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A PROBABILISTIC METHOD FOR THE RATE OF CONVERGENCE TO THE DIRICHLET PROBLEM

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The expectation $E^P(\Phi)$ approximates the solution $u(z) = E^W(\Phi)$ of the Dirichlet problem for a plane domain D with boundary conditions ϕ on the boundary γ of D, where W is Wiener measure, P is the measure generated by a random walk which approximates Brownian motion beginning at z, and Φ is the functional on paths which equals the value of ϕ at the point where the path first meets γ. This paper develops a specific rate of convergence. If γ is C^2, and P^n is generated by random walks beginning at z, with independent increments in the coordinate directions at intervals $1/n$, with mean zero, variance $1/\sqrt{n}$, and absolute third moment bounded by M, then $|E^{P^n}(\Phi) - E^W(\Phi)| \leqq (CMV/\rho(z,\gamma))n^{-1/16}(\log n)^{9/8}$, where V is the total variation of ϕ on γ, $\rho(z,\gamma)$ is the distance from z to γ, and C is a constant depending only on γ.

Assume D is a Jordan region. If $z_t = x_t + iy_t$ is Brownian motion in R^2 beginning at z_0, (cf. e.g., [5, p. 262]), and $\tau = \inf\{t: z_t \in \gamma\}$ is the first time z hits the boundary γ of D, then Φ is the functional given by $\Phi(z.) = \phi(z_\tau)$. Let $E^W(\Phi(z.)) = \int \Phi(z.)dW$ be the expectation of Φ with respect to Wiener measure W on $C([0, \infty), \mathbb{C})$. (See [8, pp. 218–19] for a definition of Brownian motion on the interval [0, 1] and the corresponding Wiener measure.)

Let $g_1^1, g_1^2, g_2^1, g_2^2, \cdots, g_k^1, g_k^2, \cdots$ be a sequence of indendent random variables with mean zero, variance 1, and absolute third moment bounded uniformly by $M < \infty$, and let

$$\xi_i^\alpha = g_i^\alpha/\sqrt{n},\ \zeta_0 = z_0,\ \zeta_k = z_0 + \sum_{i=1}^k (\xi_i^1 + \sqrt{-1}\xi_i^2),\ t_k = k/n\ .$$

Let $\xi(t)$ be the continuous random broken line which has vertices (t_k, ζ_k) and is linear between vertices. Let P^n be the measure on $C([0, \infty), \mathbb{C})$ generated by this line, i.e., $P^n(S) = P(\xi(t) \in S)$.

Now by the Central Limit Theorem $P^n(\xi^\alpha(t) \leqq \lambda) \to W(z_t^\alpha \leqq \lambda)$, $\alpha = 1, 2$, where $\xi^\alpha(t), z_t^\alpha$ are the real and imaginary parts of $\xi(t), z_t$ respectively, (cf. e.g., [1, pp. 186–7]). More exactly one has the Barry-Esseen Theorem [3, p. 521]: *For nt an integer*

$$\sup_\lambda |P(\xi^\alpha(t) \leqq \lambda) - N(\lambda/\sqrt{t})| \leqq \frac{33}{4}M/\sqrt{nt} \tag{1.1}$$

where $N(x)$ is the normal distribution. A useful generalization of the

Central Limit Theorem is that convergence also takes place for the expectation of any functional on $C[0, 1)$ which is continuous with respect to uniform convergence on $[0, 1]$ and satisfies mild growth conditions, e.g. $\Phi(x) = \int_0^1 x(t)^2 dt$, $\sup_{0 \leq t \leq 1} x_t$, etc. Rates of convergence have been calculated for some specific one-dimensional functionals Φ, (e.g., [10], [11]). For an arbitrary functional Φ satisfying a uniform Hölder condition one can get rates of convergence using Levy distance in $C[0, t]$ ([9], see also §2 of this paper). Explicit rates of convergence are of interest for various practical problems and computer applications.

Although $\Phi(z_.) = \phi(z_\tau)$ is not continuous with respect to uniform convergence, it is continuous a.s. with respect to Wiener measure, so convergence takes place. In this paper we obtain a rate of convergence.

