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ON THE UNIVALENCE OF SOME ANALYTIC FUNCTIONS

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Let

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k$$

and

$$g(z) = z + \sum_{k=n+1}^{\infty} a_k z^k$$

be analytic and satisfy

(a) $$\mathrm{Re}\,(f(z)/[\lambda f(z) + (1-\lambda)\,g(z)]) > 0$$

or

(b) $$|\, f(z)/[\lambda f(z) + (1-\lambda)\,g(z)] - 1\,| < 1$$
$$\textbf{for } |\,z\,| < 1,\ 0 \leqq \lambda < 1\ .$$

We propose to determine the values of R such that $f(z)$ is univalent and starlike for $|\,z\,| < R$ under the assumption (i) Re $(g(z)/z) > 0$, or (ii) Re $(zg'(z)/g(z)) > \alpha$, $0 \leqq \alpha < 1$.

We also consider the case when $n = 1$ and Re $(g(z)/z) > 1/2$ and show that under condition (a) $f(z)$ is univalent and starlike for $|\,z\,| < (1-\lambda)/(3+\lambda)$.

2. LEMMA 1. *If* $p(z) = 1 + b_n z^n + b_{n+1} z^{n+1} + \cdots$ *is analytic and satisfies* $\mathrm{Re}(p(z)) > \alpha$, $0 \leqq \alpha < 1$, *for* $|\,z\,| < 1$, *then*

(1) $$p(z) = [1 + (2\alpha - 1)z^n u(z)]/[1 + z^n u(z)]\ , \quad \textit{for } |\,z\,| < 1\ ,$$

where $u(z)$ *is analytic and* $|\,u(z)\,| \leqq 1$ *for* $|\,z\,| < 1$.

Proof. Let

(2) $$F(z) = [p(z) - \alpha]/(1 - \alpha) = 1 + c_n z^n + c_{n+1} z^{n+1} + \cdots\ .$$

$F(z)$ is analytic and Re $(F(z)) > 0$ for $|\,z\,| < 1$ and hence

(3) $$h(z) = [1 - F(z)]/[1 + F(z)] = d_n z^n + d_{n+1} z^{n+1} + \cdots\ ,$$

is analytic and $|\,h(z)\,| < 1$ for $|\,z\,| < 1$. Thus, by Schwarz's lemma

(4) $$h(z) = z^n u(z)\ ,$$

where $u(z)$ is analytic and $|\,u(z)\,| \leqq 1$ for $|\,z\,| < 1$. Now equations (2), (3) and (4) prove (1).

LEMMA 2. *Under the hypothesis of Lemma* 1 *we have for* $|\,z\,| < 1$

$|zp'(z)/p(z)| \leqq 2nz^n(1-\alpha)/\{(1-|z|^n)[1+(1-2\alpha)|z|^n]\}$.

Proof. Proceeding as in the proof of Lemma 1, we have in view of (3) and a result of Goluzin [1] that for $|z|<1$

$$|h'(z)| \leqq n|z|^{n-1}(1-|h(z)|^2)/(1-|z|^{2n}) . \tag{5}$$

Using (3), the inequality (5) takes the form

$$|F'(z)| \leqq 2n|z|^{n-1}\operatorname{Re}(F(z))/(1-|z|^{2n}) .$$

Hence, in view of (2),

$$|p'(z)| \leqq 2n|z|^{n-1}[\operatorname{Re}(p(z))-\alpha]/(1-|z|^{2n}) \tag{6}$$

or,

$$|zp'(z)/p(z)| \leqq 2n|z|^n(1-\alpha/(|p(z)|)/(1-|z|^{2n}) . \tag{7}$$

Equation (4) gives

$$|h(z)| \leqq |z|^n \qquad \text{for } |z|<1 , \tag{8}$$

and hence, by virtue of (3),

$$|F(z)| \leqq (1+|z|^n)/(1-|z|^n) \qquad \text{for } |z|<1 . \tag{9}$$

From (2) and (9),

$$\begin{aligned} |p(z)| &= |\alpha+(1-\alpha)F(z)| \\ &\leqq \alpha+(1-\alpha)|F(z)| \\ &\leqq [1+(1-2\alpha)|z|^n]/(1-|z|^n) . \end{aligned}$$

The inequality (7), because of the last inequality, reduces to

$$|zp'(z)/p(z)| \leqq 2n|z|^n(1-\alpha)/\{(1-|z|^n)[1+(1-2\alpha)|z|^n]\} \text{ for } |z|<1$$

and this completes the proof.

We remark that in the case $\alpha=0$, the above lemma reduces to a result of MacGregor [2; Lemma 1] and the inequality (6) with $\alpha=0$, $n=1$, gives another result of MacGregor [2, Lemma 2].

