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# THE $p$-CLASSES OF AN $H^*$-ALGEBRA

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**This paper considers a family of $*$-subalgebras of a semisimple $H^*$-algebra $A$. For $0 < p \leqq \infty$ a nonnegative extended-real value $|a|_p$ is associated with each $a$ in $A$; then the $p$-class $A_p$ is defined to be $\{a \in A: |a|_p < \infty\}$. If $1 \leqq p \leqq \infty$, $A_p$ is then a two-sided $*$-ideal of $A$ (proper only if $p < 2$), and $(A_p, |\cdot|_p)$ is a normed $*$-algebra. $(A_2, |\cdot|_2)$ is $(A, ||\cdot||)$; and for $1 \leqq p < 2$, $(A_p, |\cdot|_p)$ is a Banach $*$-algebra, for which structure theorems are given.**

**1. Introduction.** Let $A$ be a semisimple $H^*$-algebra with inner product and norm denoted by $( , )$ and $||\cdot||$, respectively. The trace class of $A$, that is, the set $\tau(A) = \{xy: x, y \in A\}$, has been studied by Saworotnow and Friedell [8], who show, first of all, that for any nonzero $a \in A$ there exists a positive element $[a] \in A$ such that $[a]^2 = a^*a$, and $a \in \tau(A)$ if and only if $[a] \in \tau(A)$. An algebra norm $\tau$ is then introduced on $\tau(A)$ by defining $\tau(a) = tr[a]$ for each $a \in \tau(A)$, where in turn the trace functional $tr$ is unambiguously defined on $\tau(A)$ by letting $tr\, xy = (x, y^*) = \Sigma(xyp_\omega, p_\omega)$, $\{p_\omega: \omega \in \Omega\}$ being any maximal family of mutually orthogonal nonzero self-adjoint idempotents. With this norm, $\tau(A)$ is actually a Banach algebra [9, Corollary to Theorem 1]. This presentation parallels that of Schatten [10] for $\tau c$, the trace class of $\sigma c$, the Schmidt class of operators on a Hilbert space.

In a somewhat similar sense our central development in §3 brings over into the present context some of the work of McCarthy [6] on the operator algebras $c_p$. We preface this with a basic spectral theorem established in §2; in §4 we study the structure of the Banach $*$-algebras $A_p$, where $1 \leqq p < 2$. Finally, in §5 we relate $A_p$ to the class $c_p$ of operators on a Hilbert space [6; 2, ch. XI. 9] and also to $\mathscr{E}_p$ spaces [3, pp. 70 ff.; 5].

**2. Preliminary spectral theory.** Throughout the remainder of this paper $A$ will continue to denote a semisimple $H^*$-algebra. By a *projection* $p$ in $A$ we shall mean a nonzero self-adjoint idempotent. A projection $p$ is *primitive* if $p$ cannot be expressed as $p = p_1 + p_2$, where $p_1$ and $p_2$ are orthogonal projections. By a *projection base* in $A$ we mean a maximal family of mutually orthogonal projections (not necessarily primitive); note that if $a \in A$ and $\{p_\omega: \omega \in \Omega\}$ is a projection base, then $a = \Sigma ap_\omega = \Sigma p_\omega a$ [1, Theorem 4.1, where primitivity of the projections is not needed to establish this point]. Finally, we shall say that an element $a$ in $A$ is *positive* if $(ax, x) \geqq 0$ for every $x \in A$; $a$ is then necessarily self-adjoint.

LEMMA 2.1. *Let $b$ be a nonzero normal element of $A$. There is a well-defined family $\{p_\omega: \omega \in \Omega\}$ of mutually orthogonal projections in $A$, and a well-defined set $\{\alpha_\omega: \omega \in \Omega\}$ of complex numbers, such that*

(1) $b = \Sigma \alpha_\omega p_\omega$

(2) $bp_\omega = p_\omega b = \alpha_\omega p_\omega$ *for each* $\omega \in \Omega$.

*The nonzero $\alpha_\omega$ are precisely the nonzero elements of the spectrum of $b$.*

*Proof.* Let $A_0$ be the intersection of all maximal commutative $*$-subalgebras of $A$ containing $b$. $A_0$ is a proper $H^*$-algebra in the inner product and involution of $A$. Let $\{p_\omega: \omega \in \Omega\}$ be the collection of projections of $A_0$ which are primitive in $A_0$; then each $p_\omega A_0$ is a minimal ideal of $A_0$, and if $\omega_1 \neq \omega_2$ we have $p_{\omega_1} p_{\omega_2} = 0$ and $(p_{\omega_1}, p_{\omega_2}) = 0$. Also, $A_0 = \Sigma p_\omega A_0$, the orthogonal direct sum of the minimal ideals $p_\omega A_0$, each of which is one-dimensional and consists of scalar multiples of $p_\omega$ [1, Corollary 4.1]. Therefore $b = \Sigma \alpha_\omega p_\omega$, where $\{\alpha_\omega: \omega \in \Omega\}$ is a set of complex numbers. Property (2) is immediate from the orthogonality of the $p_\omega$. We shall show that the nonzero $\alpha_\omega$ are the nonzero elements of $sp(b \mid A_0)$, the spectrum of $b$ relative to $A_0$. Let $\phi$ be any multiplicative linear functional on $A_0$. We have $\phi(p_\omega) = \phi(p_\omega^2) = [\phi(p_\omega)]^2$, and hence the value of $\phi$ at each projection $p_\omega$ must be either 0 or 1. $\phi$ cannot have the value 0 at every $p_\omega$ or else $\phi$ would vanish on $A_0$; nor can we have $\phi(p_{\omega_1}) = 1 = \phi(p_{\omega_2})$ if $\omega_1 \neq \omega_2$, for then $1 = \phi(p_{\omega_1})\phi(p_{\omega_2}) = \phi(p_{\omega_1} p_{\omega_3}) = \phi(0) = 0$. Therefore, each multiplicative linear functional on $A_0$ is of the form $\phi_\nu(p_\omega) = \delta_{\nu\omega}$, where $\nu \in \Omega$. We have, for each $\nu \in \Omega$, $\phi_\nu(b) = \Sigma_\omega \alpha_\omega \phi_\nu(p_\omega) = \alpha_\nu = \hat{b}(\phi_\nu)$, where $\hat{b}$ denotes the Gelfand transform of $b \in A_0$. Since the nonzero $\alpha_\omega$ are therefore the nonzero elements of the range of $\hat{b}$, they are by the Gelfand theory precisely the nonzero elements of $sp(b \mid A_0)$. However, $sp(b \mid A) = sp(b \mid A_0)$, since if $c \in A_0$ has a quasi-inverse $c^0$ in $A$, then, as is well-known, $c^0$ belongs to every maximal commutative $*$-subalgebra of $A$ containing $c$, or equivalently, $c^0 \in A_0$. Finally, it is clear that the element $b$ uniquely determines the algebra $A_0$, along with its set of primitive projections $\{p_\omega: \omega \in \Omega\}$ and the corresponding numbers $\alpha_\omega$, since $\alpha_\omega p_\omega$ is the orthogonal projection of $b$ on the closed ideal $p_\omega A_0$ of $A_0$.

LEMMA 2.2. *Let $b$ be a nonzero normal element of $A$, and let $b = \Sigma \mu_n q_n$, where $\{q_n\}$ is a countable (possibly finite) family of mutually orthogonal projections, and the $\mu_n$ are nonzero complex numbers such that $\mu_m \neq \mu_n$ if $m \neq n$. Let $h$ be any self-adjoint element of $A$ which commutes with $b$. Then for each $n$, $hq_n = q_n h$.*

*Proof.* Extend $\{q_n\}$ to a projection base $\{q_\gamma: \gamma \in \Gamma\}$. For each $\gamma$, if $q_\gamma = q_n$ for some $n$, let $\mu_\gamma = \mu_n$; otherwise, let $\mu_\gamma = 0$. (Note that $bq_\gamma = q_\gamma b = \mu_\gamma q_\gamma$ for each $\gamma \in \Gamma$.) Then for any $q_n$ we have $q_n h =$

$\Sigma_\gamma q_n h q_\gamma$. Also, since $b$ and $h$ commute, $\mu_n q_n h q_\gamma = q_n b h q_\gamma = q_n h b q_\gamma = \mu_\gamma q_n h q_\gamma$. If $q_\gamma \neq q_n$ then $\mu_\gamma \neq \mu_n$ and consequently $q_n h q_\gamma = 0$. Thus $q_n h = q_n h q_n$. Taking adjoints we have $h q_n = q_n h q_n$; therefore $h q_n = q_n h$.

