PACIFIC JOURNAL OF MATHEMATICS
Vol. 42, No. 3, 1972

UNIVERSAL COSIMPLE ISOLS

ERIK ELLENTUCK

Our results follow from a single priority scheme which we give in detail. They are (i) if an arbitrary $n+1$-ary relation over the nonnegative integers determines an n-ary function, then its canonical extension to the isols determines a function on the cosimple isols if and only if the function determined on the integers is an almost recursive combinatorial function, and (ii) every countable partially ordered set can be embedded in (a) the cosimple isols, and even more (b) the cosimple regressive isols. The remaining material generalizes and extends these results.

1. Independence. Let $\omega =$ the nonnegative integers, $P =$ the set of all subsets of ω, and $Q =$ the set of all finite subsets of ω. Use $X^k A$ for the k-fold direct power of A. The results of this section all follow from the priority method of [11]. Let $f: \omega \rightarrow X^2 Q$ be a sequence of pairs $f(s) = (\alpha_s, \beta_s)$ where $\alpha_s \cap \beta_s = \emptyset$, and let $g: \omega \rightarrow \omega$. The *requirement* $R_k = \{(\alpha_s, \beta_s) \mid g(s) = k\}$. $\xi \in P$ is said to *meet* the requirement R_k if $\alpha \subseteqq \xi$, $\xi \cap \beta = \emptyset$ for some $(\alpha, \beta) \in R_k$. With every pair (f, g) as above we associate the *priority sequence* $\xi: \omega \rightarrow Q$ constructed in stages. Stage $s = 0$, $\xi_0 = \emptyset$ and for stage $s > 0$, $\xi_s = \xi_{s-1}$ if (1), (2), or (3) below is true. Otherwise $\xi_s = \xi_{s-1} \cup \alpha_s$.

(1) $\xi_{s-1} \cap \beta_s \neq \emptyset$.

(2) there is an $r < s$, $r > 0$, $g(r) < g(s)$ such that

$$\alpha_r \not\subseteqq \xi_{r-1},\ \alpha_r \subseteqq \xi_r,\ \xi_{s-1} \cap \beta_r = \emptyset \text{ and } \alpha_s \cap \beta_r \neq \emptyset .$$

(3) there is an $r < s$, $r > 0$, $g(r) = g(s)$ such that

$$\alpha_r \not\subseteqq \xi_{r-1},\ \alpha_r \subseteqq \xi_r \text{ and } \xi_{s-1} \cap \beta_r = \emptyset .$$

The requirement R_k is met at stage s if $s > 0$, $g(s) = k$ and $\alpha_s \not\subseteqq \xi_{s-1}$ but $\alpha_s \subseteqq \xi_s$. The requirement R_k is *injured* at stage s if for some $r < s$, R_k was met at stage r, $\xi_{s-1} \cap \beta_r = \emptyset$ but $\xi_s \cap \beta_r \neq \emptyset$. The basic combinatorial content of the priority method is summarized in the fundamental

LEMMA 1. (Sacks [11]). *For each k the set of stages s where R_k is either met or injured is finite.*

Proof. First we need two facts, (4) and (5) below. They are

(4) if R_k is injured at stage s then for some $j < k$, R_j is met at stage s.

This follows because if R_k is injured at stage s then there is an $r < s$ such that R_k was met at stage r, $\xi_{s-1} \cap \beta_r = \varnothing$ but $\xi_s \cap \beta_r \neq \varnothing$. Hence (1), (2), and (3) fail at stage s and $r > 0$, $g(r) = k$, $\alpha_r \nsubseteqq \xi_{r-1}$ but $\alpha_r \subseteqq \xi_r$. If $g(r) < g(s)$ then (2) is true at stage s and if $g(r) = g(s)$ then (3) is true at stage s. Hence $g(s) < g(r) = k$. The second fact that we need is

(5) if R_k is met at stages r and s where $r < s$ then for some u, $r < u < s$ and R_k is injured at stage u.

This follows because if R_k is met at stage r then (1) fails at stage r, so $\xi_{r-1} \cap \beta_r = \varnothing$, and $\alpha_r \nsubseteqq \xi_{r-1}$ but $\alpha_r \subseteqq \xi_r$. Since $\alpha_r \cap \beta_r = \varnothing$ we have $\xi_r \cap \beta_r = \varnothing$ and since R_k is met at stage s, (3) must fail at stage s. But $r > 0$ and $g(r) = g(s)$ which implies $\xi_{s-1} \cap \beta_r \neq \varnothing$. Let u be the least integer $r < u < s$ such that $\xi_u \cap \beta_r \neq \varnothing$. Clearly R_k is injured at stage u.

