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CRITERIA FOR BANACH SPACES

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It is well known in euclidean geometry that the quadrilateral obtained from an arbitrary quadrilateral by joining its midpoints is a parallelogram. The purpose of this paper is to show that a complete metric space with a unique metric line joining any pair of its distinct points is a Banach space if and only if it has the above mentioned property.

Let p, q, r, and s be distinct points in a Banach space such that no three are linear let m_1, m_2, m_3, and m_4 be the midpoints of the algebraic segments joining p and q, q and r, r and s, and s and p, respectively. It is well known that $m_3 - m_2 = m_4 - m_1$ and $m_2 - m_1 = m_3 - m_4$. In Euclidean space one usually refers to this result by saying that the midpoints m_1, m_2, m_3, m_4 form a parallelogram. If the Banach space does not have unique segments joining pairs of distinct points, then the restriction that the $\{m_i\}$ be midpoints of algebraic segments is easily seen to be necessary. We shall say that the metric space M satisfies the quadrilateral midpoint postulate provided that if p, q, r, s are points of M such that no three are linear and if m_1, m_2, m_3, m_4 are the respective midpoints, then $m_1m_2 = m_3m_4$ and $m_2m_3 = m_1m_4$. Hereafter we shall assume that M is a complete metric space with a unique metric line joining any pair of its distinct points and show that the Quadrilateral Midpoint Postulate characterizes the class of Banach spaces among such metric spaces.

The technique will be to show that a complete metric space with a unique metric line joining any pair of its distinct points satisfies the Quadrilateral Midpoint Postulate if and only if it satisfies the Young Postulate which may be stated as follows.

The Young Postulate. If p, q, r are points of a metric space M and q' and r' are the midpoints of p and q, and p and r, respectively, then $q'r' = qr/2$.

The result will then follow, for Andalafte and Blumenthal [1] have shown that a complete metric space with a unique metric line joining any pair of its distinct points is a Banach space if and only if it satisfies the Young Postulate.

That the Young Postulate implies the Quadrilateral Midpoint Postulate is almost immediate. For if a complete metric space with a metric line joining any pair of its distinct points satisfies the Young Postulate, then it is a Banach space and consequently satisfies the Quadrilateral Midpoint Postulate.

Suppose M satisfies the Quadrilateral Midpoint Postulate and p, q, r, are non-linear points of M with m_1, m_2 the midpoints of p and q, q and r, respectively.

LEMMA 1. *There exists a number k, depending only on p and r, such that if q, m_1, m_2 are as above, then $m_1m_2 = kpr$.*

Proof. Let s be a point such that no three of p, q, r, s are collinear, and let m_3, m_4 be the midpoints of the segments joining r and s, s and p, respectively. Let $k = m_3m_4/pr$. Then since M satisfies the quadrilateral midpoint property, $m_1m_2 = m_3m_4 = kpr$. We see immediately that k does not depend on q.

LEMMA 2. *The k in Lemma 1 is 1/2.*

Proof. Let $\{x_i\}$ be a sequence of points tending to x on the segment between p and r with $p \neq x \neq r$ and such that for each i we have p, x_i, r non-collinear. Let $\{p_i\}$ and $\{r_i\}$ be the sequences such that p_i and r_i are the midpoints of the segments determined by p and x_i, r and x_i, respectively. Then $\lim p_ix_i = 1/2 \lim px_i = 1/2\, px$ and similarly $\lim r_ix_i = 1/2\, rx$. This, along with the triangle inequality $p_ix_i + x_ir_i \geqq p_ir_i = kpr$, implies $k \leqq 1/2$. However, the inequality $pr \leqq pp_i + p_ir_i + r_ir = pp_i + kpr + r_ir$ and the aforementioned limits imply $k \geqq 1/2$. Hence k is 1/2.

THEOREM. *A complete metric space with a unique line joining any two of its distinct points is a normed linear space (Banach Space) if and only if it satisfies the Quadrilateral Midpoint Postulate.*

Proof. We have shown that the Quadrilateral Midpoint Postulate implies the Young Postulate; that is, if p' and r' are midpoints of p and q, and q and r, respectively, then $p'r' = (1/2)pr$. Thus an application of the Andalafte-Blumenthal result [1] completes the proof.

REFERENCES

1. E. Z. Andalafte and L. M. Blumenthal, *Metric characterization of banach and euclidean spaces*, Fund. Math., **LV** (1964), 23-55.

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