COROLLARY 2.3. *Let $b$, $\{\mu_n\}$, and $\{q_n\}$ be as in the lemma, and let $A_0$ be, as before, the intersection of all maximal commutative $*$-subalgebras of $A$ containing $b$. Then for each $n$, $q_n \in A_0$.*

*Proof.* Let $A_1$ be any maximal commutative $*$-subalgebra of $A$ containing $b$. Since $A_1$ is a $*$-algebra, each $x \in A_1$ is of the form $x = h + ik$, where $h, k \in A_1$, and $h$ and $k$ are self-adjoint. Therefore, each $q_n$ commutes with every element of $A_1$, and by maximality of $A_1$, $q_n \in A_1$. Therefore, finally, $q_n \in A_0$.

LEMMA 2.4. *Let $b$, $\{\mu_n\}$, and $\{q_n\}$ be as in Lemma 2.2. Then each $q_n$ is a finite sum of the projections $p_\omega$ of Lemma 2.1.*

*Proof.* Each $q_n$ belongs to $A_0$, and therefore, as in the proof of Lemma 2.1, $q_n = \Sigma \beta_\omega p_\omega$ for suitable numbers $\beta_\omega$. Also, $q_n = q_n^2 = \Sigma \beta_\omega^2 p_\omega$, and therefore each $\beta_\omega$ is either 0 or 1. Only finitely many can be 1, since $\|q_n\|^2 = \Sigma \beta_\omega^2 \|p_\omega\|^2 \geqq \Sigma \beta_\omega^2$.

Now let $q_n = p_{n_1} + \cdots + p_{n_{k(n)}}$. The orthogonal projection of $b$ on the closed left ideal $Aq_n$ is $bq_n = \mu_n q_n = \mu_n(p_{n_1} + \cdots + p_{n_{k(n)}})$. From Lemma 2.1, since $b = \Sigma \alpha_\omega p_\omega$, this projection of $b$ is also $\alpha_{n_1} p_{n_1} + \cdots + \alpha_{n_{k(n)}} p_{n_{k(n)}}$. Therefore $\alpha_{n_i} = \mu_n$, $i = 1, \cdots k(n)$, and in the representation $b = \Sigma \alpha_\omega p_\omega$ we may replace the sum $\alpha_{n_1} p_{n_1} + \cdots + \alpha_{n_{k(n)}} p_{n_{k(n)}}$ by $\mu_n q_n$. If this is done for each $n$ indexing the countable set $\{q_n\}$, the procedure evidently replaces the representation $b = \Sigma \alpha_\omega p_\omega$ by $b = \Sigma \mu_n q_n$, and therefore makes use of every term $\alpha_\omega p_\omega$ except those for which $\alpha_\omega = 0$. We thus have the following spectral theorem.

THEOREM 2.5. *Let $b$ be a nonzero normal element of $A$. Then $b$ may be represented uniquely (apart from the order of the terms) as a sum*

$$(*) \qquad b = \Sigma \lambda_n e_n \,,$$

*in which*

(1) *$\{\lambda_n\}$ is a countable family of distinct nonzero complex numbers consisting of the nonzero elements of the spectrum of $b$, and*

(2) *$\{e_n\}$ is a countable family of mutually orthogonal projections. We have $be_n = e_n b = \lambda_n e_n$ for each $n$; $b$ is self-adjoint if and only if each $\lambda_n$ is real, and $b$ is positive if and only if each $\lambda_n > 0$.*

DEFINITION 2.6. Let $b$ be a nonzero normal element of $A$. A representation $(*)$ of $b$ having properties (1) and (2) of Theorem 2.5

will be called a *spectral representation* of $b$. If $b$ is a positive element of $A$, we shall refer to *the* spectral representation of $b$, meaning the one in which $\lambda_m < \lambda_n$ if $m > n$. For any nonzero normal element $b$, the set $E_b$ of mutually orthogonal projections in a spectral representation of $b$ will be called the *spectral family* of $b$.

DEFINITION 2.7. Let $b$ be a nonzero normal element of $A$, and let $E_b$ be its spectral family. A projection base $\{e_\omega: \omega \in \Omega\}$ containing every $e_n$ in $E_b$ will be called a *projection base associated with* $b$. (Note that by a simple maximality argument, $E_b$ can always be extended to a projection base associated with $b$.)

**3. The classes $A_p$ and their basic properties.** We begin this section by recalling some basic results from [8]. Corresponding to each $a$ in $A$ there is a unique positive element $[a]$ of $A$ such that $[a]^2 = a^*a$. Moreover, there is, for each nonzero $a$ in $A$, a well-defined partial isometry $W$ on $A$, having initial set $\overline{[a]A}$ and final set $\overline{aA}$, such that $a = W[a]$, $[a] = W^*a$, and $\|W\| = 1$. We shall call $W$ the *partial isometry associated with a*. We define a *left centralizer* on $A$ to be an operator $S$ in $B(A)$ such that $S(xy) = (Sx)y$ for all $x, y \in A$. (This terminology, though widely used, is not universal; the type of operator just defined is called a right centralizer in [8] and ]9], and elsewhere.) Evidently, each left multiplication operator $L_a$, $a \in A$, is a left centralizer on $A$; also, for any nonzero $a$ in $A$, the partial isometry $W$ associated with $a$ is a left centralizer (see [8, p. 97]). We note, finally, for fairly frequent use, that for any $x \in A$, $\|ax\| = \|[a]x\|$, since $\|ax\|^2 = (ax, ax) = (a^*ax, x) = ([a]^2x, x) = ([a]x, [a]x) = \|[a]x\|^2$.

DEFINITION 3.1. Let $a$ be a nonzero element of $A$, and let $[a] = \Sigma\lambda_n e_n$ be the spectral representation of $[a]$. We define

$$|a|_p = (\Sigma\lambda_n^p \|e_n\|^2)^{1/p} \text{ for } 0 < p < \infty\,,$$
$$|a|_\infty = \lambda_1.$$

For $a = 0$, we define $|a|_p = 0$, $0 < p \leqq \infty$.

DEFINITION 3.2. For $0 < p \leqq \infty$, $A_p = \{a \in A: |a|_p < \infty\}$.

REMARK 3.3. For $0 < p \leqq \infty$,

(1) $a \in A_p$ if and only if $[a] \in A_p$, since $[a] = [[a]]$ implies $|a|_p = |[a]|_p$;

(2) if $e$ is a projection, $e \in A_p$ and $|e|_p = \|e\|^{2/p}$.

REMARK 3.4. Let $\{e_\omega: \omega \in \Omega\}$ be a projection base associated with $[a]$. We shall write $[a] = \Sigma\lambda_\omega e_\omega$, always assuming that $\lambda_\omega = \lambda_n$ if $e_\omega \notin E_{[a]}$. Then $|a|_p = (\Sigma\lambda_\omega^p \|e_\omega\|^2)^{1/2}$ for $0 < p < \infty$; and we continue to write $|a|_\infty = \lambda_1$, understanding $\lambda_1$ to be $\sup\{\lambda_\omega: \omega \in \Omega\}$.

REMARK 3.5. Let $\{e_\omega: \omega \in \Omega\}$ be a projection base associated with $[a] \in A$.