We prove our lemma by induction. Suppose that for each $j < k$ the set of stages s where R_j is met or injured is finite. Choose $u < \omega$ such that for no $j < k$ and $s > u$ is R_j met or injured at stage s. Then for $s > u$, R_k cannot be injured at stage s because otherwise by (4) R_j would be met at stage s for some $j < k$. Also for at most one $s > u$ is R_k met at stage s because otherwise by (5) R_k would be injured at some stage $r > u$.

With every pair (f, g) as above we associate the *priority set* $\xi = \bigcup_{s<\omega}\xi_s$ where ξ_s is the priority sequence associated with (f, g). First note that if a requirement R_k is met at stage s and subsequently never injured then ξ meets R_k. We lift recursive notions from ω to Q via the canonical enumeration of finite sets and note that ξ_s is recursive in the pair (f, g) which implies that ξ is r.e. in the pair (f, g).

Let $j \colon X^2\omega \to \omega$ be the usual recursive one-to-one onto pairing function with first and second inverses k, l respectively. Let $\omega^n = \{j(x, n) \mid x < \omega\}$ and $\alpha^n = \alpha \cap \omega^n$. $\varphi_n(\gamma)$ is a partial recursive function which enumerates with index n all partial recursive functions whose range and domain are contained in Q. Let $(n_s, \gamma_s, \delta_s)$ be a total recursive enumeration of $\{(n, \gamma, \delta) \mid \varphi_n(\gamma) = \delta\}$. We define (f, g) as follows. $f(2s + 1) = (\alpha_{2s+1}, \beta_{2s+1})$ where $\alpha_{2s+1} = \delta_s - \gamma_s$, $\beta_{2s+1} = \gamma_s$, and $g(2s + 1) = 2n_s + 1$. $f(2s) = (\alpha_{2s}, \beta_{2s})$ where α_{2s}, β_{2s}, $\bigcup_{r<2s}(\alpha_r \cup \beta_r)$ are pairwise disjoint, both α_{2s}, β_{2s} are subsets of $\omega^{k(k(s))}$ each containing $l(k(s)) + 1$ elements, and $g(2s) = 2k(s)$. Let $\xi \colon \omega \to Q$ be the priority sequence associated with (f, g) and let $\xi = \bigcup_{s<\omega}\xi_s$ be the priority set associated with (f, g). Throughout the remainder of this section ξ_s, ξ will keep these meanings. Let $\eta = \omega - \xi$.

LEMMA 2. *η^m is infinite for each $m < \omega$.*

Proof. Suppose η^m contained n elements. Let $k = 2j(m, n)$ and let $u < \omega$ be such that for no $s > u$ and $j \leqq k$ is R_j either met or injured at stage s. Choose $s > u$ such that $g(s) = k$. Since $s > u$, $\xi_s = \xi_{s-1}$ and hence (1), (2), or (3) hold at stage s. By construction of f, both (1) and (2) fail at stage s, so (3) must hold. Hence there is an $r < s$, $r > 0$, $g(r) = g(s)$ such that $\alpha_r \not\subseteqq \xi_{r-1}$, $\alpha_r \subseteqq \xi_r$, and $\xi_{s-1} \cap \beta_r = \emptyset$. β_r contains $n + 1$ elements of ω^m, and for $t \geqq s$, $\xi_t \cap \beta_r = \emptyset$ because otherwise R_k would be injured at stage t. Hence $\beta_r \subseteqq \eta$.

Let M be the class of partial recursive functions φ whose range and domain are both included in Q, which are increasing, i.e., if $\alpha \in$ domain (φ) then $\alpha \subseteqq \varphi(\alpha)$, and which are monotone, i.e., if $\alpha, \beta \in$ domain (φ) and $\alpha \subseteqq \beta$ then $\varphi(\alpha) \subseteqq \varphi(\beta)$.

LEMMA 3. *For each $\varphi \in M$ there is a $\gamma \in Q$, $\gamma \subseteqq \eta$ such that for all $\alpha \in Q$, $\gamma \subseteqq \alpha \subseteqq \eta$ either $\alpha \notin$ domain (φ) or $\varphi(\alpha) = \alpha$ or $\varphi(\alpha) \cap \xi \neq \emptyset$.*