THEOREM. *There exists a universal constant $C^* = C^*(\gamma)$ such that*

$$|E^{P^n}(\Phi) - E^W(\Phi)| \leq \frac{C^* V(\phi) M}{\rho(z_0, \gamma)} n^{-1/16} (\log n)^{9/8} \tag{1.2}$$

where $V(\phi)$ is the total variation of ϕ on γ, M is the bound on absolute third moments defined above, z_0 is the initial point of the paths $z_.$, and $\rho(z_0, \gamma) = \inf_s |z_0 - \gamma(s)|$ is the distance from z_0 to γ.

2. **Levy distance.** We define measures P_t^n, W_t on $C([0, t], \mathbb{C})$ by

$$P_t^n(S) = P^n(\pi^{-1}S), \; W_t(S) = W(\pi^{-1}S) ,$$

where $\pi: C([0, \infty), \mathbb{C}) \to C([0, t], \mathbb{C})$ is the projection $\pi(f) = f|_{[0,t]}$. The Levy distance L between the measures P_t^n and W_t is given by

$$L(P_t^n, W_t) = \max(\varepsilon_1, \varepsilon_2) , \tag{2.1}$$

where

$$\varepsilon_1 = \inf\{\varepsilon: P_t^n(S) \leq W_t(S^{\varepsilon,t}) + \varepsilon \text{ for all closed sets } S\} ,$$
$$\varepsilon_2 = \inf\{\varepsilon: W_t(S) \leq P_t^n(S^{\varepsilon,t}) + \varepsilon \text{ for all closed sets } S\} ,$$

and

$$S^{\varepsilon,t} = \{y: \exists x \in S \ni \sup_{0 \leq s \leq t} |y(s) - x(s)| < \varepsilon\}$$

is an ε-neighborhood of S with respect to the sup-norm on $[0, t]$.

The following proposition is a direct generalization of a result of Prokhorov ([9]) to two dimensions as is its proof.

PROPOSITION 1. *There exists an absolute constant C such that*

$$L(P_1^n, W_1) \leq C M^{1/4} n^{-1/8} (\log n)^{15/8} . \tag{2.2}$$

COROLLARY.
If $t = k/n$, k *an integer, then* $L(P_t^n, W_t) \leqq C\sqrt{t}\,k^{-1/8}(\log k)^{15/8}$ *for some constant* C.

3. Boundedness of harmonic density. Fix a point γ_0 on γ, and a direction along γ, parametrize γ by arclength in the chosen direction from γ_0. Let l denote the length of γ, and take the argument s of $\gamma = \gamma(s)$ modulo l.

Since γ is C^2, there exists $R > 0$ such that any circle of radius R will meet γ in at most two points. It follows that for any two points $\gamma(a)$ and $\gamma(a + \delta)$ on γ where $0 < \delta < R$ that $\gamma([a, a + \delta])$ will lie in the intersection of the closed disks bounded by the two circles of radius R through $\gamma(a)$ and $\gamma(a + \delta)$. The case we have to eliminate is where γ is tangent to one of the circles at $\gamma(a)$ and $\gamma(a + \delta)$, but does not cross the circle, i.e., there are neighborhoods in γ of $\gamma(a)$ and $\gamma(a + \delta)$ which do not meet the closed disk bounded by the circle except at $\gamma(a)$ or $\gamma(a + \delta)$. But in this case we observe that a small rotation of the circle about one of the points $\gamma(a)$ or $\gamma(a + \delta)$ will result in three points of intersection, contradicting our assumption about γ. Furthermore, it follows from the Jordan curve theorem that the center of one of the two circles will be in D, the other center will be outside D.

We are now ready to prove the following result.

PROPOSITION 2.

$$W(z_\tau \in \gamma([a, a + \delta])) \leqq B\delta/\rho(z_0, \gamma)$$

where B *is an absolute constant depending only on* γ.

Proof. We may assume $\delta < R$ and also δ sufficiently small that

$$2(R - (R^2 - \delta^2/4)^{1/2}) < \rho(z_0, \gamma)/2 ,$$

since by addition if the proposition holds for small δ, it holds for δ in general.