LEMMA 3. *Under the hypothesis of Lemma* 1 *we have for* $|z|<1$ $\operatorname{Re}(p(z)) \geqq [1+(2\alpha-1)|z|^n]/(1+|z|^n)$.

Proof. We have from equation (3), $F(z)=[1-h(z)]/[1+h(z)]$ and also from (8), $|h(z)| \leqq |z|^n$ for $|z|<1$. Hence the image of $|z|<r\ (0<r<1)$ under $F(z)$ lies in the interior of the circle with the line segment joining the points $(1-r^n)/(1+r^n)$ and $(1+r^n)/(1-r^n)$ as a diameter. Consequently $\operatorname{Re}(F(z)) \geqq (1-|z|^n)/(1+|z|^n)$ for

$|z| < 1$. The result now follows from the last inequality involving $F(z)$ and equation (2).

LEMMA 4. ([6]). *If* $h(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \cdots$ *is analytic and* $\operatorname{Re}(h(z)) > 0$ *for* $|z| < 1$, *then*

$$[1 - \lambda |h(z)|]^{-1} \leqq (1 - |z|^n)/[(1 - |z|^n) - \lambda(1 + |z|^n)]$$

for $|z| < [(1 - \lambda)/(1 + \lambda)]^{1/n}$, *where* $0 \leqq \lambda < 1$.

3. THEOREM 1. *Suppose that* $f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \cdots$, *and* $g(z) = z + b_{n+1} z^{n+1} + b_{n+2} z^{n+2} + \cdots$ *are analytic and* $\operatorname{Re}(g(z)/z) > 0$ *for* $|z| < 1$. *If* $\operatorname{Re}(f(z)/[\lambda f(z) + (1 - \lambda) g(z)]) > 0$, $0 \leqq \lambda < 1$, *for* $|z| < 1$, *then* $f(z)$ *is univalent and starlike for* $|z| < R^{1/n}$, *where* $R = \{[(2n + \lambda - n\lambda)^2 + (1 - \lambda^2)]^{1/2} - (2n + \lambda - n\lambda)\}/(1 + \lambda)$.

Proof. Let

$$h(z) = f(z)/[\lambda f(z) + (1 - \lambda) g(z)] = 1 + c_n z^n + c_{n+1} z^{n+1} + \cdots ,$$

then $h(z)$ is analytic and $\operatorname{Re}(h(z)) > 0$ for $|z| < 1$. Now

$$f(z)[1 - \lambda h(z)] = (1 - \lambda) h(z) z p(z) , \tag{10}$$

where $p(z) = g(z)/z = 1 + b_{n+1} z^n + b_{n+2} z^{n+1} + \cdots$. Multiplying the logarithmic derivative of both sides of equation (10) by z we have

$$z f'(z)/f(z) = 1 + z p'(z)/p(z) + z h'(z)/\{h(z)[1 - \lambda h(z)]\} . \tag{11}$$

Equation (11) is valid for those z for which $1 - \lambda h(z) \neq 0$ and $|z| < 1$. Since $|h(z)| \leqq (1 + |z|^n)/(1 - |z|^n)$, $1 - \lambda h(z) \neq 0$ in particular if $|z| < [(1 - \lambda)/(1 + \lambda)]^{1/n}$. Now from equation (11), we have

$$|z f'(z)/f(z) - 1| \leqq |z p'(z)/p(z)| + |z h'(z)/h(z)| \, |1 - \lambda h(z)|^{-1}$$

and by using Lemma 2 with $\alpha = 0$ and Lemma 4, this gives

$$\begin{aligned} |z f'(z)/f(z) - 1| &\leqq \frac{2n|z|^n}{1 - |z|^{2n}} + \frac{2n|z|^n}{(1 - |z|^{2n}) - \lambda(1 + |z|^n)^2} , \\ &= \frac{2n|z|^n[(1 - |z|^n) - \lambda(1 + |z|^n) + (1 - |z|^n)]}{(1 - |z|^{2n})[(1 - |z|^n) - \lambda(1 + |z|^n)]} \end{aligned} \tag{12}$$

provided that $|z| < [(1 - \lambda)/(1 + \lambda)]^{1/n}$.

