PACIFIC JOURNAL OF MATHEMATICS
Vol. 43, No. 1, 1972

COMPLEMENTATION IN THE LATTICE OF REGULAR TOPOLOGIES

M. JEANETTE HUEBENER

The present paper is concerned with the lattice of regular topologies on a set, and establishes the following results: a complete, complemented sublattice of the lattice of regular topologies on a set is exhibited and shown to be anti-isomorphic to the lattice of equivalence relations on the set; the lattice of regular topologies on a set is shown to be nonmodular if the cardinality of the set is at least four; the problem of complementation for regular topologies is reduced to considering T_0 regular topologies without isolated points; conditions are found which are equivalent to a regular topology having a principal regular complement; then follow some conditions under which the problem can be reduced to considering connected spaces; the final section consists of constructions of complements for certain classes of regular topologies, which classes may or may not be exhaustive.

Principal regular topologies and relations. Let $(\mathscr{S}, \vee, \wedge)$ be the lattice of all topologies on a set E. $(\mathscr{S}, \vee, \wedge)$ is complete, anti-atomic, complemented, and, if $|E|$, the cardinality of E, is at least three, it is not modular, [10, pp. 384–5, 389–397]. Next, let $(\mathscr{R}, \vee, \wedge^r)$ be the lattice of all regular topologies on E. $(\mathscr{R}, \vee, \wedge^r)$ is complete but not a sublattice of $(\mathscr{S}, \vee, \wedge)$. The greatest lower bound in $\mathscr{R}$ of a collection of topologies in $\mathscr{R}$ is only the least upper bound of all the regular topologies which are weaker than the collection's greatest lower bound in $\mathscr{S}$ [8, pp. 754–755].

The anti-atoms of $\mathscr{S}$ are the ultraspaces on E; these are topologies of the form $\mathfrak{S}(x, \mathscr{U}) = P_c(x) \cup \mathscr{U}$ where $\mathscr{U}$ is an ultrafilter on E different from $\mathscr{U}(x) = \{A \subset E: x \in A\}$ and where $P_c(x) = \{A \subset E: x \notin A\}$. Frohlich [5, p. 81, Satz 3] showed that every topology τ on E is the infimum of the ultraspaces on E which are finer than τ.

The special sublattice of $(\mathscr{S}, \vee, \wedge)$, which is anti-isomorphic to the lattice of preorders on E, is called the lattice of principal topologies. From this sublattice Steiner [10, p. 383, Theorem 2.6; pp. 389–397] and van Rooij [16, p. 807] take their complements. Now an ultraspace is said to be principal if its topology is of the form $\mathfrak{S}(x, \mathscr{U}(y))$ where $x \neq y$. A topology τ is principal if $\tau = 1$, or if τ is the infimum of the principal ultratopologies finer than τ. These topologies are also characterized [10, pp. 381–2, Theorem 2.3] by the fact that they have a base of open sets which is minimal at each point, i.e. for any $x \in E$ every open set containing x must contain the open set

$$B_x = \{y \in E: \mathfrak{S}(x, \mathscr{U}(y)) \geqq \tau\} .$$

(Throughout the paper B_x in a principal topology σ will denote the σ-open set minimal at the point x.) Using this characterization it is easily seen [10, p. 382, Theorem 2.5] that the principal topologies form a sublattice of $(\mathscr{S}, \bigvee, \bigwedge)$. The mapping establishing the anti-isomorphism between this lattice and the lattice of preorders is given by

$$\eta(\tau) = G_\tau = \{(x, y): \mathfrak{S}(x, \mathscr{U}(y)) \geqq \tau\}$$

and

$$\eta^{-1}(G) = \tau_G = \bigwedge \{\mathfrak{S}(x, \mathscr{U}(p)): (x, y) \in G\} .$$

In the lattice of regular topologies there is a sublattice of the lattice of principal topologies which has a familiar structure:

THEOREM 1.1. *A principal topology τ on E is regular iff its representation satisfies the condition $\mathfrak{S}(x, \mathscr{U}(y)) \geqq \tau$ implies $\mathfrak{S}(y, \mathscr{U}(x)) \geqq \tau$ for any $x, y \in E$.*

Proof. Suppose τ is principal and regular and that $\mathfrak{S}(x, \mathscr{U}(y)) \geqq \tau$. Then $y \in B_x$ and $B_y \subset B_x$. Now $\sim B_y$ is a closed set not containing y; accordingly there exists $U \in \tau$ such that $U \supset \sim B_y$ and $U \cap B_y = \varnothing$ which implies that $U = \sim B_y \in \tau$. If $x \in \sim B_y \in \tau$, then $B_y \subset B_x \subset \sim B_y$ which is a contradiction. Hence $x \in B_y$ and $\mathfrak{S}(y, \mathscr{U}(x)) \geqq \tau$.

Conversely, in terms of the base of minimal open sets, the condition, $\mathfrak{S}(x, \mathscr{U}(y)) \geqq \tau$ implies $\mathfrak{S}(y, \mathscr{U}(x)) \geqq \tau$ for any $x, y \in E$, become $y \in B_x$ iff $x \in B_y$. Hence $B_x = B_y$ or $B_x \cap B_y = \varnothing$ for every $x, y \in E$. In which case, if $U = \bigcup \{B_y: y \in U\} \in \tau$ and $x \in \sim U$ then $B_x \cap U = \varnothing$ and it follows that $\sim U = \bigcup \{B_x: x \in \sim U\} \in \tau$. Every open set being closed implies τ is regular.