Let C be the circle of radius R through $\gamma(a)$ and $\gamma(a + \delta)$, with center not in D. Then

$$P_{z_0}(z_\tau \in \gamma([a, a + \delta])) \leqq P_{z_0}(z_{\tau(C)} \in \delta^*)$$

where $\tau(C) = \inf\{t: z_t \in C\}$, and $\delta^* = D \cap C$. Now invert the plane with respect to the circle C, sending z_0 into $I(z_0)$. Now $I(z_t)$ is Brownian motion with a time change. (P. Lévy [7, p. 254], see also [5, pp. 279–80] for another proof of this.) However, where $I(z_t)$ first hits C is independent of any time change; "*Les proprietés intrinsèques de*

la courbe C sont invariantes par une representation conforme."
Now

$$z_{\tau(C)} \in C \Longrightarrow I(z_{\tau(C)}) = z_{\tau(C)},\ I(\delta^*) = \delta^* ,$$

so $P_{z_0}(z_{\tau(C)} \in \delta^*) = P_{z_0}((I(z))_{\tau'(C)} \in \delta^*)$ where $\tau'(C) = \inf\{s: (I(z))_s \in C\}$.

But the harmonic density on a circle is given by the Poisson kernel (cf. e.g., [4, p. 361 ff.]); it is bounded,

$$P_{z_0}((I(z))_{\tau'(C)} \in \delta^*) \leqq \frac{2}{2\pi}|\delta^*|/\rho(I(z_0), C)$$

where $|\delta^*|$ is the length of δ^*. Now

$$R - \rho(I(z_0), C) = R^2/(\rho(z_0, C) + R)$$

so

$$1/\rho(I(z_0), C) = (\rho(z_0, C) + R)/R\rho(z_0, C) \leqq \frac{\Delta/R + 1}{\rho(z_0, C)} ,$$

where Δ is the diameter of D.
Now look at $\rho(z_0, C)$:

$$\rho(z_0, C) \geqq \rho(z_0, \gamma) - 2(R - (R^2 - s^2/4)^{1/2})$$

where $s = |\gamma(a + \delta) - \gamma(a)| \leqq \delta$. But δ was sufficiently small that

$$2(R - (R^2 - \delta^2/4)^{1/2}) < \rho(z_0, \gamma)/2 ,$$

and since $s \leqq \delta$,

$$2(R - (R^2 - s^2/4)^{1/2}) \leqq 2(R - (R^2 - \delta^2/4)^{1/2}) .$$

Hence

$$\rho(z_0, C) > \rho(z_0, \gamma)/2, \text{ also } |\delta^*| \leqq \frac{\pi}{2}s \leqq \frac{\pi}{2}\delta$$

and it follows that

$$W(z_\tau \in \gamma([a, a + \delta])) \leqq \frac{2}{2\pi}\,\frac{\pi}{2}\delta\cdot 2(\Delta/R + 1)/\rho(z_0, \gamma) = B\delta/\rho(z_0, \gamma) .$$

4. **Some inequalities.** We shall need the following.

LEMMA.

$$\begin{aligned} &W(\tau > t) \leqq \frac{4}{\pi}\exp(-\pi^2 t/8\Delta^2) \\ (4.1)\qquad &P^n(\tau > t) \leqq \frac{4}{\pi}\exp(-\pi^2 t/8\Delta^2) + AM(nt)^{-1/8}(\log nt)^{1/2} , \end{aligned}$$

where Δ is the diameter of D and A is an absolute constant.

Proof.

$$\begin{aligned} W(\tau > t) &\leqq \Pr(\max_{0\leqq s\leqq t} |z_s - z_0| < \Delta) \\ &\leqq \Pr(\max_{0\leqq s\leqq 1} |\mathrm{Re}\,(z_s - z_0)| < \Delta/\sqrt{t}) = T(\Delta/\sqrt{t}) \\ &\leqq \frac{4}{\pi}\exp(-\pi^2 t/8\Delta^2)\,, \end{aligned}$$

where $T(\lambda) = \Pr(\max_{0\leqq s\leqq 1} |x_s| < \lambda)$. The last inequality comes from the fact that the infinite series expansion for $T(\lambda)$ [11] is alternating, with decreasing terms.

$$\begin{aligned} P^n(\tau > t) &\leqq \Pr \max_{k\leqq nt} (|\zeta_k - z_0| < \Delta) \\ &\leqq \Pr(\max_{k\leqq nt} |\mathrm{Re}\,(\zeta_k - z_0)/\sqrt{t}| < \Delta/\sqrt{t})\,. \end{aligned}$$