The fact that $|z f'(z)/f(z) - 1| < 1$ implies that $\operatorname{Re}(z f'(z)/f(z)) > 0$, it follows from the inequality (12) that $\operatorname{Re}(z f'(z)/f(z) > 0$ if

$$|z| < [(1 - \lambda)/(1 + \lambda)]^{1/n}$$

and if

$$G(|z|^n) \equiv (1+\lambda)|z|^{3n} + (4n + 2n\lambda + \lambda - 1)|z|^{2n} + (2n\lambda - 4n - \lambda - 1)|z|^n + (1-\lambda) > 0 . \tag{13}$$

Let $|z|^n = t$ and consider the cubic polynomial $G(t)$ for $0 \leqq t \leqq 1$. $G(t)$ has at most two positive zeros. Since $G(0) = (1-\lambda) > 0$, $G[(1-\lambda)/(1+\lambda)] = -4\lambda n(1-\lambda)/(1+\lambda)^2 < 0$ and $G(1) = 4\lambda n > 0$, it follows that $G(t_1) = 0$ for some t_1 such that $0 < t_1 < (1-\lambda)/(1+\lambda)$ and $G(t) > 0$ for $0 \leqq t < t_1$ and $G(t) < 0$ for $t_1 < t < (1-\lambda)/(1+\lambda)$. Hence $\text{Re}\,(zf'(z)/f(z)) > 0$ for those z for which only the inequality (13) is true. Now the inequality (13) holds if, in particular

$$(1+\lambda)|z|^{3n} + (4n - 2n\lambda + \lambda - 1)|z|^{2n} + (2n\lambda - 4n - \lambda - 1)|z|^n + (1-\lambda) > 0$$

or,

$$(|z|^n - 1)\,[(1+\lambda)|z|^{2n} + (4n - 2n\lambda + 2\lambda)|z|^n + (\lambda - 1)] > 0$$

or,

$$(1+\lambda)|z|^{2n} + (4n - 2n\lambda + 2\lambda)|z|^n + (\lambda - 1) < 0 .$$

The last inequality holds if

$$|z|^n < \{[(2n + \lambda - n\lambda)^2 + (1-\lambda^2)]^{1/2} - (2n + \lambda - n\lambda)\}/(1+\lambda) . \tag{14}$$

Since $f(z)$ is univalent and starlike for those z for which

$$\text{Re}\,(zf'(z)/f(z)) > 0 ,$$

we have that $f(z)$ is univalent and starlike for $|z| < R^{1/n}$, where R is the right side of (14).

If we put $\lambda = 0$ in Theorem 1 we obtain the following result which, when $n = 1$, reduces to a result of Ratti [5, Theorem 1].

COROLLARY 1. *Suppose that* $f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2}\cdots$, *and* $g(z) = z + b_{n+1}z^{n+1} + b_{n+2}z^{n+2} + \cdots$ *are analytic and* $\text{Re}\,(g(z)/z) > 0$ *for* $|z| < 1$. *If* $\text{Re}\,(f(z)/g(z)) > 0$ *for* $|z| < 1$ *then* $f(z)$ *is univalent and starlike for* $|z| < [(4n^2+1)^{1/2} - 2n]^{1/n}$.

The functions $f(z) = z(1-z^n)^2/(1+z^n)^2$ and $g(z) = z(1-z^n)/(1+z^n)$ satisfy the hypothesis of Corollary 1 and it is easy to see that the derivative of $f(z)$ vanishes at $z = [(4n^2+1)^{1/2} - 2n]^{1/n}$ and hence $[(4n^2+1)^{1/2} - 2n]^{1/n}$ is in fact the radius of univalence for such functions $f(z)$. This shows that Corollary 1 is sharp and hence Theorem 1 is sharp at least for $\lambda = 0$.

THEOREM 2. *Suppose* $f(z) = z + a_2z^2 + \cdots$, *and*

$$g(z) = z + b_2 z^2 + \cdots$$

are analytic for $|z| < 1$ *and* $\mathrm{Re}\,(g(z)/z) > 1/2$ *for* $|z| < 1$. *If*

$$\mathrm{Re}\,(f(z)/[\lambda f(z) + (1-\lambda)g(z)]) > 0 \qquad \text{for } |z| < 1$$

then $f(z)$ *is univalent and starlike for* $|z| < (1-\lambda)/(3+\lambda)$.

Proof. Let $h(z) = f(z)/[\lambda f(z) + (1-\lambda)g(z)] = 1 + c_1 z + c_2 z^2 + \cdots$. Now $h(z)$ is analytic and $\mathrm{Re}\,(h(z)) > 0$ for $|z| < 1$ and

$$f(z)\,[1 - \lambda h(z)] = (1-\lambda)h(z)g(z)\,. \tag{15}$$

If we let $g(z) = zp(z)$, then by applying Lemma 1 with $\alpha = 1/2$ and $n = 1$ we have that $p(z) = [1 + zu(z)]^{-1}$, where $u(z)$ is analytic and $|u(z)| \leqq 1$ for $|z| < 1$. Equation (15) now reduces to

$$f(z)\,[1 - \lambda h(z)] = (1-\lambda)zh(z)/[1 + zu(z)]\,.$$

Hence

$$\frac{zf'(z)}{f(z)} = \frac{1 - z^2u'(z)}{1 + zu(z)} + \frac{zh'(z)}{h(z)\,[1 - \lambda h(z)]}$$

and

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) \geqq \mathrm{Re}\left(\frac{1 - z^2u'(z)}{1 + zu(z)}\right) - \frac{|zh'(z)/h(z)|}{|1 - \lambda h(z)|}\,.$$

Using Lemmas 2 and 4 with $n = 1$, we get

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) \geqq \mathrm{Re}\left(\frac{1 - z^2u'(z)}{1 + zu(z)}\right) - \frac{2|z|}{(1 - |z|^2) - \lambda(1 + |z|)^2}$$

for $|z| < (1-\lambda)/(1+\lambda)$.