Proof. Let φ have index n in the enumeration of partial recursive functions above and let $k = 2n + 1$. Choose $u < \omega$ such that for no $s > u$ and $j \leqq k$ is R_j either met or injured at stage s. Choose $\gamma \in Q$ such that $\eta \cap \bigcup_{r \leqq u} \beta_r \subsetneqq \gamma \subseteqq \eta$. Suppose there were an $\alpha \in Q$, $\gamma \subseteqq \alpha \subseteqq \eta$ such that $\alpha \in$ domain (φ) and $\alpha \subsetneqq \varphi(\alpha)$. Then for some s, $\alpha_s = \varphi(\alpha) - \alpha$, $\beta_s = \alpha$, and $g(s) = k$. $s > u$ for otherwise $\alpha \subsetneqq \gamma$. Hence $\xi_s = \xi_{s-1}$ so that (1), (2), or (3) must hold at stage s. Since $\xi_{s-1} \subseteqq \xi$ and $\alpha \subseteqq \eta$, (1) fails at stage s. If there were an $r < s$, $r > 0$, $g(r) = j < k$ such that R_j is met at stage r and $\xi_{s-1} \cap \beta_r = \emptyset$ then $\beta_r \subseteqq \eta$ since otherwise R_j would be injured at some stage $t > u$. Hence $\beta_r \subseteqq \alpha$ and so (2) fails at stage s. Hence (3) holds at stage s and there is an $r < s$, $r > 0$, $g(r) = g(s) = k$ such that $\alpha_r \not\subseteqq \xi_{r-1}$, $\alpha_r \subseteqq \xi_r$ and $\xi_{s-1} \cap \beta_r = \emptyset$. $\beta_r \subseteqq \eta$ for otherwise R_k would be injured at some stage $t > u$. $\emptyset \neq \alpha_r = \varphi(\beta_r) - \beta_r \subseteqq \xi$ and by monotonicity we have $\varphi(\alpha) \cap \xi \neq \emptyset$.

Let $\Lambda =$ the isols and for $\alpha \in P$ let Req $(\alpha) =$ the recursive equivalence type of α. Λ_z (the cosimple isols) $=$ the set of those isols which can be represented as Req (α) where $\omega - \alpha$ is r.e. and $\Lambda_z^\infty = \Lambda_z - \omega$, i.e., the infinite cosimple isols. For the rest of this section we fix the following notation $S = \{y_m \mid m < \omega\}$ where $y_m =$ Req (η^m) for $m < \omega$.

THEOREM 1. *S is an infinite subset of Λ_z^∞.*

Proof. First we show that η is immune. η is infinite by Lemma 2. Suppose that η contained an infinite r.e. subset β which we assume is enumerated by a total recursive function $b(n)$. Define a recursive function φ: $Q \to Q$ by letting $\varphi(\alpha) = \alpha \cup \{b\}$ where b is the first element in the enumeration $b(n)$ which does not occur in α. Since $\varphi \in M$ there is a $\gamma \in Q$, $\gamma \subseteqq \eta$ such that for all $\alpha \in Q$, $\gamma \subseteqq \alpha \subseteqq \eta$ either $\alpha \notin$ domain (φ) or $\varphi(\alpha) = \alpha$ or $\varphi(\alpha) \cap \xi \neq \emptyset$. Since all three of these possibilities are ruled out by our construction, η is immune. ξ is r.e. since it is r.e. in (f, g), the latter clearly being recursive. Each ξ^m is r.e. since it is the intersection of two r.e. sets and each η^m is immune since it is an infinite subset of an immune set. Thus $S \subseteqq \Lambda_z^\infty$. Next we show that if $m < n < \omega$ then $y_m \neq y_n$. Otherwise there would be a one-one partial recursive function p which maps η^m onto η^n. Define a recursive function $\varphi: Q \rightarrow Q$ by letting $\varphi(\alpha) = \alpha \cup p(\alpha^m)$. Since $\varphi \in M$ there is a $\gamma \in Q$, $\gamma \subseteqq \eta$ which satisfies the conclusion of Lemma 3. By our construction $\varphi(\alpha) = \alpha$ for each $\alpha \in Q$, $\gamma \subseteqq \alpha \subseteqq \eta$. Choose $x \in \eta^m - p^{-1}(\gamma)$ so that $p(x) \in \varphi(\gamma \cup \{x\}) = \gamma \cup \{x\}$. Since $p(x) \notin \gamma$ we have $p(x) = x$ so that $\eta^m \cap \eta^n \neq \emptyset$, a contradiction.

A set A of isols is called *strongly recursively independent* if for each $n > 0$, $R \subseteqq X^{n+1}\omega$ satisfying $(\forall v_0, \cdots, v_{n-1} \in \omega)(\exists!\ v_n \in \omega)(v_0, \cdots, v_n) \in R$, and distinct elements $x_0, \cdots, x_{n-1} \in A$, if there is an $x_n \in \Lambda$ such that $(x_0, \cdots, x_n) \in R_\Lambda$ then R is the graph of an eventually recursive combinatorial function.

