKS. The author has compiled table of the constants $K(n, p, q, \alpha)$ for $q = 1$ and ∞ and of the Green's functions for values of n as large as 7.

The reader may have observed that some of the results can be easily generalized to allow weighted L^p norms and/or arbitrary but fixed location of n zeros in some interval $[a, b]$.

Applications of this paper to the area of polynomial interpolation problems are immediate. The author will consider applications to the field of disconjugate differential equations in a forthcoming paper.

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