PACIFIC JOURNAL OF MATHEMATICS
Vol. 42, No. 3, 1972

LOCALLY HOLOMORPHIC SETS AND THE LEVI FORM

L. R. HUNT

Suppose we have a real k-dimensional $\mathscr{C}^2$ manifold M embedded in C^n. If M has a nondegenerate complex tangent bundle of positive rank at some point $p \in M$, then the vanishing or nonvanishing of the Levi form on M near p determines whether or not M is locally holomorphic at p. We show that if M is locally holomorphic at p, then the Levi form vanishes near p, the converse being a known result. In addition we prove a $C-R$ extendibility theorem for a certain case when M is $\mathscr{C}^\infty$ and has a nonzero Levi form at $p \in M$.

1. Introduction. In the study of holomorphic extendibility and holomorphic convexity we often want to know whether a set is a holomorphic set or not. For instance a totally real submanifold of a Stein manifold is a holomorphic set (see [5]). If a real k-dimensional $\mathscr{C}^2$ manifold M is embedded in C^n in such a way that M has a nondegenerate complex tangent bundle at some point p, the property of being locally holomorphic at p depends on the Levi form on M near p. It has been shown that if the Levi form vanishes near p then M is locally holomorphic at p. The converse has been proved only in the generic case when $k > n$ and M is $\mathscr{C}^\infty$ ([1] or [6]). It is the purpose of this paper to prove the converse in all cases. For a particular case (which we call pseudo-hypersurface) we combine a lemma of Nirenberg [4] with the compactly supported solutions to the $C-R$ equations to prove a $C-R$ extension theorem.

In §2 we define exceptional points, the Levi form, and the concept of local holomorphicity. Section 3 contains a discussion of the relation of the Levi form to the local equations of the embedded manifold. In §4 we show that to prove theorems about local holomorphicity, we need only consider open sets of C^n with $\mathscr{C}^\infty$ boundaries. We show that a locally holomorphic set has a vanishing Levi form, if the Levi form can be defined. In §5 we define the concept of a pseudo-hypersurface and prove that if the Levi form does not vanish on a pseudo-hypersurface, all $C-R$ functions are extendible to an open set in C^n.

2. Definitions. Let M be a real k-dimensional $\mathscr{C}^2$ manifold embedded in C^n, $k, n \geqq 2$. Let $T_x(M)$ be the real tangent space to M at x and $H_x(M) = T_x(M) \cap iT_x(M)$. Then $H_x(M)$ is the maximal complex subspace of C^n contained in $T_x(M)$, called the vector space

of *holomorphic tangent vectors* to M at x. It is well known that

$$\max(k-n,0) \leqq \dim_C H_x(M) \leqq \left[\frac{k}{2}\right].$$

The real tangent bundle of M is denoted by $T(M)$.

If f is the embedding of M into C^n, then $J(f)$ denotes the complex Jacobian of f (as a C-linear map from $T(M) \otimes C \to C^n$). If $q = \min(n, k)$, a point p in M is said to be an *exceptional point of order* l, $0 \leqq l \leqq [k/2] - \max(k-n, 0)$ if the complex rank of $J(F)|_p$ is equal to $q - l$.

A point p in M is *generic* if p is an exceptional point of order 0. The manifold M is *locally generic* at p if every point in some open neighborhood of p is generic, and is *locally* $C - R$ at p if every point in some open neighborhood of p is an exceptional point of the same order.

Suppose M is locally $C - R$ at $p \in M$ and $H_p(M)$ is nonzero. Then we define the Levi form at any x near p

$$L_x(M)\colon H_x(M) \longrightarrow \frac{T_x(M) \otimes C}{H_x(M) \otimes C}$$

by $L_x(M)(t) = \pi_x\{[Y, \bar{Y}]\}_x$, where Y is a local section of the fiber bundle $H(M)$ (with fiber $H_x(M)$) such that $Y_x = t$, $[Y, \bar{Y}]_x$ is the Lie bracket evaluated at x, and

$$\pi_x\colon T_x(M) \otimes C \longrightarrow \frac{T_x(M) \otimes C}{H_x(M) \otimes C}$$

is the projection.