Hence $\mathrm{Re}\,(zf'(z)/f(z)) > 0$ if $|z| < (1-\lambda)/(1+\lambda)$ and

$$T(|z|)\,\mathrm{Re}\,[(1 - z^2u'(z))(1 + \overline{zu(z)})] - 2|z|\,\mathrm{Re}\,[(1 + zu(z))(1 + \overline{zu(z)})] > 0\,,$$

where $T(|z|) = (1 - |z|^2) - \lambda(1 + |z|)^2$. The last inequality holds if

$$\begin{aligned} &T(|z|)\,\mathrm{Re}\,(1 + \overline{zu(z)}) - T(|z|)\,\mathrm{Re}\,[z^2u'(z)(1 + \overline{zu(z)})] \\ &\quad + 2|z|\,\mathrm{Re}\,[(1 - zu(z))(1 + \overline{zu(z)})] - 4|z|\,\mathrm{Re}\,(1 + \overline{zu(z)}) > 0\,, \end{aligned}$$

or if

$$\begin{aligned} &[4|z| - T(|z|)]\,\mathrm{Re}\,(1 + \overline{zu(z)}) + T(|z|)\,\mathrm{Re}\,[z^2u'(z)(1 + \overline{zu(z)})] \\ &< 2|z|\,(1 - |z|^2\,|u(z)|^2) \end{aligned}$$

or

$$| 4 | z | - T(| z |) | (1 + | z | | u(z) |) + T(| z |) | z |^2 | u'(z) | (1 + | z | | u(z) |) < 2 | z | (1 - | z |^2 | u(z) |^2) .$$

This inequality holds, in view of (5) with $n = 1$ if

$$| 4 | z | - T(| z |) | + T(| z |) | z |^2 (1 - | u(z) |^2)(1 - | z |^2)^{-1} < 2 | z | (1 - | z | | u(z) |) . \tag{16}$$

Two cases arise according as $4 | z | - T(| z |)$ is nonnegative or not.

Case 1. $4 | z | - T(| z |) \geqq 0$, i.e. $| z | \geqq [(4\lambda + 5)^{1/2} - (\lambda + 2)]/(1 + \lambda)$. Since $[(4\lambda + 5)^{1/2} - (\lambda + 2)] < (1 - \lambda)$ for $0 \leqq \lambda < 1$, it follows, in view of inequality (16), that $\operatorname{Re}(zf'(z)/f(z)) > 0$ for those z for which $[(4\lambda + 5)^{1/2} - (\lambda + 2)]/(1 + \lambda) \leqq | z | < (1 - \lambda)/(1 + \lambda)$ and

$$4 | z | - T(| z |) + T(| z |) | z |^2 (1 - | u(z) |^2)(1 - | u(z) |^2)^{-1} < 2 | z | (1 - | z | | u(z) |) .$$

The last inequality holds, because of the original value of $T(| z |)$, if

$$2 | z | + 2 | z |^2 - 1 + \lambda(1 + | z |)^2 - \lambda | z |^2(1 + | z |)/(1 - | z |) < | z |^2 | u(z) |^2 - \lambda| z |^2 | u(z) |^2 (1 + | z |)/(1 - | z |) - 2 | z |^2 | u(z) | . \tag{17}$$

Since $| u(z) | \leqq 1$, the right side of inequality (17)

$$\geqq | z |^2 | u(z) |^2 - 2 | z |^2 | u(z) | - \lambda | z |^2 (1 + | z |)/(1 - | z |) .$$

Hence inequality (17) holds, if in particular

$$2 | z | + 2 | z |^2 - 1 + \lambda(1 + | z |)^2 < | z |^2 | u(z) |^2 - 2 | z |^2 | u(z) | . \tag{18}$$