(1) $|a|_2^2 = |[a]|_2^2 = \Sigma\lambda_\omega^2 \|e_\omega\|^2 = \Sigma\|\lambda_\omega e_\omega\|^2 = \Sigma\|[a]e_\omega\|^2 = \Sigma\|ae_\omega\|^2 = \|a\|^2$. Hence $|a|_2 = \|a\|$ and $A_2 = A$.

(2) $|a|_1 = |[a]|_1 = \Sigma\lambda_\omega \|e_\omega\|^2 = \Sigma(\lambda_\omega e_\omega, e_\omega) = \Sigma([a]e_\omega, e_\omega) = tr\,[a] = \tau(a)$ [8, Lemma 3]. Hence $|a|_1 = \tau(a)$ and $A_1 = \tau(A)$, the trace class of $A$.

DEFINITION 3.6. Let $b$ be a nonzero positive element of $A$, with spectral representation $b = \Sigma\lambda_n e_n$. For $0 < p < \infty$, $b^p = \Sigma\lambda_n^p e_n$, provided that this sum exists in $A$.

REMARK 3.7. From [8, Lemma 3] we have that $a \in A_p$ if and only if $[a]^p \in A_1 = \tau(A)$. This occurs if and only if $[a]^{p/2}$ exists in $A$; we then have $|a|_p^p = \Sigma\lambda_n^p \|e_n\|^2 = \tau([a]^p) = |[a]^p|_1 = \|[a]^{p/2}\|^2 = \Sigma([a]^p p_\omega, p_\omega)$ for any projection base $\{p_\omega: \omega \in \Omega\}$.

REMARK 3.8. For $0 < p \leqq \infty$, clearly $|a|_p \geqq 0$, and $|a|_p = 0$ if and only if $a = 0$. Also, since $[\alpha a] = |\alpha|\,[a]$ for any complex number $\alpha$, we have $|\alpha a|_p = |\alpha|\,|a|_p$.

LEMMA 3.9. *For any $a \in A$ and $0 < p < \infty$, $|a|_\infty \leqq |a|_p$.*

*Proof.* For $a = 0$ the result is obvious. Otherwise, using the spectral representation of $[a]$, we have $|a|_\infty^p = \lambda_1^p \leqq \Sigma\lambda_n^p \|e_n\|^2 = |a|_p^p$.

LEMMA 3.10. *For any $a \in A$, $\|ax\| \leqq |a|_\infty \|x\|$.*

*Proof.* For $a \neq 0$, let $\{e_\omega: \omega \in \Omega\}$ be a projection base associated with $[a]$. Then $[a]x = \Sigma\lambda_\omega e_\omega x$ and $\|[a]x\|^2 = \Sigma\lambda_\omega^2 \|e_\omega x\|^2 \leqq \lambda_1^2 \Sigma\|e_\omega x\|^2 = \lambda_1^2 \|x\|^2$. Hence $\|ax\| = \|[a]x\| \leqq |a|_\infty \|x\|$.

COROLLARY 3.11. *For any $a \in A$, $|a|_\infty = \|L_a\|$.*

*Proof.* For $a, x \neq 0$, $\|ax\|/\|x\| \leqq |a|_\infty$, by the lemma. But $\|ae_1\|/\|e_1\| = \|[a]e_1\|/\|e_1\| = \lambda_1 = |a|_\infty$.

PROPOSITION 3.12. *For $a \in A$ and $0 < p < q \leqq \infty$, $|a|_q \leqq |a|_p$.*

*Hence* $A_p \subset A_q$, *and if* $2 \leqq p \leqq \infty$ *then* $A_p = A$.

*Proof.* Using the spectral representation of $[a]$, we have $|a|_q^q = \Sigma\lambda_n^q \|e_n\|^2 = \Sigma\lambda_n^{q-p}\lambda_n^p\|e_n\|^2 \leqq \lambda_1^{q-p}\Sigma\lambda_n^p\|e_n\|^2 = |a|_\infty^{q-p}|a|_p^p \leqq |a|_p^q$, by Lemma 3.9.

REMARK 3.13. By 3.7, $a \in A_{2p}$ $(0 < p < \infty)$ if and only if $[a]^p$ exists in $A$. For $1 \leqq p < \infty$, $A_{2p} = A$ and hence $[a]^p$ is defined.

PROPOSITION 3.14. *If $A$ is infinite-dimensional, then for* $0 < p < q \leqq 2$, $A_q$ *is properly larger than* $A_p$.

*Proof.* From the structure theory of $H^*$-algebras [1], we see that if $A$ is infinite-dimensional then $A$ contains a countably infinite set $\{e_n : n \in N\}$ of mutually orthogonal projections. Choose $r$ such that $p < r < q$; then the series $\sum_{n=1}^\infty n^{-1/r}\|e_n\|^{-2/q} e_n$ converges to a positive element of $A$ (since the squares of the norms of its terms have a finite sum). Denoting this element by $a$, we observe that the given series (or one obtained from it by grouping and rearranging terms) is the spectral representation of $a$. Thus $a \in A_q$, since $|a|_q^q = \sum_{n=1}^\infty n^{-q/r} < \infty$; however $a \notin A_p$, since $|a|_p^p = \sum_{n=1}^\infty n^{-p/r}\|e_n\|^{2-(2p/q)} \geqq \sum_{n=1}^\infty n^{-p/r} = \infty$.

Some elements of the following lemma appear in [8, p. 96]. For most of it, however, the author is indebted to M. Kervin.

LEMMA 3.15. *Let $a$ be any nonzero element of $A$, and let* $[a] = \Sigma\lambda_n e_n$ *be the spectral representation of* $[a]$. *For each $n$, let* $f_n = \lambda_n^{-2} a e_n a^*$. *Then* $[a^*] = \Sigma\lambda_n f_n$ *is the spectral representation of* $[a^*]$, *and* $\|f_n\| = \|e_n\|$ *for each $n$.*

*Proof.* Clearly, the $\lambda_n$ are distinct positive numbers and the $f_n$ are self-adjoint. We recall, first of all, that $[a]^2 = \Sigma\lambda_n^2 e_n = a^*a$, and therefore $a^*ae_n = e_n a^*a = \lambda_n^2 e_n$. Thus $f_m f_n = (\lambda_m^{-2} a e_m a^*)(\lambda_n^{-2} a e_n a^*) = \lambda_m^{-2}\lambda_n^{-2} a e_m (a^*ae_n)a^* = \lambda_m^{-2} a e_m e_n a^* = \delta_{mn} f_n$. Therefore, the $f_n$ are mutually orthogonal idempotents. Also, $\lambda_n^2\|f_n\|^2 = \lambda_n^{-2}(ae_na^*, ae_na^*) = (e_na^*, e_na^*) = \lambda_n^2\|e_n\|^2$, and therefore $\|f_n\| = \|e_n\|$ and the $f_n$ are nonzero. Now we wish to show that $[a^*] = \Sigma\lambda_n f_n$. We shall show first that $a = \Sigma a e_n$. Extend the family $E_{[a]}$ to a projection base $\{e_\omega : \omega \in \Omega\}$. Then $a = \Sigma a e_\omega$ and $a^*a = \Sigma a^*ae_\omega$. But if $e_\alpha \notin E_{[a]}$ then $a^*ae_\alpha = 0$, since $a^*a = \Sigma\lambda_n^2 e_n = \Sigma a^*ae_n$. Therefore, for $e_\alpha \notin E_{[a]}$ we have $e_\alpha a^*ae_\alpha = 0 = (ae_\alpha)^*(ae_\alpha)$, and thus $ae_\alpha = 0$ [1, Lemma 2.2]. We conclude that $a = \Sigma a e_n$. Finally, $(\Sigma\lambda_n f_n)^2 = \Sigma\lambda_n^2 f_n = \Sigma a e_n a^* = aa^*$, and therefore $\Sigma\lambda_n f_n$ is the (unique) positive square root of $aa^*$; that is, $\Sigma\lambda_n f_n = [a^*]$.