A compact set K in a complex manifold X is a *holomorphic set* (also called a S_δ set) if there is a sequence of open Stein manifolds $X_i \subset X$ such that $X_{i+1} \subset X_i$ and

$$K = \bigcap_{i=1}^{\infty} X_i .$$

A set K is *locally holomorphic at* $p \in K$ if there exists a compact neighborhood N of p such that $N \cap K$ is holomorphic.

3. Local equations and the Levi form.

Again let M be a real k-dimensional $\mathscr{C}^2$ manifold embedded in C^n, $k, n \geqq 2$. Suppose M is locally $C - R$ at p and p is an exceptional point of order l. If $k > n$ the local equations of M in a neighborhood of p are (after a suitable coordinate change)

$$
(1)\quad
\begin{aligned}
z_1 &= x_1 + ih_1(x_1, \cdots, x_{2(n-l)-k}, w_1, \cdots, w_{k-n+l})\\
&\vdots\\
z_{2(n-l)-k} &= x_{2(n-l)-k} + ih_{2(n-l)-k}(x_1, \cdots, x_{2(n-l)-k}, w_1, \cdots, w_{k-n+l})\\
z_{2(n-l)-k+1} &= u_1 + iv_1 = w_1\\
&\vdots\\
z_{n-l} &= u_{k-n+l} + iv_{k-n+l} = w_{k-n+l}\\
z_{n-l+1} &= g_1(x_1, \cdots, x_{2(n-l)-k}, w_1, \cdots, w_{k-n+l})\\
&\vdots\\
z_n &= g_l(x_1, \cdots, x_{2(n-l)-k}, w_1, \cdots, w_{k-n+l}) ,
\end{aligned}
$$

where $x_1, \cdots, x_{2(n-l)-k}, u_1, v_1, \cdots, u_{k-n+l}, v_{k-n+l}$ are local coordinates for M in a neighborhood of p vanishing at p, and $z_1, \cdots, z_n$ are coordinates for C^n vanishing at p. The real-valued functions $h_1, \cdots h_{2(n-l)-k}$ as well as the complex-valued functions $g_1, \cdots, g_l$ vanish to order 2 at p. Because M is locally $C-R$ at p, the functions $g_1, \cdots, g_l$ must be complex-analytic functions of $w_1, \cdots, w_{k-n+l}$ (see [3]).

Letting $g_j = g_j' + ig_j'', j = 1, \cdots l$, we find in [7] that the Levi form vanishes at p if and only if the complex Hessians at p of each of the functions $h_1, \cdots, h_{2(n-l)-k}, g_1', g_1'', \cdots, g_l', g_l''$ with respect to the variables $w_1, \cdots w_{k-n+l}$ all have zero eigenvalues.

Fix $x_1, \cdots, x_{2(n-l)-k}$ and expand each g_j in a Taylor series in $w_1, \cdots, w_{k-n+l'}$

$$g_j = \sum_\alpha a_{j,\alpha} w^\alpha ,$$

where $w = (w_1, \cdots, w_{k-n+l})$ and $\alpha = (\alpha_1, \cdots, \alpha_{k-n+l})$. Replacing z_{n-l+j} by $z_{n-l+j} - \sum_\alpha a_{j,\alpha} w^\alpha$, we have that $z_{n-l+1} = 0, \cdots, z_n = 0$ in our new local equations. Thus the Levi form vanishes at p if and only if the complex Hessians at p of each of the functions $h_1, \cdots, h_{2(n-l)-k}$ are all zero matrices.

If $k \leqq n$ the local equations of M in a neighborhood of p are (after a suitable coordinate change)

$$
(2)\quad
\begin{aligned}
z_1 &= x_1 + ih_1(x_1, \cdots, x_{k-2l}, w_1, \cdots, w_l)\\
&\vdots\\
z_{k-2l} &= x_{k-2l} + ih_{k-2l}(x_1, \cdots, x_{k-2l}, w_1, \cdots, w_l)\\
z_{k-2l+1} &= u_1 + iv_1 = w_1\\
&\vdots\\
z_{k-l} &= u_l + iv_l = w_l\\
z_{k-l+1} &= g_1(x_1, \cdots, x_{k-2l}, w_1, \cdots, w_l)\\
&\vdots\\
z_n &= g_{n-k+l}(x_1, \cdots, x_{k-2l}, w_1, \cdots, w_l) ,
\end{aligned}
$$