THEOREM 2. *S is strongly recursively independent.*

Proof. Let $n, R, x_0, \cdots, x_{n-1}$ be as in the hypothesis of the above definition. Using Theorem 3.1 of [10] ($(R \times S)_\Lambda = R_\Lambda \times S_\Lambda$, and $\omega_\Lambda = \Lambda$) there is no loss of generality in assuming that $x_m = y_m$ for $m < n$. Suppose there is a $z \in \Lambda$ such that $(y_0, \cdots, y_{n-1}, z) \in R_\Lambda$. For $\alpha \in P$ let $\bar{\alpha} = (\alpha^0, \cdots \alpha^{n-1})$. Then there is a $\zeta \in z$ and a recursive R-frame F such that $(\bar{\eta}, \zeta) \in \mathscr{A}(F)$. If $\alpha = (\alpha_0, \cdots, \alpha_{n-1})$, where possibly some of the α_i are themselves tuples, and $C_F(\alpha)$ is defined, define $C_F^{\surd}(\alpha_0, \cdots, \alpha_i^{\surd}, \cdots, \alpha_{n-1}) = \beta_i$ where $C_F(\alpha) = (\beta_0, \cdots, \beta_{n-1})$ and $\alpha_j \leqq \beta_j$ for $j < n$ (here $\surd$ occurs as a superscript to exactly one α_i and $\leqq$ denotes componentwise inclusion). Now define a partial recursive function φ whose range and domain are included in Q as follows. $\varphi(\alpha) = \alpha \cup \beta$ where $\bar{\beta} = C_F^{\surd}(\bar{\alpha}^{\surd}, \emptyset)$. Since $\varphi \in M$, Lemma 3 yields a $\gamma \in Q$, $\gamma \subseteqq \eta$ such that for all $\alpha \in Q$, $\gamma \subseteqq \alpha \subseteqq \eta$, either $\alpha \notin$ domain (φ) or $\varphi(\alpha) = \alpha$ or $\varphi(\alpha) \cap \xi \neq \emptyset$. Since $(\bar{\eta}, \zeta) \in \mathscr{A}(F)$ only the second of these alternatives can occur, i.e., $\varphi(\alpha) = \alpha$. But this implies that $C_F(\alpha, \emptyset) = (\alpha, C_F^{\surd}(\alpha, \emptyset^{\surd}))$ for each $\alpha \in A = \{\alpha \in X^nQ \mid \bar{\gamma} \leqq \alpha \leqq \bar{\eta}\}$. If $\alpha \in A$ let $\psi(\alpha) = C_F^{\surd}(\alpha, \emptyset^{\surd})$ so that $(\alpha, \psi(\alpha)) \in F$. Let $\wedge$ denote componentwise intersection and observe that $(\alpha \wedge \alpha', \emptyset) \leqq (\alpha, \psi(\alpha)) \wedge (\alpha', \psi(\alpha') = (\alpha \wedge \alpha', \psi(\alpha) \cap \psi(\alpha'))$ so that $\psi(\alpha \wedge \alpha') \subseteqq \psi(\alpha) \cap \psi(\alpha')$. But then our hypothesis that R is the graph of a function give (i) $\psi(\alpha \wedge \alpha') = \psi(\alpha) \cap \psi(\alpha')$ and (ii) $\alpha \sim \alpha'$ componentwise implies $\psi(\alpha) \sim \psi(\alpha')$ (here $\sim$ denotes equal cardinality) for every $\alpha, \alpha' \in A$. For $m < n$ let p_m be a one-to-one function mapping ω onto $\eta^m - \gamma^m$. No recursive condition is placed on the p_m's. For $\alpha = (\alpha_0, \cdots, \alpha_{n-1}) \in X^n\omega$ let $p(\alpha) = (p_0(\alpha_0), \cdots, p_{n-1}(\alpha_{n-1}))$ and define a function φ: $X^nQ \to Q$ by $\varphi(\alpha) = \psi(\bar{\gamma} \vee p(\alpha))$ where $\vee$ denotes componentwise union. ψ inherits properties (i) and (ii) from ψ and is therefore a combinatorial operator inducing a combinatorial function r: $X^n\omega \to \omega$ such that if γ^m has c_m elements then $(x_0 + c_0, \cdots, x_{n-1} + c_{n-1}, r(x_0, \cdots, x_{n-1})) \in R$ for every $(x_0, \cdots, x_{n-1}) \in X^n\omega$. Thus R is the graph of an eventually combinatorial function. Let $B = \{(\alpha, \beta) \in (X^nQ) \times Q \,|\, \alpha \wedge \bar{\gamma} = \varnothing$ and $C_F(\alpha \vee \bar{\gamma}, \varnothing) = (\alpha \vee \bar{\gamma}, \beta)\}$. B is r.e. and if we let $|\cdot|$ denote componentwise cardinality, $\tilde{R} = \{x \in X^{n+1}\omega \,|\, (\exists\, \alpha)(x = |\alpha|$ and $\alpha \in B)\}$ is an r.e. set which is the graph of r. Thus R is the graph of an eventually recursive combinatorial function.