COROLLARY 3.16. *For any* $a \in A$ *and* $0 < p \leqq \infty$, $|a|_p = |a^*|_p$. *Hence* $a \in A_p$ *if and only if* $a^* \in A_p$.

In order to arrive at the results announced in our opening synopsis, we shall need to establish several crucial inequalities. Lemmas 3.17, 3.18, and 3.22 are adapted from [6, Lemmas 2.1, 2.2].

LEMMA 3.17. *For* $0 < p < \infty$, *let* $b$ *be a positive element of* $A_{2p}$ *(so that* $b^p$ *exists in* $A$*). Then for any nonzero* $x \in A$,

(1) $(b^p x, x) \geqq (bx, x)^p ||x||^{2(1-p)}$ *if* $1 \leqq p < \infty$,

(2) $(b^p x, x) \leqq (bx, x)^p ||x||^{2(1-p)}$ *if* $0 < p \leqq 1$.

*Proof.* (1) Suppose $1 \leqq p < \infty$. Let $\{e_\omega : \omega \in \Omega\}$ be a projection base associated with $b$, where, as usual, we take $\lambda_\omega = \lambda_n$ if $e_\omega = e_n \in E_b$, and $\lambda_\omega = 0$ if $e_\omega \notin E_b$. We have, by Hölder's inequality,

$$\begin{aligned}(bx, x) &= \Sigma\lambda_\omega(e_\omega x, x)\\ &\leqq [\Sigma\lambda_\omega^p(e_\omega x, x)]^{1/p}[\Sigma(e_\omega x, x)]^{1-(1/p)}\\ &= [(\Sigma\lambda_\omega^p e_\omega x, x)]^{1/p}[\Sigma ||e_\omega x||^2]^{(p-1)/p}\\ &= (b^p x, x)^{1/p} ||x||^{2(p-1)/p} .\end{aligned}$$

Hence $(b^p x, x) \geqq (bx, x)^p ||x||^{2(1-p)}$.

(2) Suppose $0 < p \leqq 1$. Replace the element $b$ in (1) by $b^p$ and the exponent $p$ by $1/p$ to obtain the desired inequality.

LEMMA 3.18. *Let* $a \in A$, *and let* $\{q_\omega : \omega \in \Omega\}$ *be a projection base for* $A$. *Then*

(1) $|a|_p^p \leqq \Sigma ||aq_\omega||^p ||q_\omega||^{2-p}$ *if* $1 \leqq p \leqq 2$,

(2) $|a|_p^p \geqq \Sigma ||aq_\omega||^p ||q_\omega||^{2-p}$ *if* $2 \leqq p < \infty$.

*In each case, equality holds if* $\{q_\omega : \omega \in \Omega\}$ *is a projection base associated with* $[a]$.

*Proof.* We note first that $[a]^p$ exists, since $p \geqq 1$.

(1) Suppose $1 \leqq p \leqq 2$. By (2) of Lemma 3.17 we have for each $q_\omega$,

$$\begin{aligned}([a]^p q_\omega, q_\omega) &= (([a]^2)^{p/2} q_\omega, q_\omega)\\ &\leqq ([a]^2 q_\omega, q_\omega)^{p/2} ||q_\omega||^{2-p}\\ &= ||aq_\omega||^p ||q_\omega||^{2-p} .\end{aligned}$$

Summing over $\Omega$ gives, by 3.7,

$$|a|_p^p = \Sigma([a]^p q_\omega, q_\omega) \leqq \Sigma ||aq_\omega||^p ||q_\omega||^{2-p} .$$

If $\{q_\omega\}$ is a projection base associated with $[a]$, then by 3.4 we have

$$\begin{aligned}\Sigma \| aq_\omega \|^p \| q_\omega \|^{2-p} &= \Sigma \| [a] q_\omega \|^p \| q_\omega \|^{2-p} \\ &= \Sigma \lambda_\omega^p \| q_\omega \|^p \| q_\omega \|^{2-p} \\ &= \Sigma \lambda_\omega^p | q_\omega \|^2 \\ &= | a |_p^p .\end{aligned}$$

(2) is proved similarly, using (1) of Lemma 3.17.

PROPOSITION 3.19. *For $1 \leqq p \leqq \infty$, let $a \in A_p$, and let $S$ be a left centralizer on $A$. Then $Sa \in A_p$, and $|Sa|_p \leqq \|S\| \, |a|_p$.*

*Proof.* The result is standard for $p = \infty$. Suppose $1 \leqq p \leqq 2$; let $\{e_\omega : \omega \in \Omega\}$ be a projection base associated with $[a]$. By Lemma 3.18 (1), $|Sa|_p^p \leqq \Sigma \|(Sa)e_\omega\|^p \|e_\omega\|^{2-p} = \Sigma \|S(ae_\omega)\|^p \|e_\omega\|^{2-p} \leqq \|S\|^p \Sigma \|ae_\omega\|^p \|e_\omega\|^{2-p} = \|S\|^p |a|_p^p$. Now suppose $2 \leqq p < \infty$, and this time let $\{e_\omega : \omega \in \Omega\}$ be a projection base associated with $[Sa]$. We have, using (2) of Lemma 3.18, $|Sa|_p^p = \Sigma \|(Sa)e_\omega\|^p \|e_\omega\|^{2-p} = \Sigma \|S(ae_\omega)\|^p \|e_\omega\|^{2-p} \leqq \|S\|^p \Sigma \|ae_\omega\|^p \|e_\omega\|^{2-p} \leqq \|S\|^p |a|_p^p$.

COROLLARY 3.20. *For $1 \leqq p \leqq \infty$, let $a \in A_p$, $x \in A$. Then $xa$ and $ax$ belong to $A_p$, and $|xa|_p \leqq |x|_\infty |a|_p$, $|ax|_p \leqq |a|_p |x|_\infty$.*

*Proof.* By Corollary 3.11 the statements about $xa$ are immediate, since $L_x$ is a left centralizer. We also have, by Corollary 3.16, $|ax|_p = |(ax)^*|_p = |x^* a^*|_p \leqq |x^*|_\infty |a^*|_p = |a|_p |x|_\infty$.

COROLLARY 3.21. *For $1 \leqq p \leqq \infty$, let $a, b \in A_p$. Then $|ab|_p \leqq |a|_p |b|_p$.*

In our next lemma we shall make use of a special operator decomposition given by McCarthy [6, p. 250]. Suppose $T \in B(A)$; then $T = (TT^*)^{1/4} U (T^* T)^{1/4}$, where $U$ is a partial isometry with $\|U\| = 1$.

LEMMA 3.22. *Suppose $1 \leqq p < \infty$. Let $a \in A$, and let $\{q_\omega : \omega \in \Omega\}$ be any projection base for $A$. Then $\Sigma |(aq_\omega, q_\omega)|^p \|q_\omega\|^{2(1-p)} \leqq |a|_p^p$.*