where $x_1, \cdots, x_{k-2l}, u_1, v_1, \cdots, u_l, v_l$ are the local coordinates for M in a neighborhood of p vanishing at p, and $z_1, \cdots, z_n$ are coordinates for $\boldsymbol{C}^n$ vanishing at p. The real-valued functions $h_1, \cdots, h_{k-2l}$ as well as the complex-valued functions $g_1, \cdots, g_{n-k+l}$ vanish to order 2 at p. Since we are assuming M is locally $C - R$ at p, the functions $g_1, \cdots, g_{n-k+l}$ are complex-analytic functions of $w_1, \cdots, w_l$. An argument similar to the previous one gives us that the Levi form on M vanishes at p if and only if the complex Hessians at p of each of the functions $h_1, \cdots, h_{k-2l}$ with respect to the variables $w_1, \cdots, w_l$ are all zero matrices.

4. Certain open sets. We wish to show that to discuss local holomorphicity we need only consider open sets with $\mathscr{C}^\infty$ boundaries.

Suppose our manifold M is locally $C - R$ at $p \in M$ and Ω is a pseudoconvex open set of $\boldsymbol{C}^n$ containing a compact neighborhood N of p in M. Let K be a compact subset of Ω also containing N, and let W be an open neighborhood of $\hat{K}_{\Omega,P}$, the plurisubharmonic hull of K in Ω. Since $\hat{K}_{\Omega,P}$ is compact in Ω, we may take $W \subset\subset \Omega$, where $\subset\subset$ denotes relative compactness. We need the following theorem [2].

THEOREM 1. *Let Ω, K, and W be as stated. Then there exists a function $u \in \mathscr{C}^\infty(\Omega)$ such that*

(a) *u is strictly plurisubharmonic for every $z \in \Omega$.*

(b) *$u < 0$ in K but $u > 0$ in $\Omega \cap \{$complement of $W\}$.*

(c) *$\{z, z \in \Omega, u(z) < c\} \subset\subset \Omega$ for every $C \in \boldsymbol{R}$.*

Since $\bar{W}$ is a compact subset of Ω and u is a continuous real-valued function on Ω, there exists a real number $c' \geqq 0$ such that $u \leqq c'$ in $\bar{W}$. Letting $\Omega' = \{z; z \in \Omega; u(z) < c'\}$, we have that $\Omega' \subset\subset \Omega$ and $\partial\Omega' = \{z; z \in \Omega; u(z) = c'\}$. Thus Ω' is an open manifold in $\boldsymbol{C}^n$ such that

(3)
(i) Ω' contains N,
(ii) Ω' has a $\mathscr{C}^\infty$ boundary,
(iii) $\bar{\Omega}'$ is a compact subset of Ω, and
(iv) the function defining $\partial\Omega'$ has a positive definite Hessian at every point of $\partial\Omega'$.

We shall restrict ourselves to the case $k > n$ since discussions for $k \leqq n$ follow in an analogous manner. Thus M has local equations (1) near p. Let us assume $L_p(M) \neq 0$, and arbitrarily that $\partial^2 h_1/\partial w_1 \partial \bar{w}_1(p) = -1$ with the Hessian of h_1 in diagonal form. Coordinate changes on $\boldsymbol{C}^n$ (see [2], p. 51) allow us to write

$$y_1 - h_1(z) = y_1 - \sum_{j=1}^{n-l} \frac{\partial^2 h_1}{\partial z_j \partial \bar{z}_j}(p) z_i \bar{z}_j + 0(|z|^3) .$$

Using continuity arguments and properties of superharmonic functions in the variable w_1 we find there exists a closed neighborhood A of p in M such that h_1, as a function of $w_1, \bar{w}_1$ with the other variables fixed, cannot assume a relative minimum at any interior point of A. Thus for fixed $x_1, \cdots, x_{2(n-l)-k}, w_2, \cdots, w_{k-n+l}$, the minimum of h_1 as a function of $w_1, \bar{w}_1$ is attained on the boundary of A. We may assume this minimum is $\leqq -\varepsilon, \varepsilon > 0$, for every $x_1, \cdots, x_{2(n-l)-k}, w_2, \cdots, w_{k-n+l}$ with $(x_1, \cdots, x_{2(n-l)-k}, w_1, \cdots, w_{k-n+l}) \in A$.