Now the theorem of Rosencrantz [10] applies [11] and we have

$$\begin{aligned} &\Pr(\max_{k\leqq nt} |\mathrm{Re}\,(\zeta_k - z_0)/\sqrt{t}| < \Delta/\sqrt{t}) \\ &\leqq A\cdot M(\log nt)^{1/2}(nt)^{-1/8} + T(\Delta/\sqrt{t}) \end{aligned}$$

where A is an absolute constant. But we saw above that

$$T(\Delta/\sqrt{t}) \leqq \frac{4}{\pi}\exp(-\pi^2 t/8\Delta^2)\,,$$

so

$$P^n(\tau > t) \leqq \frac{4}{\pi}\exp(-\pi^2 t/8\Delta^2) + AM(nt)^{-1/8}(\log nt)^{1/2}\,.$$

Now we need more notation. Let $K_\lambda = \gamma([0, \lambda])$, let $(z_\tau \in K_\lambda)^{\varepsilon,\tau} \subset C([0, \infty), \mathbb{C})$ be defined by $y \in (z_\tau \in K_\lambda)^{\varepsilon,\tau}$ iff $\exists z$ such that $z_\tau \in K_\lambda$ and (for $\tau = \tau(z)$) $\sup_{0\leqq s\leqq \tau} |y_s - z_s| < \varepsilon$. Let $\delta = \sqrt{\varepsilon}$, and let $K_\lambda^\delta = \gamma([0, \lambda + \delta]) \cup \gamma(l - \delta, l])$, where l is the length of γ.

PROPOSITION 3.

$$W((z_\tau \in K_\lambda)^{\varepsilon,\tau} \cap (z_\tau \notin K_\lambda^\delta)) \leqq G\sqrt{\varepsilon}$$

where G is a constant depending only on γ.

Proof. Let $\tau(\partial\varepsilon) = \inf\{t: \rho(z_t, K_\lambda) < \varepsilon\}$ where $\rho(z_t, K_\lambda)$ is the distance from z_t to K_λ. Then

$$\begin{aligned} &W((z_\tau \in K_\lambda)^{\varepsilon,\tau} \cap (z_\tau \notin K_\lambda^\delta)) \\ &\quad \leqq W(\tau(\partial\varepsilon) < \tau,\, z_\tau \notin K_\lambda^\delta) + W(\tau(\partial\varepsilon) > \tau,\, z_\tau \notin K_\lambda^\delta,\, \tau(\partial\varepsilon) < \tau(s\varepsilon)) \\ &\quad = E^W(\chi_{\tau(\delta\varepsilon)<\tau} P_{z_{\tau(\partial\varepsilon)}}(z_\tau \notin K_\lambda^\gamma) \\ &\qquad + \chi_{[\tau(\partial\varepsilon)>\tau, z_\tau \notin K_\lambda]} P_{z_\tau}(\tau(\partial\varepsilon) < \tau(s\varepsilon))) \end{aligned}$$

by the strong Markov property [1, p. 268], where $\tau(s\varepsilon) = \inf\{t: \rho(z_t, D) > \varepsilon\}$. We estimate $P_{z_{\tau(\partial\varepsilon)}}(z_\tau \notin K_\lambda^\delta)$:

Let $\gamma(a)$ be a point in K_λ of distance ε from $z_{\tau(\partial\varepsilon)}$, let T be the tangent to γ at $\gamma(a)$. Let S_i $(i = 1, 2)$ be lines perpendicular to T through the points $\gamma(a - \delta)$ and $\gamma(a + \delta)$. The distance d_i from $z_{\tau(\partial\varepsilon)}$ to each of the lines S_i will be less than $\delta + \varepsilon$ (less than δ unless $\gamma(a)$ is an endpoint of K_λ; let $d = \min(d_1, d_2)$. Let T' be parallel to T, at a distance $\varepsilon \cdot \sup|\gamma''|$ on the opposite side of T from $z_{\tau(\partial\varepsilon)}$. I now claim $\gamma([a - \delta, a + \delta]) \cap T' = \varnothing$ if $2\delta < 1/\sup|\gamma''|$. Choose coordinates such that $\gamma(a) = 0$, $\gamma'(a) > 0$. Then by Taylor's Theorem, for each h there exists θ such that

$$\begin{aligned} \operatorname{Im}\gamma(a + h\delta) &= \operatorname{Im}\gamma(a) + \operatorname{Im}\gamma'(a)\cdot h\delta + \operatorname{Im}\gamma''(a + \theta h\delta)\cdot h^2\delta^2/2 \\ &= \operatorname{Im}\gamma''(a + \theta h\delta)\cdot h^2\delta^2/2\,. \end{aligned}$$

Hence for $|h| \leqq 1$, $|\operatorname{Im}\gamma(a + h\delta)| \leqq \sup|\delta''|\cdot\delta^2/2 < \varepsilon\cdot\sup|\gamma''|$ and $\gamma([a - \delta, a + \delta])$ does not meet T'.