*Proof.* We use the operator decomposition just mentioned: $L_a = (L_a L_a^*)^{1/4} U (L_a^* L_a)^{1/4} = L_{[a^*]}^{1/2} U L_{[a]}^{1/2}$. We have, by two applications of the Schwarz inequality,

$$\begin{aligned}&\Sigma |(aq_\omega, q_\omega)|^p \|q_\omega\|^{2(1-p)} = \Sigma |(UL_{[a]}{}^{1/2} q_\omega, L_{[a^*]}{}^{1/2} q_\omega)|^p \|q_\omega\|^{2(1-p)} \\ &\leqq \Sigma \|L_{[a]}^{1/2} q_\omega\|^p \|L_{[a^*]}^{1/2} q_\omega\|^p \|q_\omega\|^{2(1-p)} \\ &= \Sigma (\|L_{[a]}^{1/2} q_\omega\|^p \|q_\omega\|^{1-p})(\|L_{[a^*]}^{1/2} q_\omega\|^p \|q_\omega\|^{1-p}) \\ &\leqq [\Sigma \|L_{[a]}^{1/2} q_\omega\|^{2p} \|q_\omega\|^{2(1-p)}]^{1/2} [\Sigma \|L_{[a^*]}^{1/2} q_\omega\|^{2p} \|q_\omega\|^{2(1-p)}]^{1/2}\end{aligned}$$

$$= [\Sigma(L_{[a]}^{1/2}q_\omega, L_{[a]}^{1/2}q_\omega)^p ||q_\omega||^{2(1-p)}]^{1/2}[\Sigma(L_{[a^*]}^{1/2}q_\omega, L_{[a^*]}^{1/2}q_\omega)^p ||q_\omega||^{2(1-p)}]^{1/2}$$
$$= [\Sigma([a]q_\omega, q_\omega)^p ||q_\omega||^{2(1-p)}]^{1/2}[\Sigma([a^*]q_\omega, q_\omega)^p ||q_w||^{2(1-p)}]^{1/2}$$
$$\leqq [\Sigma([a]^p q_\omega, q_\omega)]^{1/2}[\Sigma([a^*]^p q_\omega, q_\omega)]^{1/2} \text{ by Lemma 3.17 (1)}$$
$$= |a|_p^{p/2}|a^*|_p^{p/2} \text{ by 3.7}$$
$$= |a|_p^p.$$

PROPOSITION 3.23. *For* $1 \leqq p \leqq \infty$, *let* $a, b \in A_p$. *Then* $|a + b|_p \leqq |a|_p + |b|_p$.

*Proof.* The result is well-known for $p = \infty$. For $1 \leqq p < \infty$, let $\{e_\omega: \omega \in \Omega\}$ be a projection base associated with $[a + b]$, and let $W$ be the partial isometry associated with $a + b$. Then

$$|a + b|_p = [\Sigma([a + b]e_\omega, e_\omega)^p ||e_\omega||^{2(1-p)}]^{1/p}$$
$$= [\Sigma |([a + b]e_\omega, e_\omega)|^p ||e_\omega||^{2(1-p)}]^{1/p}$$
$$= [\Sigma |((W^*(a + b))e_\omega, e_\omega)|^p ||e_\omega||^{2(1-p)}]^{1/p}$$
$$= [\Sigma |((W^*a)e_\omega, e_\omega)||e_\omega||^{2(1-p)/p} + ((W^*b)e_\omega, e_\omega)||e_\omega||^{2(1-p)/p}|^p]^{1/p}$$
$$\leqq [\Sigma |((W^*a)e_\omega, e_\omega)|^p ||e_\omega||^{2(1-p)}]^{1/p} + [\Sigma |((W^*b)e_\omega, e_\omega)|^p ||e_\omega||^{2(1-p)}]^{1/p}$$

by Minkowski's inequality

$$\leqq |W^*a|_p + |W^*b|_p \qquad \text{by Lemma 3.22}$$
$$\leqq ||W^*|| \, |a|_p + ||W^*|| \, |b|_p \qquad \text{by Proposition 3.19}$$
$$= |a|_p + |b|_p.$$

COROLLARY 3.24. *For* $1 \leqq p \leqq \infty$, $A_p$ *is a normed linear space. Hence* $A_p$ *is a two-sided* $*$*-ideal of* $A$ *and* $(A_p, |\cdot|_p)$ *is a normed algebra.*

Now for $1 \leqq p \leqq \infty$ we wish to investigate the relationship between $A_p$ and the dual space of $A_q$, where $(1/p) + (1/q) = 1$. In what follows we shall omit proofs for the cases $p = 1, q = \infty$ and $p = \infty, q = 1$; these are given in [9].

LEMMA 3.25. *Let* $(1/p) + (1/q) = 1$, *where* $1 \leqq p, q \leqq \infty$. *Let* $a \in A_p, b \in A_q$. *Then* $|tr\ ab| = |tr\ ba| \leqq |a|_p |b|_q$.

*Proof.* We shall assume with no loss of generality that $1 < p \leqq 2$ and hence $2 \leqq q < \infty$. Let $\{e_\omega: \omega \in \Omega\}$ be a projection base associated with $[a]$. Then $|tr\ ab| = |tr\ ba| = |\Sigma(bae_\omega, e_\omega)| \leqq \Sigma |(ae_\omega, b^*e_\omega)| \leqq \Sigma ||ae_\omega|| \, ||b^*e_\omega|| = \Sigma ||ae_\omega|| \, ||e_\omega||^{(2-p)/p} ||b^*e_\omega|| \, ||e_\omega||^{(2-q)/q}$, since $((2 - p)/p) + ((2 - q)/q) = 0$. By Hölder's inequality, the last sum does not exceed $[\Sigma ||ae_w||^p ||e_\omega||^{2-p}]^{1/p}[\Sigma ||b^*e_\omega||^q ||e_\omega||^{2-q}]^{1/q}$. But the first sum in brackets is $|a|_p^p$, and the second is less than or equal to $|b^*|_q^q$, by Lemma 3.18

(2). Hence $|tr\, ab| \leqq |a|_p |b|_q$.

For each $a \in A_p$ we now define $\phi_a(x) = tr\, xa$ for all $x \in A_q$. From the linearity of $tr$ on the trace class $\tau(A)$, it is evident that $\phi_a$ is a linear functional on $A_q$; moreover, $\phi_a$ is bounded and $\|\phi_a\| \leqq |a|_p$, by Lemma 3.25. We shall show that the opposite inequality holds as well.

PROPOSITION 3.26. *For $1 \leqq p \leqq \infty$, the mapping $a \rightarrow \phi_a$ is a linear isometry of $A_p$ into $A'_q$, the dual space of $A_q$.*

*Proof.* Again using the linearity of $tr$ on $\tau(A)$ one easily verifies that the mapping is linear. In view of our above remarks, therefore, we need only prove that $|a|_p \leqq \|\phi_a\|$. Let $[a] = \Sigma \lambda_n e_n$ be the spectral representation of $[a]$, and let $w_k = \Sigma_{n=1}^k \lambda_n^{p-2} e_n a \in A_q$. We shall compute $|w_k|_q$. First of all, $w_k^* w_k = (\sum_{m=1}^k \lambda_m^{p-2} a e_m)(\sum_{n=1}^k \lambda_n^{p-2} e_n a^*) = \sum_{m,n=1}^k \lambda_m^{p-2} \lambda_n^{p-2} a e_m e_n a^* = \sum_{n=1}^k \lambda_n^{2(p-2)} a e_n a^* = \sum_{n=1}^k \lambda_n^{2(p-1)} \lambda_n^{-2} a e_n a^* = \sum_{n=1}^k \lambda_n^{2(p-1)} f_n$, where $f_n = \lambda_n^{-2} a e_n a^*$. Since, by Lemma 3.15, the $f_n$ are mutually orthogonal projections with $\|f_n\| = \|e_n\|$, we have $[w_k] = \sum_{n=1}^k \lambda_n^{p-1} f_n$, and $|w_k|_q = [\sum_{n=1}^k \lambda_n^{q(p-1)} \|f_n\|^2]^{1/q} = [\sum_{n=1}^k \lambda_n^p \|e_n\|^2]^{1/q}$. We also have $\sum_{n=1}^k \lambda_n^p \|e_n\|^2 = \sum_{n=1}^k \lambda_n^p tr\, e_n = |tr(\sum_{n=1}^k \lambda_n^p e_n)| = |tr(\sum_{n=1}^k \lambda_n^{p-2} e_n a^* a)| = |tr\, w_k a| = |\phi_a(w_k)| \leqq \|\phi_a\|\, |w_k|_q = \|\phi_a\| [\sum_{n=1}^k \lambda_n^p \|e_n\|^2]^{1/q}$. Thus $[\sum_{n=1}^k \lambda_n^p \|e_n\|^2]^{1/p} \leqq \|\phi_a\|$, and since $\sum_{n=1}^k \lambda_n^p \|e_n\|^2 \leqq \|\phi_a\|^p$ for every $k$, we have $|a|_p^p \leqq \|\phi_a\|^p$.