If we let $F(x) = x^2 | z |^2 - 2x | z |^2$, where $x = | u(z) |$, $0 \leqq x \leqq 1$, then $F(x)$ is a decreasing function of x for $0 \leqq x \leqq 1$, and hence

$$F(x) \geqq F(1) = -| z |^2 \qquad \text{for } 0 \leqq x \leqq 1 .$$

Hence inequality (18) holds if $2 | z | + 2 | z |^2 - 1 + \lambda(1 + | z |)^2 < -| z |^2$ or $(3 | z | - 1)(| z | + 1) + \lambda(1 + | z |)^2 < 0$ or $3 | z | - 1 + \lambda(1 + | z |) < 0$ or if $| z | < (1 - \lambda)/(3 + \lambda)$. Since $(1 - \lambda)/(3 + \lambda) < (1 - \lambda)/(1 + \lambda)$, we have shown that

$$\operatorname{Re}(zf'(z)/f(z)) > 0 \quad \text{for } [(4\lambda + 5)^{1/2} - (\lambda + 2)]/(1 + \lambda) \leqq | z | < (1 - \lambda)/(3 + \lambda) . \tag{19}$$

Case 2. $4 | z | - T(| z |) < 0$, i.e. $| z | < [(4\lambda + 5)^{1/2} - (\lambda + 2)]/(1 + \lambda)$. We intend to show that $\operatorname{Re}(zf'(z)/f(z)) > 0$ in this case also. Since $f(z)$ and $g(z)$ satisfy, in particular, the hypothesis of Theorem 1 with $n = 1$, it follows from Theorem 1 that

$$\mathrm{Re}\,(zf'(z)/f(z)) > 0 \quad \text{for } |z| < [(5-\lambda^2)^{1/2} - 2]/(1+\lambda)\,.$$

It is easy to see that

$$[(4\lambda+5)^{1/2} - (\lambda+2)] \leqq (5-\lambda^2)^{1/2} - 2 \quad \text{for } 0 \leqq \lambda \leqq 1$$

and hence in particular

$$\mathrm{Re}\,(zf'(z)/f(z)) > 0 \quad \text{for } |z| < [(4\lambda+5)^{1/2} - (\lambda+2)]/(1+\lambda)\,.$$

In view of the above and (19), it now follows that $f(z)$ is univalent and starlike for $|z| < (1-\lambda)/(3+\lambda)$ and this completes the proof.

For $\lambda = 0$ the above result reduces to a result of Ratti [5, Theorem 2] and improves a result of MacGregor [2, Theorem 4] since $\mathrm{Re}\,(g(z)/z) > 1/2$ does not necessarily imply that $g(z)$ is convex [7]. The functions $f(z) = z(1-z)/(1+z)^2$ and $g(z) = z/(1+z)$ satisfy the hypothesis of Theorem 2 with $\lambda = 0$ and $f(z)$ is univalent in no circle $|z| < r$ with $r > 1/3$ since $f'(z)$ vanishes at $z = 1/3$. This shows that Theorem 2 is sharp at least for $\lambda = 0$.

A function $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ is said to be starlike of order α, $0 \leqq \alpha < 1$, for $|z| < 1$ if $\mathrm{Re}\,(zf'(z)/f(z)) > \alpha$ for $|z| < 1$, we now prove the following result.

THEOREM 3. *Let* $f(z) = z + \sum_{k=n+1}^{\infty} b_k z^k$ *and* $g(z) = z + \sum_{k=n+1}^{\infty} b_k z^k$ *be analytic for* $|z| < 1$ *and* $g(z)$ *be starlike of order* α, $0 \leqq \alpha < 1$, *for* $|z| < 1$. *If* $\mathrm{Re}\,(f(z)/[\lambda f(z) + (1-\lambda)g(z)]) > 0$ *for* $|z| < 1$, *then* $f(z)$ *is univalent and starlike for*

(i) $\qquad |z| < [(1-\lambda)/(1+\lambda+2n)]^{1/n} \qquad$ *if* $\alpha = 1/2$;

and

(ii) $\qquad |z| < R^{1/n}\,, \qquad$ *if* $\alpha \neq 1/2$,

where

$$R = \{[A^2 + 4(1-\lambda^2)(2\alpha-1)]^{1/2} - A\}/[2(1+\lambda)(2\alpha-1)]$$

with $A = 2n + \lambda + 1 - (2\alpha-1)(1-\lambda)$.

Proof. Proceeding as in the proof of Theorem 1 we get

$$\mathrm{Re}\,(zf'(z)/f(z)) \geqq \mathrm{Re}\,(zg'(z)/g(z)) - |zh'(z)/h(z)|\,|1-\lambda h(z)|^{-1}\,.$$

Applying Lemma 3 (to $zg'(z)/g(z)$) and Lemmas 2 and 4 we get,

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) \geqq \frac{1+(2\alpha-1)|z|^n}{1+|z|^n} - \frac{2n|z|^n}{(1-|z|^{2n}) - \lambda(1+|z|^n)^2} \tag{20}$$

provided that $|z| < [(1-\lambda)/(1+\lambda)]^{1/n}$.