2. Applications.

COROLLARY 1. *Let $n > 0$ and $R \subseteq X^{n+1}\omega$ the graph of a function r. Then* (i) *r is eventually recursive combinatorial if and only if for each $x \in X^n\Lambda_z^\infty$ there is exactly one $y \in \Lambda$ such that $(x, y) \in R_\Lambda$,* (ii) *r is almost recursive combinatorial if and only if for each $x \in X^n\Lambda_z$ there is exactly one $y \in \Lambda$ such that $(x, y) \in R_\Lambda$. This solves the problem left open on the middle of page* 247 *of* [7].

A set A of isols is called *recursively independent* if for each $n > 0$, $R \subseteq X^n\omega$, and distinct elements $x_0, \cdots, x_{n-1} \in A$, if $(x_0, \cdots, x_{n-1}) \in R_\Lambda$ then there exists $\alpha \in Q$ such that $X^n(\omega - \alpha) \subseteq R$.

THEOREM 3. *If A is a strongly recursively independent set of isols, then A is recursively independent.*

Proof. For $R \subseteq X^n\omega$ let $\tilde{R} = \{(x, 0) \,|\, x \in R\} \cup \{(x, 1) \,|\, x \in X^n\omega - R\}$. $\tilde{R}$ is the graph of the characteristic function r of R and satisfies $(\forall x)(x \in R \to (x, 0) \in \tilde{R})$ in ω and hence in Λ. Let $x_0, \cdots, x_{n-1}$ be distinct elements of A and suppose that $(x_0, \cdots, x_{n-1}) \in R_\Lambda$. Then $(x_0, \cdots, x_{n-1}, 0) \in \tilde{R}_\Lambda$ which implies that r is eventually combinatorial. Since eventually combinatorial functions are eventually monotone, r is eventually equal to 0 or eventually equal to 1. If r is eventually equal to 1 there is an $a < \omega$ such that $(\forall x_0, \cdots, x_{n-1})(x_0 + a, \cdots, x_{n-1} + a) \notin R$ holds in ω and hence in Λ. This implies that at least one of the x_i's in A is finite, in fact $< a$. Clearly no member of a strongly recursively independent set can be finite so r must be eventually equal to 0.

COROLLARY 2. *S is recursively independent.*

A set A of isols is called *independent* if for each $n > 1$ and distinct elements $x_0, \cdots, x_{n-1} \in A$, $x_0 \not\leqq x_1 + \cdots + x_{n-1}$.

THEOREM 4. *If A is a recursively independent set of isols then A is independent.*

Proof. Fix n and let $R = \{(x_0, \cdots, x_{n-1}) \in X^n\omega \,|\, x_0 \leqq x_1 + \cdots + x_{n-1}\}$. Then $(\forall x_0, \cdots, x_{n-1}, y)(x_0 + y = x_1 + \cdots + x_{n-1} \rightarrow (x_0, \cdots, x_{n-1}) \in R)$ holds in ω and hence in Λ. Thus if $x_0, \cdots, x_{n-1}$ are distinct elements of A and $x_0 \leqq x_1 + \cdots + x_{n-1}$ then $(x_0, \cdots, x_{n-1}) \in R_\Lambda$, a contradiction.

COROLLARY 3. *S is independent.*

Let A be a set partially ordered by a relation R. We say that $\langle A, R\rangle$ can be *embedded* in Λ if there exists an order isomorphism $h: A \rightarrow \Lambda$ with respect to the usual ordering of Λ. The following theorem is of interest when compared with results in [5].

THEOREM 5. *Every countable partially ordered set can be embedded in the cosimple isols.*

Proof. By a result of [9] there is a recursive relation R which partially orders ω in such a way that every countable partial ordering can be embedded in $\langle \omega, R\rangle$. We prove our theorem by embedding $\langle \omega, R\rangle$ in Λ_z. Assume that R is reflexive and define $h(n) = \bigcup \{\eta^m \,|\, mRn\}$. $h(n)$ is an immune set whose complement in ω is $\bigcup \{\omega^m \,|\, \sim mRn\} \cup (\xi \cap \bigcup \{\omega^m \,|\, mRn\})$ which clearly is an r.e. set. Thus $\mathrm{Req}\,(h(n)) \in \Lambda_z$. If $n_0 R n_1$ then $h(n_0) \subseteqq h(n_1)$ and $h(n_0), h(n_1) - h(n_0)$ are recursively separated by the recursive set $\bigcup \{\omega^m \,|\, mRn_0\}$. Thus $\mathrm{Req}\,(h(n_0)) \leqq \mathrm{Req}\,(h(n_1))$ by the identity function. If $\sim mRn$ but $\mathrm{Req}\,(h(m)) \leqq \mathrm{Req}\,(h(n))$ then for some one-one partial recursive function p, $\eta^m \subseteqq$ domain (p) and p maps η^m into $\bigcup \{\eta^j \,|\, j \neq m\}$. Define a partial recursive function φ whose range and domain are included in Q by $\varphi(\alpha) = \alpha \cup p(\alpha^m)$. Then $\varphi(\alpha) = \alpha$ for all $\alpha \in Q$, $\gamma \subseteqq \alpha \subseteqq \eta$ where γ is given by Lemma 3. If $x \in \eta^m - p^{-1}(\gamma)$ then $p(x) \in \varphi(\gamma \cup \{x\}) = \gamma \cup \{x\}$. Since $p(x) \notin \gamma$ we have $p(x) = x$ so $\eta^m \cap \bigcup \{\eta^j \,|\, j \neq m\} \neq \emptyset$, a contradiction. Thus $\mathrm{Req}\,(h(m)) \not\leqq \mathrm{Req}\,(h(n))$ and we see that $\mathrm{Req}\,(h(n))$ is the required order isomorphism.