THEOREM 3.27. *For $1 \leqq p \leqq 2$, the mapping $a \rightarrow \phi_a$ is a linear isometry of $A_p$* onto *$A'_q$.*

*Proof.* Let $\phi$ be any bounded linear functional on $A_q$. Then for all $x \in A_q (= A)$, $|\phi(x)| \leqq \|\phi\| |x|_q \leqq \|\phi\|\, \|x\|$, by Proposition 3.12. Therefore $\phi$ is a bounded linear functional on $A$, and by the Riesz representation theorem there exists $a \in A$ such that $\phi(x) = (x, a^*) = tr\, xa$ for all $x \in A$. We need only show that $a \in A_p$. But if we again consider the spectral representation $[a] = \Sigma \lambda_n e_n$ and define $w_k$ as in the preceding proof, the same computations show that $\sum_{n=1}^k \lambda_n^p \|e_n\|^2 \leqq \|\phi\|^p$ for every $k$, and hence $\Sigma \lambda_n^p \|e_n\|^2 < \infty$ and $a \in A_p$.

COROLLARY 3.28. *For $1 \leqq p \leqq 2$, $(A_p, |\cdot|_p)$ is a Banach $*$-algebra.*

We conclude this section with an example to show that if $2 < p \leqq \infty$ and $A (= A_p)$ is infinite-dimensional, then $(A^p, |\cdot|_p)$ is incomplete. First of all, if $(A_p, |\cdot|_p)$ is complete, then from the inverse mapping theorem and the fact that $|\cdot|_p$ is dominated by $\|\cdot\|$, we can conclude that these two norms are equivalent on $A$. But this is not so if $A$ is infinite-dimensional, for if $\{e_n : n \in N\}$ is a countably infinite set of mutually orthogonal projections in $A$ and we let $s_k = \sum_{n=1}^k n^{-1/2} \|e_n\|^{-2/p} e_n$, then $\{s_k\}$ is a Cauchy sequence in the $|\cdot|_p$-topology but not in the

$\|\cdot\|$-topology.

**4. The structure of the Banach $*$-algebras $A_p$.** In this section we shall confine our attention mainly to the algebras $A_p$, where $1 \leqq p \leqq 2$, although some of our results hold for $p > 2$ as well. Unless otherwise indicated, therefore, we shall assume throughout that $1 \leqq p < 2$. We begin by observing that for these vealues of $p$, $A_p$ is a quite special instance of an $IP$-algebra, as introduced and studied by Yood in [12]; hence the entire theory of that paper is at our disposal. Furthermore, it is readily verified that $(A_p, \|\cdot\|)$ is a (normed) Hilbert algebra; we shall immediately note some properties of this Hilbert algebra. Our first lemma is a simple consequence of the $\|\cdot\|$-continuity of multiplication.

LEMMA 4.1. *If $R$ is any right ideal of $A_p$, then $\bar{R}$, the closure of $R$ in $A$, is a closed right ideal of $A$.*

LEMMA 4.2. *If $R$ is a right ideal of $A_p$ and $P$ is the orthogonal projection operator of $A$ onto $\bar{R}$, the closure of $R$ in $A$, then for any $a \in A_p$, $Pa \in A_p$. In particular, if $R$ is relatively $\|\cdot\|$-closed in $A_p$ then $Pa \in R$.*

*Proof.* This is immediate from Proposition 3.19, inasmuch as $P$ is a left centralizer on $A$.

PROPOSITION 4.3. *If $R$ is a relatively $\|\cdot\|$-closed right ideal of $A_p$, then $A_p = R \oplus R^\perp$, where $R^\perp$ is the orthogonal complement of $R$ in $A_p$.*

*Proof.* Considering the closures in $A$ of these right ideals, we have, for any $a \in A_p$, $a = a_1 + a_2$, where $a_1 \in \bar{R}$, $a_2 \in \overline{R^\perp}$. But by Lemma 4.2, $a_1 \in R$ and $a_2 \in R^\perp$.

REMARK 4.4. For a closed right ideal $R$ in any Hilbert algebra, we have $\mathscr{L}(R) = R^{\perp *}$, where $\mathscr{L}(R)$ is the left annihilator of $R$. This is readily established by the argument used for an $H^*$-algebra [5, Theorem 12]. Combining this fact with Proposition 4.3 we obtain the following.

COROLLARY 4.5. *$(A_p, \|\cdot\|)$ is a dual Hilbert algebra.*

Our next proposition, along with the known structure theory of $H^*$-algebras [1, Theorem 4.2], enables us to obtain a structure theorem for the Hilbert algebras $A_p$.

PROPOSITION 4.6. *Let $I$ be a closed two-sided ideal of $A$ (and therefore an $H^*$-algebra). Then $I \cap A_p = I_p$, the $p$-class of $I$.*

*Proof.* If $a \in I_p$ then $[a]$, as an element of the $H^*$-algebra $I$, has a spectral decomposition $[a] = \Sigma\lambda_n e_n$, where $e_n \in I$ for each $n$, and $\Sigma\lambda_n^p \|e_n\|^2 < \infty$. This is therefore the (unique) spectral decomposition of $[a]$ in $A$, and therefore $a \in I \cap A_p$. Conversely, suppose $a \in I \cap A_p$. Since $a \in I$, $[a]$ has a spectral decomposition $[a] = \Sigma\lambda_n e_n$ in $I$, and again this is its unique spectral decomposition in $A$. Since $a \in A_p$ we have $\Sigma\lambda_n^p \|e_n\|^2 < \infty$, and therefore $a \in I_p$.

REMARK 4.7. Let $J$ be a relatively $\|\cdot\|$-closed two-sided ideal of $A_p$. Then $J$ is a minimal closed ideal of $A_p$ if and only if $\bar{J}$, the closure of $J$ in $A$, is a minimal closed ideal of $A$. If the latter condition holds (so that $\bar{J}$ is a topologically simple $H^*$-algebra), then $J$ is a topologically simple Hilbert algebra.

We use these results and Lemma 4.2 to obtain our structure theorem for $A_p$ as a Hilbert algebra.

THEOREM 4.8 *The Hilbert algebra $(A_p, \|\cdot\|)$ is the direct topological sum of its minimal closed two-sided ideals, which are mutually orthogonal. Each of these is a topologically simple Hilbert algebra and is the $p$-class of a minimal closed two-sided ideal of $A$.*

For the remainder of this section we consider the Banach $*$-algebras $(A_p, |\cdot|_p)$. Our aim in the following development is twofold: (1) to investigate the $|\cdot|_p$-closed right ideals of $A_p$; (2) to obtain a structure theorem for $(A_p, |\cdot|_p)$ analogous to Theorem 4.8.

LEMMA 4.9. *Let $I$ be any $\|\cdot\|$-closed two-sided ideal of $A$. For any $a \in A$, let $a_1$ denote the orthogonal projection of $a$ on $I$. Then*
(1) $(a^*)_1 = (a_1)^*$,
(2) $[a]_1 = [a_1]$.

*Proof.* Let $a = a_1 + a_2$, where $a_2 \in I^\perp$, the orthogonal complement of $I$ in $A$. Then $a^* = a_1^* = (a_1)^* + (a_2)^*$. (1) follows readily from the fact that $I$ and $I^\perp$ are closed under the involution. To establish (2), we first note that $a^*a = a_1^*a_1 + a_2^*a_2$. Then, letting $[a] = [a]_1 + [a]_2$, we have $a^*a = [a]^2 = [a]_1^2 + [a]_2^2$, and hence $[a]_1^2 = a_1^*a^1$, by the uniqueness of the decomposition. If we show that $[a]_1$ is positive, then $[a]_1 = [a_1]$ by the definition of $[a_1]$. For any $x \in A$, let $x = x_1 + x_2$, where $x_1 \in I$, $x_2 \in I^\perp$. Then $([a]_1x, x) = ([a]_1x_1 + [a]_1x_2, x_1 + x_2) = ([a]_1x_1, x_1) = ([a]_1x_1 + [a]_2x_1, x_1) = ([a]x_1, x_1) \geqq 0$.