Hence $\operatorname{Re}(zf'(z)/f(z)) > 0$ for those z for which $|z| < [1-\lambda)/(1+\lambda)]^{1/n}$ and the right side of inequality (20) is greater than zero. The latter holds if

$$\begin{aligned} G(|z|^n) \equiv (1+\lambda)(2\alpha-1)|z|^{2n} \\ + [2n+\lambda+1-(2\alpha-1)(1-\lambda)]|z|^n - (1-\lambda) < 0 \,. \end{aligned} \tag{21}$$

Let $|z|^n = t$ and consider the quadratic $G(t)$ for $0 \leqq t \leqq 1$. Since $G(0) = \lambda - 1 < 0$, $G[(1-\lambda)/(1+\lambda)] = 2n(1-\lambda)/(1+\lambda) > 0$, it follows that $G(t_1) = 0$ for some t_1 such that $0 < t_1 < (1-\lambda)/(1+\lambda)$ and $G(t) < 0$ for $0 \leqq t < t_1$ and $G(t) > 0$ for $t_1 < t < (1-\lambda)/(1+\lambda)$. Hence $f(z)$ is univalent and starlike for those z for which only the inequality (21) holds. Now the inequality (21) holds if

$$|z| < [(1-\lambda)/(1+\lambda+2n)]^{1/n}$$

when $\alpha = 1/2$ and

$$|z| < \{[A^2 + 4(1-\lambda^2)(2\alpha-1)]^{1/2} - A\}^{1/n}/[2(1+\lambda)(2\alpha-1)]^{1/n}$$

when $\alpha \neq 1/2$, where $A = 2n + \lambda + 1 - (2\alpha-1)(1-\lambda)$ and this completes the proof.

If we put $\lambda = 0, n = 1$ and $\alpha = 0$ in the above result then we see that $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ under the modified hypothesis is univalent and starlike for $|z| < 2 - \sqrt{3}$, a result obtained by MacGregor [2, Theorem 3]. On the other hand if $\lambda = 0$ and $n = 1$, Theorem 3 reduces to a result of Ratti [5, Theorem 3]. The functions

$$f(z) = z(1-z^n)/(1+z^n)^{\frac{2-2\alpha}{n}+1} \quad \text{and} \quad g(z) = z/(1+z^n)^{\frac{2-2\alpha}{n}}$$

show that Theorem 3 is sharp at least for $\lambda = 0$ and arbitrary n, since the derivative of $f(z)$ vanishes at

$$z = \{[(n+1-\alpha) - ((n+1-\alpha)^2 - (1-2\alpha))^{1/2}]/(1-2\alpha)\}^{1/n}$$

for $\alpha \neq 1/2$ and at $z = -1/(2n+1)$ when $\alpha = 1/2$.

4. Let $S(R)$ denote the functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ which are analytic and satisfy $|zf'(z)/f(z) - 1| < 1$ for $|z| < R$. Obviously every member of $S(R)$ is univalent and starlike for $|z| < R$. We now prove the following result.

THEOREM 4. *Let* $f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \cdots$, *and* $g(z) = z + b_{n+1}z^{n+1} + b_{n+2}z^{n+2} + \cdots$ *be analytic and satisfy* $\operatorname{Re}(g(z)/z) > 0$ *for* $|z| < 1$. *If* $|f(z)/[\lambda f(z) + (1-\lambda)g(z)] - 1| < 1$, $0 \leqq \lambda < 1$, *for* $|z| < 1$, *then* $f(z) \in S(R^{1/n})$, *where R is the smallest positive root of the equation* $(2n\lambda + \lambda - n - 1)R^2 - (3n + \lambda - 2n\lambda)R + (1-\lambda) = 0$.

Proof. Let

$$h(z) = f(z)/[\lambda f(z) + (1-\lambda)g(z)] - 1 = c_n z^n + c_{n+1}z^{n+1} + \cdots . \tag{22}$$

By hypothesis, $h(z)$ is analytic and $|h(z)| < 1$ for $|z| < 1$ and hence by a result of Goluzin [1] we have that for $|z| < 1$

$$|h'(z)| \leqq n|z|^{n-1}(1 - |h(z)|^2)/(1 - |z|^{2n}) \tag{23}$$

and by Schwarz's lemma for $|z| < 1$

$$|h(z)| \leqq |z|^n . \tag{24}$$

If we let $g(z) = zp(z)$, then we have from (22)

$$f(z)[1 - \lambda - \lambda h(z)] = (1-\lambda)zp(z)[1 + h(z)] .$$

Hence,

$$\frac{zf'(z)}{f(z)} = 1 + \frac{zp'(z)}{p(z)} + \frac{zh'(z)}{[1 + h(z)][1 - \lambda - \lambda h(z)]}$$

and this gives

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leqq \left|\frac{zp'(z)}{p(z)}\right| + \frac{|zh'(z)|}{|1 + h(z)||1 - \lambda - \lambda h(z)|} .$$