Let $\tau_{T'}$ be the first time (after $\tau(\partial\varepsilon)$) that z_t hits the line T', τ_S the first time (after $\tau(\partial\varepsilon)$) that z_t hits $S_1 \cup S_2$, and $c = \rho(z_{\tau(\partial\varepsilon)}, T') \leqq \varepsilon\cdot(\sup|\gamma''| + 1)$. Note that $\tau_{T'}$ and τ_S are independent, since the components of Brownian motion in the direction of S_i and T' are independent. We can write

$$\begin{aligned} P_{z_{\tau(\delta\varepsilon)}}(z_\tau \notin K_\lambda^\delta) &< P_{z_{\tau(\delta\varepsilon)}}(\tau_T{}' > \tau_S) + O(\delta) \\ &= \int_0^\infty P_{z_{\tau(\partial\varepsilon)}}(\tau_S < t)\, d_t P_{z_{\tau(\partial\varepsilon)}}(\tau_{T'} \leqq t) + O(\delta)\,. \end{aligned}$$

Now

$$\begin{aligned} P_{z_{\tau(\partial\varepsilon)}}(\tau_{T'} \leqq t) &= P(\sup_{0\leqq s\leqq t} x_s \geqq c) = P(\sup_{0\leqq s\leqq 1} x_s \geqq c/\sqrt{t}\,) \\ &= \sqrt{2/\pi}\int_{c/\sqrt{t}}^\infty e^{-u^2/2}\,du \end{aligned}$$

(cf. e.g., [1, p. 287] and [8, p. 227]).
Hence

$$\begin{aligned} P_{z_{\tau(\delta\varepsilon)}}(\tau_{T'} > \tau_S) &= \int_0^\infty P_{z_{\tau(\partial\varepsilon)}}(\tau_S < t)\sqrt{2/\pi}\frac{1}{2}ct^{-3/2}e^{-c^2/t}\,dt \\ &\leqq \int_0^\infty P(\sup_{0\leqq s\leqq t}|x_s| > d)(c/\sqrt{2\pi})t^{-3/2}e^{-c^2/2t}\,dt \\ &\leqq 2\int_0^\infty P(\sup_{0\leqq s\leqq 1} x_s > d/\sqrt{t}\,)(c/\sqrt{2\pi})t^{-3/2}e^{-c^2/2t}\,dt \end{aligned}$$

which by a straightforward computation, is bounded by $2(d^2/c^2+1)^{-1/2}$.

But I claim $d \sim \delta$: choose coordinate such that $\gamma(a)=0$, $\gamma'(a)=1$. Using Taylor's Theorem we get $\delta \geqq d \geqq \delta - (\sup|\gamma''|/2)\delta^2 - \varepsilon$, so $d \sim \delta$. And $\delta = \sqrt{\varepsilon}$, $c \leqq \varepsilon(\sup|\gamma''|+1)$, so

$$2(d^2/c^2+1)^{-1/2} \leqq G_1\varepsilon/\delta = G_1\sqrt{\varepsilon}$$

for some constant G_1.

Now the same argument can be applied to estimate $P_{z_\tau}(\partial\varepsilon) < \tau(s\varepsilon))$ (i.e., the probability that a Brownian path will move a distance $d \sim \delta = \sqrt{\varepsilon}$ in the direction tangent to the curve before it moves a distance $c = O(\varepsilon)$ in the direction normal to the curve).
Hence $P_{z_\tau}(\tau(\partial\varepsilon) < \tau(s\varepsilon)) \leqq G_2\sqrt{\varepsilon}$ for some constant G_2 and our proposition follows.