PROPOSITION 4.10. *Let $\{J_\gamma: \gamma \in \Gamma\}$ be a family of mutually orthogonal relatively $||\cdot||$-closed two-sided ideals of $A_p$. Let $a_\gamma \in J_\gamma$ for each $\gamma$, and let $a = \Sigma a_\gamma$ (in the $||\cdot||$-topology). Then $|a|_p^p = \Sigma |a_\gamma|_p^p$, and hence $a \in A_p$ if and only if $\Sigma |a_\gamma|_p^p < \infty$.*

*Proof.* Clearly, each $a_\gamma$ is the orthogonal projection of $a$ on $J_\gamma$, and hence, by the preceding lemma, $[a] = \Sigma[a_\gamma]$. Now for each $\gamma$, let $[a_\gamma] = \Sigma_n \lambda_{\gamma_n} e_{\gamma_n}$ be the spectral representation of $[a_\gamma]$ in the $H^*$-algebra $\bar{J}_\gamma$,t he $||\cdot||$-closure of $J_\gamma$ in $A$. Then $|a_\gamma|_p^p = \Sigma_n \lambda_{\gamma_n}^p ||e_{\gamma_n}||^2$. Also, $[a] = \Sigma_\gamma \Sigma_n \lambda_{\gamma_n} e_{\gamma_n}$, and since in this sum there cannot be infinitely many equal coefficients, the spectral represention of $[a]$ is obtained by merely grouping the terms of the series having the same coefficient, and then rearranging the terms, if necessary. Hence $|a|_p^p = \Sigma_\gamma \Sigma_n \lambda_{\gamma_n}^p ||e_{\gamma_n}||^2 = \Sigma_\gamma |a_\gamma|_p^p$.

REMARK 4.11. This proposition also holds for $2 \leqq p < \infty$. Also, it is easily seen that $|a|_\infty = \sup_\gamma |a_\gamma|_\infty$.

LEMMA 4.12. *Let $a \in A_p$ and let $\varepsilon$ be any positive number. Then there exist projections $e$ and $f$ in $A_p$ such that $|a - ae|_p < \varepsilon$ and $|a - fa|_p < \varepsilon$.*

*Proof.* Let $A_0$ be the intersection of all maximal commutative $*$-subalgebras of $A$ containing $[a]$. Then, as in Lemma 2.1, we have a representation $[a] = \Sigma \alpha_n p_n$ (each $\alpha_n \neq 0$), which, by grouping and rearranging of terms, yields the spectral representation of $[a]$; hence $|a|_p^p = \Sigma \alpha_n^p ||p_n||^2$. (Note that $[a] \in (A_0)_p$.) We may write $[a] = ([a] - \Sigma_{n=1}^k \alpha_n p_n) + (\sum_{n=1}^k \alpha_n p_n)$, where $\sum_{n=1}^k \alpha_n p_n$ belongs to the relatively $||\cdot||$-closed two-sided ideal $\sum_{n=1}^k (A_0)_p p_n$ of $(A_0)_p$, and $([a] - \sum_{n=1}^k \alpha_n p_n)$ belongs to the orthogonal complement of this ideal in $(A_0)_p$. By Proposition 4.10, $|a|_p^p = |[a]|_p^p = |[a] - \sum_{n=1}^k \alpha_n p_n|_p^p + |\sum_{n=1}^k \alpha_n p_n|_p^p$. But this last term is $\sum_{n=1}^k \alpha_n^p ||p_n||^2$, which has the limit $|a|_p^p$ as $k \to \infty$. We therefore have $\lim_{k\to\infty} |[a] - \sum_{n=1}^k \alpha_n p_n|_p^p = 0 = \lim_{k\to\infty} |[a] - [a]\sum_{n=1}^k p_n|_p^p$.

Hence for sufficiently large $k$ there is a projection $e = \sum_{n=1}^k p_n$ such that $|[a] - [a]e|_p < \varepsilon$. Taking $W$ to be the partial isometry associated with $a$, we have, using Proposition 3.19, $|a - ae|_p = |W[a] - (W[a])e|_p = |W[a] - W([a]e)|_p \leqq ||W||\,|[a] - [a]e|_p < \varepsilon$. There is likewise a projection $f$ such that $|a^* - a^*f|_p < \varepsilon$; hence $|a - fa|_p = |(a - fa)^*|_p < \varepsilon$.

COROLLARY 4.13. *For any $a \in A_p$, $a \in \overline{aA_p} \cap \overline{A_p a}$, where the closure is in the $|\cdot|_p$-topology.*

We remarked at the beginning of this section that $A_p$ is a special case of an $IP$-algebra, and now that we have established the result of Corollary 4.13, we immediately have the following from [12, Theorems 3.5 and 4.9].

COROLLARY 4.14. *$(A_p, |\cdot|_p)$ has dense socle, and is the direct topological sum of its minimal closed two-sided ideals.*

COROLLARY 4.15. *$(A_p, |\cdot|_p)$ is a dual algebra.*

A simple consequence of Corollary 4.15 is the following.

PROPOSITION 4.16. *Let $R$ be a right ideal of $A_p$. $R$ is closed in the $|\cdot|_p$-topology if and only if $R$ is relatively closed in the $||\cdot||$-topology.*

*Proof.* Since $||a|| \leqq |a|_p$ for every $a \in A_p$, by Proposition 3.12, it is clear that every relatively $||\cdot||$-closed subset of $A_p$ is $|\cdot|_p$-closed. Moreover, if the right ideal $R$ is $|\cdot|_p$-closed, then it is an annihilator ideal, by Corollary 4.15, and therefore is relatively $||\cdot||$-closed, by the $||\cdot||$-continuity of multiplication.

REMARK 4.17. This result holds for $2 \leqq p \leqq \infty$. In this case, $R$ is clearly $||\cdot||$-closed if it is $|\cdot|_p$-closed. But if $R$ is a $||\cdot||$-closed right ideal of $A_p(= A)$, we have $R = R^{\perp\perp} = \mathscr{L}(R^{\perp})^*$, by 4.4. By the $|\cdot|_p$-continuity of multiplication, $\mathscr{L}(R^{\perp})$ is $|\cdot|_p$-closed.

We combine Proposition 4.16 with Proposition 4.3 to obtain the following.

COROLLARY 4.18. *$(A_p, |\cdot|_p)$ is a right complemented algebra (in the sense of [11]).*

More can be said about the manner in which $A_p$ is the direct topological sum of its minimal closed two-sided ideals. In order to do so, we obtain a converse of Proposition 4.10, which leads to our final structure theorem.

PROPOSITION 4.19. *Let $\{J_\gamma : \gamma \in \Gamma\}$ be a family of mutually orthogonal closed two-sided ideals of $A_p$. Let $a_\gamma \in J_\gamma$ for each $\gamma$, and suppose that $\Sigma |a_\gamma|_p^p < \infty$. Then there exists $a \in A_p$ such that $a = \Sigma a_\gamma$, where the sum may be taken in the $|\cdot|_p$-topology or the $||\cdot||$-topology.*

*Proof.* Considering only the nonzero $a_\gamma$, which we denote as $a_n$, let $s_k = \sum_{n=1}^k a_n$. Then, by Proposition 4.10, for $k > m$ we have $|s_k - s_m|_p^p = |\sum_{n=m+1}^k a_n|_p^p = \sum_{n=m+1}^k |a_n|_p^p \rightarrow 0$ as $k, m \rightarrow \infty$. The Cauchy sequence $\{s_k\}$ thus has a limit $a$ in the Banach algebra $(A_p, |\cdot|_p)$, and $a =$

$\Sigma a_n = \Sigma a_\gamma$ in the $|\cdot|_p$-topology. (A standard argument shows that the limit is independent of the order of summation.) By Proposition 3.12, the sum is the same in the $||\cdot||$-topology.