There are several ways to extend Theorem 5. If R' is a partial ordering of Λ say that $\langle A, R\rangle$ is *R'-embedded in Λ* if the mapping $h: A \rightarrow \Lambda$ is an $R - R'$ order isomorphism. For R' we are interested in $<_\Lambda$, the canonical extension to Λ of $R = \{(x, y) \in X^2\omega \,|\, x < y\}$, and $\leqq_\Lambda$, the canonical extension to Λ of $R = \{(x, y) \in X^2\omega \mid x \leqq y\}$. Then we have

COROLLARY 4. *Every countable partially ordered set can be both* $<_\Lambda$ *and* $\leqq_\Lambda$ *embedded in* Λ_z.

Proof. Let R, h be as in the proof of Theorem 5. First we show that if mRn and $m \neq n$ then $\mathrm{Req}\,(h(m)) <_\Lambda \mathrm{Req}\,(h(n))$. Since mRn and $m \neq n$, $h(m) \subsetneqq h(n)$. Let $x \in h(n) - h(m)$ and let $F = \{(\alpha, \beta) \in X^2Q \mid x \notin \alpha$ and $\alpha \cup \{x\} \subseteqq \beta\}$. F is a recursive $<$-frame from which $(h(m), h(n))$ is attainable. Next we show that if $\sim mRn$ then $\sim \mathrm{Req}\,(h(m)) \leqq_\Lambda \mathrm{Req}\,(h(n))$. For suppose that F is a recursive $\leqq$-frame and $(h(m), h(n)) \in \mathscr{A}(F)$. Since $\sim mRn$, $\eta^m \subseteqq h(m)$ and $\eta^m \cap h(n) = \varnothing$. Now define a partial recursive function φ whose range and domain are included in Q as follows. $\varphi(\alpha) = \alpha \cup \beta$ where $\beta = \beta_0 \cup \beta_1$ and $C_F(\alpha^m, \varnothing) = (\beta_0, \beta_1)$. Since $\varphi \in M$ Lemma 3 gives the required γ. If $\alpha \subseteqq \eta^m$ then $\varphi(\alpha \cup \gamma) = \alpha \cup \gamma$ which implies that if $C_F(\alpha^m \cup \gamma^m, \varnothing) = (\beta_0, \beta_1)$ then $\beta_1 \subseteqq \gamma$. Since $\alpha \subseteqq \eta^m$ can be made as large as we please, F is not a $\leqq$-frame. Hence $(h(m), h(n)) \notin \mathscr{A}(F)$. To complete our proof recall that for $u, v \in \Lambda$, $u < v \rightarrow u <_\Lambda v \rightarrow u \leqq_\Lambda v$, $u \leqq v \rightarrow u \leqq_\Lambda v$, and $u = v \rightarrow u \leqq_\Lambda v$, all implied by a metatheorem.

Another way to extend Theorem 5 is to try to embed partially ordered sets into the cosimple regressive isols Λ_{rz}. Let τ be an infinite retraceable set enumerated in increasing order as $\{t_n \mid n < \omega\}$ where $t_n < t_{n+1}$ for $n < \omega$ and let p be a partial recursive special (cf. [3]) retracing function for τ. τ is called *T-retraceable* if for every partial recursive function h there is $u < \omega$ such that for all $n > u$ either $t_n \notin$ domain (h) or $h(t_n) < t_{n+1}$. By a result of [8] T-retraceable sets with r.e. complement exist and they are necessarily immune. Fix τ in the following discussion as T-retraceable with r.e. complement, p is a total recursive function (cf. P.5 of [3]) and $p^*(x) = (\mu n)(p^n(x) = p^{n+1}(x))$.

LEMMA 4. *If* $\alpha, \beta \subseteqq \tau$ *are infinite recursively equivalent sets then* $\alpha \cap \beta \neq \varnothing$.