5. Proof of the theorem. We are now ready to prove our theorem.

$$\begin{aligned} &|E^{P^n}(\Phi) - E^W(\Phi)| \\ (5.1)\qquad &= |E^{P^n}(\Phi\chi_{\tau\leqq t}) - E^W(\Phi\chi_{\tau\leqq t}) + E^{P^n}(\Phi\chi_{\tau>t}) - E^W(\Phi\chi_{\tau>t})| \\ &\leqq E^{P^n}(\Phi\chi_{\tau\leqq t}) - E^W(\Phi\chi_{\tau\leqq t}) + \sup_\gamma|\phi|(P^n(\tau>t)+W(\tau>t)) . \end{aligned}$$

Looking at the first term,

$$\begin{aligned} &|E^{P^n}(\Phi\chi_{\tau\leqq t}) - E^W(\Phi\chi_{\tau\leqq t})| \\ &= \left|\int_0^l \phi(\gamma(\lambda))(P^n(z_\tau\in\gamma(d\lambda), \tau\leqq t) - W(z_\tau\in\gamma(d\lambda), \tau\leqq t))\right| \\ &\leqq |\phi(\gamma(0))|(P(\tau>t)+W(\tau>t)) + \int_0^l |P^n(z_\tau\in K_\lambda, \tau\leqq t) \\ &\qquad - W(z_\tau\in K_\lambda, \tau\leqq t)|\cdot|d\phi(\lambda)| . \end{aligned}$$

We estimate the integrand:

The event $(z_\tau\in K_\lambda, \tau\leqq t)$ is determined by the behavior of the path up to time t, so

$$\begin{aligned} P^n(z_\tau\in K_\lambda, \tau\leqq t) - W(z_\tau\in K_\lambda, \tau\leqq t) &= P^n_t(z_\tau\in K_\lambda, \tau\leqq t) \\ &\quad - W_t(z_\tau\in K_\lambda, \tau\leqq t) . \end{aligned}$$

We can use the corollary of Proposition 1 to get

$$\begin{aligned} P^n_t(z_\tau\in K_\lambda, \tau\leqq t) &\leqq W_t((z_\tau\in K_\lambda, \tau\leqq t)^{\varepsilon,t}) + \varepsilon \\ &\leqq W_t(z_\tau\in K_\lambda, \tau\leqq t) + W_t((z_\tau\in K_\lambda, \tau\leqq t)^{\varepsilon,t} \\ &\qquad - (z_\tau\in K_\lambda^\delta, \tau\leqq t)) + W_t(z_\tau\in K_\lambda^\delta - K_\lambda, \tau\leqq t) + \varepsilon , \end{aligned}$$

where $\varepsilon = \varepsilon(n,t) = CM^{1/4}n^{-1/8}t^{3/8}(\log nt)^{15/8}$.

Now $y \in (z_\tau\in K_\lambda, \tau\leqq t)^{\varepsilon,t}$ means $\exists z$ such that $\tau\leqq t$, $z_\tau\in K_\lambda$,

$\sup_{0\leqq s\leqq t}|y_s - z_s| < \varepsilon$. As this condition does not depend on y_s for $s > t$,

$$\begin{aligned} &W_t((z_\tau \in K_\lambda,\ \tau \leqq t)^{\varepsilon,t} - (z_\tau \in K_\lambda^\delta,\ \tau \leqq t)) \\ &\quad \leqq W((z_\tau \in K_\lambda,\ \tau \leqq t)^{\varepsilon,\tau} - (z_\tau \in K_\lambda^\delta,\ \tau \leqq t)) \\ &\quad \leqq W((z_\tau \in K_\lambda))^{\varepsilon,\tau} - (z_\tau \in K_\lambda^\delta)) + W(\tau > t) . \end{aligned}$$

Applying Propositions 2 and 3, we then have

$$\begin{aligned} &P^n(z_\tau \in K_\lambda,\ \tau \leqq t) - W(z_\tau \in K_\lambda,\ \tau \leqq t) \\ &\qquad \leqq (G + 2B/\rho(z_0, \gamma))\sqrt{\varepsilon} + W(\tau > t) . \end{aligned}$$

We apply the above argument to the complement $\gamma - K_\lambda$ of K_λ in γ.

$$\begin{aligned} &P^n(z_\tau \in \gamma - K_\lambda,\ \tau \leqq t) - W(z_\tau \in \gamma - K_\lambda,\ \tau \leqq t) \\ &\quad \leqq (G + 2B/\rho(z_0, \gamma))\sqrt{\varepsilon} + W(\tau > t) . \end{aligned}$$