THEOREM 4.20. *The Banach $*$-algebra $(A_p, |\cdot|_p)$ is the $p$-direct sum of its minimal closed two-sided ideals $J_\lambda$. The $J_\lambda$ are mutually orthgonal and each is a topologically simple Banach $*$-algebra. $A_p$ is the "$p$-direct sum" in that it consists precisely of all sums $\Sigma a_\lambda$, $a_\lambda \in J_\lambda$, such that $\Sigma |a_\lambda|_p^p < \infty$, where $a = \Sigma a_\lambda$ may be understood as a limit in either the $|\cdot|_p$-topology or the $||\cdot||$-topology, and $|a|_p = (\Sigma |a_\lambda|_p^p)^{1/p}$.*

5. **Relationship to other systems.** If $A$ is a topologically simple $H^*$-algebra, then there is a $*$-isomorphism $x \to X$ of $A$ onto the Schmidt class $\sigma c$ of operators on the Hilbert space $H = l_2(\Gamma)$, where $\Gamma$ is the index set of a maximal family $\{q_\gamma\}$ of mutually orthogonal primitive projections in $A$ [1, Theorem 4.3]. Under this isomorphism, $||x|| = \alpha\sigma(X)$, where $\sigma(X)$ denotes the Schmidt norm of the operator $X$ and $\alpha \geqq 1$ is the norm of each of the projections $q_\gamma$ (actually, *all* primitive projections in $A$ have the same norm [7, Corollary 5.9]). Now if $x$ is any nonzero element of $A$ and $[x] = \Sigma\lambda_n e_n$ is the spectral representation of $[x]$, then we may replace the nonprimitive projections among the $e_n$ by finite sums of primitive projections to obtain a new representation

$$(*) \qquad [x] = \Sigma \mu_n p_n ,$$

where $\mu_m \leqq \mu_k$ if $m > k$. For a given coefficient $\mu_n$ in $(*)$, we shall call the number of primitive projections having $\mu_n$ as coefficient the multiplicity of $\mu_n$ in this representation, denoted by $m(\mu_n)$. We have, for $0 < p < \infty$, $|x|_p = (\Sigma \mu_n^p ||p_n||^2)^{1/p} = \alpha^{2/p}(\Sigma \mu_n^p)^{1/p}$. Also, $|x|_\infty = \mu_1$. Since the $\mu_n$ are the nonzero elements of the spectrum of $[x]$, and since the corresponding operator $[X]$ is compact, these numbers are the nonzero characteristic values of $[X]$. Now for each $\mu_n$, let $M(\mu_n)$ denote the multiplicity of $\mu_n$ as a characteristic value of the operator $[X]$; that is, the dimension of the subspace of $H$ spanned by the characteristic vectors of $[X]$ corresponding to $\mu_n$. We shall show that $m(\mu_n) = M(\mu_n)$.

LEMMA 5.1. *Let $p$ be a primitive projection in the topologically simple $H^*$-algebra $A$. Then the corresponding projection $P$ in $\sigma c$ is one-dimensional on $H$.*

*Proof.* If $P$ is not one-dimensional, let $P = Q + R$, where $Q$ and $R$ are projections onto orthogonal nonzero subspaces of $P(H)$. Letting

$q$ and $r$ be the corresponding elements of $A$, we see that $q$ and $r$ are orthogonal projections in $A$ with $p = q + r$. Thus $p$ is not primitive.

LEMMA 5.2. *For any* $\mu_n$ *in* $(*)$, $m(\mu_n) = M(\mu_n)$.

*Proof.* Let $p_{n_1}, \cdots, p_{n_k}$ be the projections in $(*)$ having coefficient $\mu_n$. Then $m(\mu_n) = k$. Also, letting $P_{n_1}, \cdots, P_{n_k}$ be the corresponding projections in $\sigma c$, we have, using the preceding lemma, $\dim (P_{n_1} + \cdots + P_{n_k})(H) = k$; therefore $M(\mu_n) \geqq k$. Suppose $M(\mu_n) > k$, and let $h$ be a nonzero element of $H$ such that $[X]h = \mu_n h$ and $h$ is orthogonal to $(P_{n_1} + \cdots + P_{n_k})(H)$. Let $Q$ be the orthogonal projection onto the one-dimensional subspace of $H$ spanned by $\{h\}$. $Q \in \sigma c$, and $[X]Q = \mu_n Q$. Now let $q$ be the corresponding projection in $A$; then $[x]q = \mu_n q$. For $i = 1, \cdots, k$, $p_{n_i} q = 0$ since $P_{n_i} Q = 0$; and for $m \neq n_1, \cdots, n_k$, $p_m[x]q = \mu_m p_m q = \mu_n p_m q$, so that $p_m q = 0$, since $\mu_m \neq \mu_n$. Thus $q$ is orthogonal to all the $p_n$, which means that $[x]q = 0$, a contradiction. We conclude that $m(\mu_n) = k = M(\mu_n)$.

Now we observe that the coefficients $\mu_n$ in $(*)$ are the nonzero characteristic values of $[X]$ enumerated according to their multiplicity $M(\mu_n)$. Thus, for $0 < p < \infty$, $|X|_p = (\Sigma \mu_n^p)^{1/p}$ and also $|X|_\infty = \mu_1$, where $|\cdot|_p$ here denotes the $c_p$ norm of $X$ as an operator on $H$. Finally, we have $|x|_p = \alpha^{2/p}|X|_p$ for $0 < p \leqq \infty$, and therefore the mapping $x \rightarrow X$ is a bicontinuous isomorphism of $A_p$ into $c_p(H)$. Since $c_2 = \sigma c$ [2, p. 1093] and $c_p \subset c_2$ for $0 < p \leqq 2$, the isomorphism is onto $c_p$ for these values of $p$.

Now let $A$ be any proper $H^*$-algebra, and let $\{I_\lambda : \lambda \in \Lambda\}$ be the family of minimal closed two-sided ideals of $A$. Each $I_\lambda$ is a topologically simple $H^*$-algebra and $A$ is the Hilbert space direct sum $\Sigma I_\lambda$. For each $\lambda \in \Lambda$, let $\Gamma_\lambda$ be the index set of a maximal family $\{e_{\lambda_\gamma} : \gamma \in \Gamma_\lambda\}$ of mutually orthogonal primitive projections in $I_\lambda$, and let $\alpha_\lambda$ be the norm $||e_{\lambda_\gamma}||$ of each of the $e_{\lambda_\gamma}$ in $I_\lambda$. For each $x_\lambda \in I_\lambda$ let $X_\lambda$ be the corresponding Schmidt class operator on $H_\lambda = l_2(\Gamma_\lambda)$. Then, as we have noted above, $|x_\lambda|_p = \alpha_\lambda^{2/p}|X_\lambda|_p$, $0 < p \leqq \infty$, where $|X_\lambda|_p$ is the $c_p$ norm of the operator $X_\lambda$. Then, by Proposition 4.10, we have $|x|_p = (\Sigma |x_\lambda|_p^p)^{1/p} = (\Sigma \alpha_\lambda^2 |X_\lambda|_p^p)^{1/p}$ for $0 < p < \infty$, and, by 4.11, $|x|_\infty = \sup_\lambda |x_\lambda| = \sup_\lambda |X_\lambda|$. Thus, again, as in Proposition 4.10, $x \in A_p$ if and only if each $x_\lambda \in (I_\lambda)_p = I_\lambda \cap A_p$ and $\Sigma |x_\lambda|_p^p < \infty$. These conditions in turn imply that each corresponding operator $X_\lambda \in c_p(H_\lambda)$ and $\Sigma \alpha_\lambda^2 |X_\lambda|_p^p < \infty$. For $1 \leqq p \leqq 2$, it has been established that the last-mentioned implication is an equivalence; for these values of $p$, therefore, in the special situation in which each $H_\lambda$ is finite-dimensional, we have shown that the algebras $A_p$ are instances of the $\mathscr{C}_p$ spaces studied in [3, pp. 70 ff.] and [5].

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