Proof. Let h be a partial recursive one-to-one function mapping α onto β. By T-retraceability there is a $u < \omega$ such that for all $n > u$, if $t_n \in \alpha$ then $h(t_n) \leqq t_n$ (since $h(t_n) < t_{n+1}$ and $h(t_n) \in \beta \subseteqq \tau$), and if $t_n \in \beta$ then $h^{-1}(t_n) \leqq t_n$. Recast the second conclusion as $t_n \leqq h(t_n)$ if $h(t_n) \in \beta$ and $p^*(h(t_n)) > u$. Choose $u' > u$ such that $p^*(h(t_n)) > u$ for

$n > u'$. Thus if $n > u'$ and $t_n \in \alpha$ then $h(t_n) \leqq t_n \leqq h(t_n)$, i.e., $t_n \in \alpha \cap \beta$.

THEOREM 6. *Every countable partially ordered set can be embedded in the cosimple regressive isols.*

Proof. Let R be as in the proof of Theorem 5, $\sigma = \omega - \tau$, and for $m < \omega$ let $\mu^m = \{x < \omega \mid k(p^*(x)) = m\}$. For any set α let $\alpha^m = \alpha \cap \mu^m$. Our embedding is defined by $h(n) = \cup \{\tau^m \mid mRn\}$. To show that $h(n)$ is retraceable note that $h(n)$ is recursively separated from $\tau - h(n)$ by the recursive set $\cup \{\mu^m \mid mRn\}$ and then use P.5 of [2] to conclude that $h(n)$ is retraceable. $h(n)$ is an immune set whose complement in ω is $\cup \{\mu^m \mid \sim mRn\} \cup (\sigma \cap \cup \{\mu^m \mid mRn\})$ which is clearly r.e. Thus $\mathrm{Req}\,(h(n)) \in \Lambda_{rz}$. If $n_0 R n_1$ then $h(n_0) \subseteqq h(n_1)$ and $h(n_0)$, $h(n_1) - h(n_0)$ are recursively separated by the recursive set $\cup \{\mu^m \mid mRn_0\}$. Therefore $\mathrm{Req}\,(h(n_0)) \leqq \mathrm{Req}\,(h(n_1))$. If $\sim mRn$ but $\mathrm{Req}\,(h(m)) \leqq \mathrm{Req}\,(h(n))$ then for some one-to-one partial recursive function q, $\tau^m \subseteqq$ domain (q) and q maps τ^m into $\cup \{\tau^j \mid j \neq m\}$. Since this is ruled out by Lemma 4 $\mathrm{Req}\,(h(n))$ is the required isomorphism.

Let $S_r = \{z_m \mid m < \omega\}$ where $z_m = \mathrm{Req}\,(\tau^m)$ for $m < \omega$.

COROLLARY 5. *S_r is an infinite independent but not recursively independent subset of Λ_{rz}^{∞}.*

Proof. We only show that S_r is not recursively independent. The rest is obvious from the proof of Theorem 6. If $n > m$ then $j(n, x) > j(m, x)$. Define a function q on μ^n by $q(x) = (\mu y)(\exists s \leqq p^*(x))$ $(y = p^s(x)$ and $p^*(y) = j(m, l(p^*(x))))$. q is readily seen to be a many one partial recursive function which maps τ^n one-to-one onto τ^m. Hence $\mathrm{Req}\,(\tau^n) \leqq^* \mathrm{Req}\,(\tau^m)$ and therefore $\mathrm{Req}\,(\tau^n) \leqq_\Lambda \mathrm{Req}\,(\tau^m)$ by Theorem 2.1 of [1] (which asserts that $\leqq^*$ agrees with $\leqq_\Lambda$ on Λ_r.

Corollary 5 explains why in Theorem 6 our method did not give $\leqq_\Lambda$ or $<_\Lambda$ embeddings. In [6] it is shown that $\mathrm{Req}\,(\tau)$ where τ is T-retraceable is *universal*, i.e., $\{\mathrm{Req}\,(\tau)\}$ is recursively independent. Call $x \in \Lambda$ *strongly universal* if for each $R \subseteqq X^2\omega$ satisfying $(\forall v_0)(\exists!\, v_1)(v_0, v_1) \in R$, if there is a $y \in \Lambda$ such that $(x, y) \in R_\Lambda$ then R is the graph of an eventually recursive increasing function.