We define a new set S with local equations at p given by

$$\begin{aligned}
z_1 &= x_1 + iy_1 \\
z_2 &= x_2 \\
&\vdots \\
z_{2(n-l)-k} &= x_{2(n-l)-k} \\
z_{2(n-l)-k+1} &= u_1 + iv_1 = w_1 \\
&\vdots \\
z_{n-l} &= u_{k-n+l} + iv_{k-n+l} = w_{k-n+l} \\
z_{n-l+1} &= 0 \\
&\vdots \\
z_n &= 0
\end{aligned} \tag{4}$$

where $-\varepsilon \leqq y_1 \leqq h_1(x_1, \cdots, x_{2(n-l)-k}, w_1, \cdots, w_{k-n+l})$, and where $(x_1, \cdots, x_{2(n-l)-k}, w_1, \cdots, w_{k-n+l})$ are in A.

Denote the projection of C^n onto the set

$$\{z \in C^n; y_2 = 0, \cdots, y_{2(n-l)-k} = 0, z_{n-l+1} = 0, \cdots, z_n = 0\} \text{ by } \pi.$$

Under this projection A goes onto the subset of S with $y_1 = h_1(x_1, \cdots, x_{2(n-l)-k}, w_1, \cdots, w_{k-n+l})$.

Let $\{\Omega_i\}_{i=1}^{\infty}$ be domains of holomorphy in C^n such that M is locally holomorphic at $p = 0$ with respect to these domains, and each contains A. Denote by $\{\Omega'_i\}_{i=1}^{\infty}$ the sets described in (3), and let u'_i be the C^∞ defining function for $\partial\Omega'_i$. As i gets large $\partial\Omega'_i$ tends to M, so there exists an integer i and a nonempty open set U of points in S such that $\pi(\partial\Omega'_i)$ does not contain U. Since $\partial\Omega'_i$ cannot intersect M there exists a point $q \in \Omega'_i$ such that $\pi(q) \in S - \pi(A)$, $\partial u'_i/\partial y'_1(q) \neq 0$, and y_1 as a function of $w_1, \bar{w}_1$ (with $x_1, z_2, \cdots, z_{2(n-l)+k}, w_2, \cdots, w_{k-n+l}, z_{n-l+1}, \cdots, z_n$ held fixed) has a relative maximum at q. However y_1 is a (strictly) subharmonic function of $w_1, \bar{w}_1$ since u'_i is and we have our contradiction. Geometrically we have, for fixed $x_1, z_2, \cdots, z_{2(n-l)+k}, w_2, \cdots, w_{k-n+l}, z_{n-l+1}, \cdots, z_n$, a real 2-dimensional manifold (with coordinates $w_1, \bar{w}_1$) which lies in or below the tangent plane to $\partial\Omega'_i$ at q. This implies the complex Hessian of u_i' at q has a nonpositive eigenvalue, also a contradiction.

We have the following result, the sufficiency being found in [6].

THEOREM 2. *Let M be a real k-dimensional $\mathscr{C}^2$ submanifold of C^n which is locally $C - R$ at $p \in M$ with complex dimension $H_p(M) = m > 0$. Then M is locally holomorphic at p if and only if the Levi form vanishes identically near p.*

5. Odd codimensional submanifolds. Suppose M is a real $(2n - 2m + 1)$-dimensional $\mathscr{C}^\infty$ submanifold of C^n, where m is a positive integer and $(2n - 2m + 1) \geqq 2$.

DEFINITION 1. The manifold M is a *local pseudo-hypersurface at* $p \in M$ if M is locally $C - R$ at $p \in M$ and p is an exceptional point of the highest possible order. The manifold M is a *pseudo-hypersurface* if it is a local pseudo-hypersurface at every one of its points.

The reasons for the name pseudo-hypersurface are the following:

(i) In the local equations for M at a point p there is only the function h_1 in (1), and if M is indeed real-analytic, coordinate changes can be made so that M appears locally as a hypersurface in some possibly lower dimensional complex Euclidean space,

(ii) if the Levi form on M does not vanish at p, we shall show that all $C - R$ functions on M extend to holomorphic functions on some open set in C^n, similar to the hypersurface case.