It follows that

$$\begin{aligned} |P^n(z_\tau \in K_\lambda,\ \tau \leqq t) - W(z_\tau \in K_\lambda,\ \tau \leqq t)| &\leqq (G + 2B/\rho(z_0, \gamma))\sqrt{\varepsilon} \\ &\quad + W(\tau > t) + P^n(\tau > t) . \end{aligned}$$

We can now estimate the integral

$$\begin{aligned} (5.3) \qquad &\int_0^l |P^n(z_\tau \in K_\lambda,\ \tau \leqq t) - W(z_\tau \in K_\lambda,\ \tau \leqq t)|\cdot|d\phi(\lambda)| \\ &\quad \leqq ((G + 2B/\rho(z_0, \gamma))\sqrt{\varepsilon} + W(\tau > t) + P^n(\tau > t))V(\phi) \end{aligned}$$

where $V(\phi)$ is the total variation of ϕ on γ.

Combining the results of (5.1), (5.2), and (5.3), (we have)

$$\begin{aligned} |E^{P^n}(\Phi) - E^W(\Phi)| &\leqq \phi(\gamma(0))(P^n(\tau > t) + W(\tau > t) \\ &\quad + V(\phi)(G + 2B/\rho(z_0, \gamma))\sqrt{\varepsilon} + W(\tau > t) + P^n(\tau > t)) \\ &\quad + \sup_\gamma |\phi|(P^n(\tau > t) + W(\tau > t)) \\ &\leqq V(\phi)(G + 2\beta/\rho(z_0, \gamma))\sqrt{\varepsilon} + (V(\phi) + 2\sup_\gamma|\phi|)\cdot(P^n(\tau > t) \\ &\quad + W(\tau > t)) . \end{aligned}$$

This estimate is minimized by choosing t so as to balance the factors $\sqrt{\varepsilon}$ and $(P^n(\tau > t) + W(\tau > t))$. So setting

$$t = \min\left\{s\colon s \geqq \frac{1}{2}(\Delta/\pi)^2 \log n,\ sn \text{ an integer}\right\} ,$$

we get

$$\begin{aligned} &P^n(\tau > t) + W(\tau > t) \\ &\leqq \frac{8}{\pi} n^{-1/16} + A\cdot M n^{-1/8}\left(\frac{1}{2}(\Delta/\pi)^2 \log n\right)^{1/8}(\log nt)^{1/2} \\ &\leqq A_1 M n^{-1/16}(\log n)^{5/8} , \end{aligned}$$

and

$$\sqrt{\varepsilon} = \sqrt{C}\, M^{1/8} n^{-1/16} t^{3/16} (\log nt)^{15/16}$$
$$\leqq \sqrt{C}\, M^{1/8} n^{-1/16} A_2 (\log n)^{9/8}$$

where A_1, A_2 are absolute constants. Hence

$$|E^{P^n}(\Phi) - E^W(\Phi)| \leqq V(\phi) M \frac{G\Delta + 2B}{\rho(z_0, \gamma)} \sqrt{C}\, A_2 n^{-1/16} (\log n)^{9/8}$$
$$+ (V(\phi) + 2 \sup_\gamma |\phi|) A_1 M n^{-1/16} (\log n)^{5/8}$$
$$\leqq (3V(\phi) + 2 \inf_\gamma |\phi|) M \left(\frac{G\Delta + 2B}{\rho(z_0, \gamma)} \sqrt{C}\ A_2 + A_1 \right) n^{-1/16} (\log n)^{9/8} .$$

But integration is linear, so we may assume $\phi(p) = 0$ for some p in γ, as we are taking the difference of expectations.

Letting $C^* = 3((G\Delta + 2B)\sqrt{C}\, A_2 + \Delta A_1)$ we have

$$|E^{P^n}(\Phi) - E^W(\Phi)| \leqq \frac{C^* V(\phi) M}{\rho(z_0, \gamma)} n^{-1/16} (\log n)^{9/8} .$$

COROLLARY. *If O is any subset of γ consisting of a finite number k of intervals, then*

$$|P^n_{z_0}(z_\tau \in O) - W_{z_0}(z_\tau \in O)| \leqq \frac{2kC^* M}{\rho(z_0, \gamma)} n^{-1/16} (\log n)^{9/8} .$$

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