THEOREM 7. *Req (τ) is strongly universal.*

Proof. Let R be as in the hypothesis of the above definition. Assume there is an isolated $\zeta \subseteqq \omega$ and recursive R-frame F such that (τ, ζ) is attainable from F. For $x < \omega$ let $s(x) = \{p^n(x) \mid n \leqq$

$p^*(x)\}$ and let $\gamma = \{x<\omega \mid (s(x), \emptyset) \in F^*\}$. Clearly $s(x)$ is a recursive function and γ is an r.e. set. For $x \in \gamma$ let $h(x) =$ the maximum element of $C_F^{\surd}(s(x)^{\surd}, \emptyset)$. $h(x)$ is partial recursive and since $\tau \subseteqq \gamma$, defined on all of τ and maps τ into τ. Hence there is a $u < \omega$ such that $h(t_n) \leqq t_n$ for $n > u$. This implies that $C_F^{\surd}(s(x)^{\surd}, \emptyset) = s(x)$ for all $x \in \tau$ with $p^*(x) > u$, and consequently for such x we have $C_F(s(x), \emptyset) = (s(x), C_F^{\surd}(s(x), \emptyset^{\surd})) \in F$. Clearly the set $\{(m, n) \mid (\exists x)(x \in \tau$ and $p^*(x) > u$ and $m = |s(x)|$ and $n = |C_F^{\surd}(s(x), \emptyset^{\surd})|)\}$ is the graph of an increasing function contained in R and has a recursive extension $\{(m, n) \mid (\exists \alpha, \beta)(\alpha, \beta) \in F$ and $m = |\alpha|$ and $n = |\beta|\}$ so R itself is the graph of an eventually recursive increasing function.

COROLLARY 6. Req (τ) *is universal.*

Proof. Just as in the proof of Theorem 3 using the fact that a bounded eventually increasing function is eventually constant. q.e.d.

Using a somewhat different priority method, we have been able to extend the results of this paper to the cosimple regressive isols. They are stated below without proof for comparison with Corollary 1. If $f: \omega \to \omega$ let $\Delta f(x) = f(x+1) - f(x)$. If $f: X^n\omega \to \omega$ and $i < n$ let Δ_i be just like Δ except that it is applied to the ith argument of f. Let $\Delta =$ the composition $\Delta_0 \cdots \Delta_{n-1}$. Then Δf is well defined and is consistent with the notation for the one variable case. f is called *recursive increasing* if f is recursive and $\Delta \hat{f}(x) \geqq 0$ for all $x \in X^n\omega$. where $\hat{f}(x_0 + 1, \cdots, x_{n-1} + 1) = f(x_0, \cdots, x_{n-1})$ and $\hat{f}(x) = 0$ otherwise. *Eventually recursive increasing* and *almost recursive increasing* are canonically defined.

THEOREM 8. *Let $n > 0$ and $R \subseteqq X^{n+1}\omega$ the graph of a function r. Then r is almost recursive increasing if and only if for each $x \in X^n\Lambda_{rz}$ there is a $y \in \Lambda$ such that $(x, y) \in R_\Lambda$.*

A function $f: X^n\omega \to \omega$ is called *recursive regular* if there are unary recursive increasing functions g_i for $i < n$ such that $f(x_0, \cdots, x_{n-1}) =$ minimum $\{g_0(x_0), \cdots, g_{n-1}(x_{n-1})\}$. *Eventually recursive regular* and *almost recursive regular* are canonically defined.

THEOREM 9. *Let $n > 0$ and $R \subseteqq X^{n+1}\omega$ the graph of a function r. Then r is almost recursive regular if and only if for each $x \in X^n\Lambda_{rz}$ there is a $y \in \Lambda_r$ such that $(x, y) \in R_\Lambda$.*

In both of these theorems the y in question, when it exists, is actually an element of Λ_z.

REFERENCES

1. J. Barback, *Two notes on recursive functions and regressive isols*, TAMS, **144** (1969), 77-94.

2. J. C. E. Dekker, *The minimum of two regressive isols*, Math. Zeitschr., **83** (1964), 345-366.

3. J. C. E. Dekker and J. R. Myhill, *Retraceable sets*, Canad. J. Math., **10** (1958), 357-373.

4. E. Ellentuck, *On the form of fnnctions which preserve regressive isols*, submitted to Composito Math.

5. ———, *Uncountable suborderings of the isols*, submitted to Composito Math.

6. J. Gersting, *A rate of growth criterion for universality of regressive isols*, Pacific J. Math., **31** (1969), 669-677.

7. L. Hay, *The cosimple isols*, Ann. of Math., **83** (1966), 231-256.

8. T. G. McLaughlin, *Hereditarily retraceable isols*, Bull. Amer. Math. Soc., **73** (1967), 113-115.

9. A. Mostowski, *Über gewisse universelle Relationen*, Ann. Soc. Polon. Math., **17** (1938), 117-118.

10. A. Nerode, *Extensions to isols*, Ann. of Math., **73** (1961), 362-403.

11. G. E. Sacks, *Degrees of Unsolvability*, Ann. of Math. Studies 55 (1963), Princeton.

Received April 27, 1971. Research supported in part by NSF contract GP-11509.

RUTGERS UNIVERSITY