Applying Lemma 2, with $\alpha = 0$, we get, in view of (23), for $|z| < 1$

$$\begin{aligned}\left|\frac{zf'(z)}{f(z)} - 1\right| &\leqq \frac{2n|z|^n}{1 - |z|^{2n}} + \frac{n|z|^n(1 - |h(z)|^2)}{(1 - |z|^{2n})\,|1 + h(z)|\,|1 - \lambda - \lambda h(z)|} \\ &\leqq \frac{2n|z|^n}{1 - |z|^{2n}} + \frac{n|z|^n(1 + |h(z)|)}{(1 - |z|^{2n})\,|1 - \lambda - \lambda h(z)|}\end{aligned}$$

by using (24), we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leqq \frac{2n|z|^n}{1 - |z|^{2n}} + \frac{n|z|^n}{(1 - |z|^n)(1 - \lambda - \lambda|z|^n)}$$

valid for $|z| < [(1-\lambda)/\lambda]^{1/n}$. Hence $|zf'(z)/f(z) - 1| < 1$ if

$$|z| < [(1-\lambda)/\lambda]^{1/n}$$

and

$$2n|z|^n(1 - \lambda - \lambda|z|^n) + n|z|^n(1 + |z|^n) < (1 - |z|^{2n})(1 - \lambda - \lambda|z|^n) .$$

The last inequality holds if

$$\begin{aligned} G(|z|^n) \equiv \lambda|z|^{3n} &+ (2n\lambda + \lambda - n - 1)|z|^{2n} \\ &- (3n + \lambda - 2n\lambda)|z|^n + (1 - \lambda) > 0 .\end{aligned} \tag{25}$$

Let $|z|^n = t$ and consider the cubic polynomial $G(t)$ for $0 \leqq t \leqq 1$.

$G(t)$ has at most two positive zeros. Since $G(0) = (1 - \lambda) > 0$ and $G((1-\lambda)/\lambda) = -(n(1-\lambda)/\lambda^2 < 0$, it follows that $G(t_1) = 0$ for some t_1 such that $0 < t_1 < (1-\lambda)/\lambda$ and $G(t) > 0$ for $0 \leqq t < t_1$ and $G(t) < 0$ for some values of t between t_1 and $(1-\lambda)/\lambda$. Hence

$$|zf'(z)/f(z) - 1| < 1$$

for those values of z for which only the inequality (25) holds. Now inequality (25) holds if, in particular

$$(2n\lambda + \lambda - n - 1)|z|^{2n} - (3n + \lambda - 2n\lambda)|z|^n + (1-\lambda) > 0$$

and this completes the proof.

If we set $\lambda = 0$ and $n = 1$ in the above result we have the following.

COROLLARY 2. *Suppose* $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ *and* $g(z) = z + b_2z^2 + b_3z^3 + \cdots$ *are analytic and satisfy* $\operatorname{Re}(g(z)/z) > 0$ *for* $|z| < 1$. *If* $|f(z)/g(z) - 1| < 1$ *for* $|z| < 1$, *then* $|zf'(z)/f(z) - 1| < 1$ *for* $|z| < 1/4(\sqrt{17} - 3)$.

It may be noted that Corollary 2 implies, in particular, that $f(z)$ is univalent and starlike for $|z| < 1/4(\sqrt{17} - 3)$ and hence includes a result of Ratti [5, Theorem 4]. If we take $f(z) = z(1-z^n)^2/(1+z^n)$ and $g(z) = z(1-z^n)/(1+z^n)$, it is easy to see that these functions satisfy the hypothesis of Theorem 4 with $\lambda = 0$. We see that $f'(z)$ vanishes at $z_0 = [-3n + (9n^2 + 4n + 4)^{1/2}]/(2n + 2)$ and hence

$$|z_0f'(z_0)/f(z_0) - 1| = 1 .$$

This shows that Theorem 4 is sharp for at least $\lambda = 0$ and also that Corollary 2 is sharp.

THEOREM 5. *Let* $f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \cdots$ *and* $g(z) = z + b_{n+1}z^{n+1} + b_{n+2}z^{n+2} + \cdots$ *be analytic for* $|z| < 1$ *and* $g(z)$ *be starlike of order* α *for* $|z| < 1$, $0 \leqq \alpha < 1$. *If*

$$|f(z)/[\lambda f(z) + (1-\lambda)g(z)] - 1 < 1 , \ 0 \leqq \lambda < 1 , \ \textit{for} \ |z| < 1 ,$$

then $f(z)$ *is univalent and starlike for* $|z| < R^{1/n}$, *where* R *is the smallest positive root of the equation*

$$\begin{aligned}(2\alpha - 1)\lambda R^3 &- (n + 2\alpha - 1 - \lambda)R^2 \\ &+ (2\alpha - 2 - 2\alpha\lambda + \lambda - n)R + (1-\lambda) = 0 .\end{aligned} \tag{26}$$