Let M be a real k-dimensional $\mathscr{C}^\infty$ manifold embedded in C^n, k, $n \geqq 2$. Suppose $f \in \mathscr{C}^\infty(M)$. We say that f is a *$C - R$ function at* $p \in M$ if $\bar{X}f(y) = 0$, for y near p and X any section of $H(M)$. If M is locally $C - R$ at p it suffices to verify the equality just for X in a local basis for $H(M)$ at p. We note that our manifold need not be globally $C - R$. Thus we may have points which are not locally $C - R$, but obviously the set of such points is nowhere dense in M. The function f is a *$C - R$ function on M* if f is a $C - R$ function at each point of M. The $C - R$ functions are denoted by $CR(M)$.

We say that M is *$C - R$ extendible* to a connected set $K = M \cup K'$ where $K' \neq \phi$, if for every $f \in CR(M)$ there exists a continuous $F: M \cup K' \to C$ so that $F|M = f$ and $F|K' \in \mathscr{O}(K')$. (By $\mathscr{O}(K')$ we denote those functions holomorphic in some open neighborhood of K').

If M is a local pseudo-hypersurface at $p \in M$, the local equations of M in an open neighbhorhood of our point p are

$$
\begin{aligned}
z_1 &= x_1 + ih_1(x_1, w_1, \cdots, w_{n-m}) \\
z_2 &= u_1 + iv_1 = w_1 \\
&\vdots \\
z_{n-m+1} &= u_{n-m} + iv_{n-m} = w_{n-m} \\
z_{n-m+z} &= g_1(x_1, w_1, \cdots, w_{n-m}) \\
&\vdots \\
z_n &= g_{m-1}(x_1, w_1, \cdots, w_{n-m}) .
\end{aligned} \tag{5}
$$

We have by Lemma 2.6.2 of [4] that every function which is $C - R$ in an open neighborhood of p in M is also $C - R$ in an open neighborhood of p in the hypersurface with local equations at p

$$
\begin{aligned}
z_1 &= x_1 + ih_1(x_1, w_1, \cdots, w_{n-m}) \\
z_2 &= u_1 + iv_1 = w_1 \\
&\vdots \\
z_n &= u_{n-1} + iv_{n-1} = w_{n-1} .
\end{aligned} \tag{6}
$$

However this is a hypersurface which is $C - R$ extendible over an open subset of C^n if the Levi form is nonvanishing at p (see Theorem 2.6.13 in [2]). We have shown the following result.

THEOREM 3. *Suppose M is a local pseudo-hypersurface at $p \in M$ and the Levi form on M is nonvanishing at p. Then M is $C - R$ extendible over an open subset of C^n.*

REMARK 1. Nirenberg [4] proves a much more general theorem (in the sense of dimensions and genericity) than the one above, but requires in the pseudo-hypersurface case that the complex Hessian of h_1 with respect to $w_1, \cdots, w_{n-m}$ have either two eigenvalues of opposite signs or all nonzero eigenvalues of the same sign. It is hoped that $C - R$ extendibility theorems depend only on the vanishing or nonvanishing of the Levi form.

REMARK 2. Since a compact pseudo-hypersurface has a peak point where the Levi form does not vanish (see [3]), it is $C - R$ extendible over an open set in C^n.

REMARK 3. This is the first example of a lower dimensional extendibility theorem without using the work of Bishop.

REFERENCES

1. S. J. Greenfield, *Cauchy-Riemann equations in several variables*, Annali Scuola Norm. Sup. di Pisa, **22** (1968), 275-314.

2. L. Hormander, *An introduction to complex analysis in several variables,* Van Nostrand, Princeton, N. J., 1966.
3. L. R. Hunt, *Real-analytic submanifolds of complex manifolds,* Proc. Amer, Math. Soc., **29** (1971), 69-74.
4. R. Nirenberg, *On the H. Lewy extension phenomenon,* Trans. Amer. Math. Soc., (1972), 337-356.
5. R. Nirenberg and R. O. Wells, Jr., *Approximation theorems on differentiable submanifolds of a complex manifold,* Trans. Amer. Math. Soc., **142** (1969), 15-35.
6. R. O. Wells, Jr., *Holomorphic hulls and holomorphic convexity of differentiable submanifolds,* Trans. Amer. Math. Soc., **132** (1968), 245-262.
7. ———, *Concerning the envelope of holomorphy of a compact differentiable submanifold of a complex manifold,* Annali Scuola Norm Sup. di Pisa, **23** (1969), 347-361.

Received May 7, 1971 and in revised form October 29, 1971.

TEXAS TECH UNIVERSITY