Proof. Proceeding as in the proof of Theorem 4 we have

$$\frac{zf'(z)}{f(z)} = \frac{zg'(z)}{g(z)} + \frac{zh'(z)}{[1 + h(z)]\,[1 - \lambda - \lambda h(z)]}\ .$$

Hence,

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \geqq \operatorname{Re}\left(\frac{zg'(z)}{g(z)}\right) - \frac{|\,zh'(z)\,|}{|\,1 + h(z)\,|\,|\,1 - \lambda - \lambda h(z)\,|}\ .$$

Since $\operatorname{Re}(zg'(z)/g(z)) > \alpha$ and $zg'(z)/g(z) = 1 + c_n z^n + c_{n+1}z^{n+1} + \cdots$, we have by Lemma 3 and inequalities (23) and (24) that

$$\begin{aligned}\operatorname{Re}(zf'(z)/f(z)) \geqq\ & [1 + (2\alpha - 1)\,|\,z\,|^n]/(1 + |\,z\,|^n) \\ & - n\,|\,z\,|^n/[(1 - |\,z\,|^n)\,(1 - \lambda - \lambda\,|\,z\,|^n)]\end{aligned} \tag{27}$$

valid for $|\,z\,| < [(1 - \lambda)/\lambda]^{1/n}$.

Hence $\operatorname{Re}(zf'(z)/f(z)) > 0$ if $|\,z\,| < [(1 - \lambda)/\lambda]^{1/n}$ and if (in view of inequality (27))

$$\begin{aligned}G(|\,z\,|^n) \equiv\ & (2\alpha - \ )\,\lambda\,|\,z\,|^{3n} \\ & - (n + 2\alpha - 1 - \lambda)\,|\,z\,|^{2n} \\ & + (2\alpha - 2 - 2\alpha\lambda + \lambda - n)\,|\,z\,|^n \\ & + (1 - \lambda) > 0\ .\end{aligned} \tag{28}$$

Let $|\,z\,| = t$ and consider the cubic polynomial $G(t)$ for $0 \leqq t \leqq 1$. Since $G(0) = 1 - \lambda > 0$ and $G((1 - \lambda)/\lambda) = (-n(1 - \lambda))/\lambda^2 < 0$, it follows that $G(t_1) = 0$ for some t_1 such that $0 < t_1 < (1 - \lambda)/\lambda$ and $G(t) > 0$ for $0 \leqq t < t_1$ and $G(t) < 0$ for some t between t_1 and $(1 - \lambda)/\lambda$. Hence $f(z)$ is starlike and univalent for $|\,z\,| < R^{1/n}$, in view of inequality (28), where R is the smallest positive root of the equation (26).

The case when $\lambda = 0$ in Theorem 5 is of special interest. In this case equation (26) becomes

$$(n + 2\alpha - 1)R^2 - (2\alpha - 2 - n)R - 1 = 0$$

which gives $R = 1/3$ in case $\alpha = 0$ and $n = 1$ and

$$R = \{(2\alpha - 2 - n) + [(2\alpha - 2 - n)^2 + 4(n + 2\alpha - 1)]^{1/2}\}/[2(n + 2\alpha - 1)] \tag{29}$$

if $\alpha \neq 0$. This proves the following result, which includes a result of Ratti [5, Theorem 6].

COROLLARY 3. *Suppose* $f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \cdots$ *and* $g(z) = z + b_{n+1}z^{n+1} + b_{n+2}z^{n+2} + \cdots$ *are analytic for* $|\,z\,| < 1$ *and* $g(z)$ *is starlike of order* α *for* $|\,z\,| < 1$, $0 \leqq \alpha < 1$. *If* $|\,f(z)/g(z) - 1\,| < 1$ *for* $|\,z\,| < 1$ *then* $f(z)$ *is univalent and starlike for*

(i) $|\,z\,| < 1/3$ *if* $\alpha = 0$ *and* $n = 1$

(ii) $|z| < R^{1/n}$, *where* R *is given by* (29) *if* $\alpha \neq 0$.

It is easy to see that the functions $f(z) = z(1 - z^n)/(1 + z^n)^{(2-2\alpha)/n}$ and $g(z) = z/(1 + z^n)^{(2-2\alpha)/n}$ satisfy the hypothesis of Corollary 3 and also that the derivative of $f(z)$ vanishes at $z = 1/3$ if $\alpha = 0$ and $n = 1$, and at $z = \{[(n + 2 - 2\alpha)^2 + 4(n + 2\alpha - 1)]^{1/2} - (n + 2 - 2\alpha)\}^{1/n}/[2(n + 2\alpha - 1)]^{1/n}$ if $\alpha \neq 0$. This shows that Corollary 3 is sharp.

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