COROLLARY 1.2. *A principal topology τ is regular iff G_τ is an equivalence relation.*

That the lattice of equivalence relations is complemented is proven *mot a mot* as in Steiner [10, p. 389, Theorem 5.1].

COROLLARY 1.3. *The lattice of principal regular topologies on E is a complete sublattice of $(\mathscr{R}, \bigvee, \bigwedge^r)$ and $(\mathscr{S}, \bigvee, \bigwedge)$.*

Finally, for $|E| \leqq 3$ the lattice $(\mathscr{R}, \vee, \wedge^r)$ is a modular sublattice of $(\mathscr{S}, \vee, \wedge)$. If $|E| \geqq 4$, then the lattice $(\mathscr{R}, \vee, \wedge^r)$ is not modular: Let a, b, c, d be distinct points of E. Define each of the following principal regular topologies by its base of minimal open sets

$$\begin{array}{ll} \tau_{(ab)} & \{a, b\}, \{c\}, \{d\} \text{ and } \{x\} \text{ for } x \neq a, b, c, d \\ \tau_{(ab)(cd)} & \{a, b\}, \{c, d\} \text{ and } \{x\} \text{ for } x \neq a, b, c, d \\ \tau_{(ad)(cb)} & \{a, d\}, \{c, b\} \text{ and } \{x\} \text{ for } x \neq a, b, c, d \\ \tau_{(abcd)} & \{a, b, c, d\} \text{ and } \{x\} \text{ for } x \neq a, b, c, d\,. \end{array}$$

Then we have the following diagram of least upper bounds and greatest lower bounds in $(\mathscr{R}, \vee, \wedge^r)$.

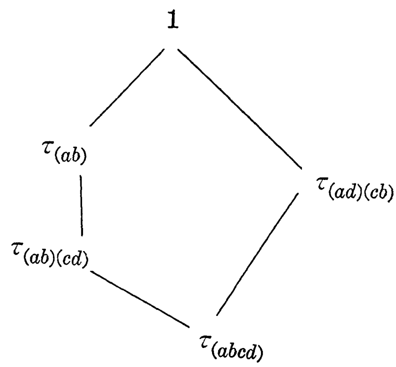

FIGURE 1

Greatest lower Bounds in $\mathscr{R}$ and continuous functions. In a paper in 1968 [14, p. 1087, Theorem 1], J. Pelham Thomas characterized the strongest regular topology on a set weaker than a given topology on that set: If τ is a topology on E, then there is a unique regular topology τ^* weaker than τ, such that, if Y is any regular space, then the continuous maps $(E, \tau) \rightarrow Y$ are the continuous maps $(E, \tau^*) \rightarrow Y$. Furthermore τ^* is the least upper bound of the regular topologies weaker than τ. In this vein we have the following lemmas.

LEMMA 2.1. *A function $f: (E, 0) \rightarrow (Y, \rho)$ is continuous where (Y, ρ) is a regular space iff $f(E) \subset \mathrm{cl}_\rho(f(x))$ for every $x \in E$.*

LEMMA 2.2. *If, for every regular T_0 space (Y, ρ), every continuous function $f: (E, \nu) \rightarrow (Y, \rho)$ is constant, then, for every regular space (Y, ρ), every continuous function $f: (E, \nu) \rightarrow (Y, \rho)$ satisfies the condition $f(E) \subset \mathrm{cl}_\rho(f(x))$ for every $x \in E$.*

Using the Thomas result we conclude that

COROLLARY 2.3. *In order for* $\sigma \wedge^r \tau = 0$ *it is necessary and sufficient that every continuous function on* $(E, \sigma \wedge \tau)$ *to a regular* T_0 *space be constant.*

It is now possible to reduce the problem to T_0 regular topologies. Let τ be a regular topology on E and E^* the set of point closures $\{\mathrm{cl}_\tau(x): x \in E\}$. Then E^* is a set of equivalence classes of E and $\varphi: E \to E^*$ given by $\varphi(x) = \mathrm{cl}_\tau(x)$ is the canonical map. If τ^* is the quotient topology relative to φ and τ, that is, the finest topology on E^* such that φ is continuous relative to (E, τ), then τ^* is a regular T_0 topology, lattice-isomorphic to τ [15, p. 92, Theorem 14.2]; further, $\varphi: (E, \tau) \to (E^*, \tau^*)$ is open and closed [9, p. 155, Theorem 9.3.6], and (E^*, τ^*) is called the T_0 quotient of (E, τ).

THEOREM 2.4. *If the* T_0 *quotient* (E^*, τ^*) *of a regular space* (E, τ) *has a (principal) complement in the lattice of regular topologies on* E^*, *then* (E, τ) *has a (principal) complement in the lattice of regular topologies on* E.

Proof. Let f be a choice function on the subsets of E, σ^* the regular complement for τ^* and $S = \{y \in E: y \neq f(\mathrm{cl}_\tau(y))\}$. Define σ to be the topology on E with the following base

$$\{(\varphi^{-1}B^*) - S: B^* \in \sigma^*\} \cup \{\{y\}: y \in S\} .$$

The topology σ is, in fact, regular. Suppose F is closed in (E, σ) and $x \notin F$. Then $\sim F = (\varphi^{-1}B^* - S) \cup A$ for some $A \subset S$ and some $B^* \in \sigma^*$. If $x \in S$, then $\{x\} \in \sigma$ and $F \subset E - \{x\} \in \sigma$. If $x \notin S$, then $\varphi x \in B^* \in \sigma^*$ and there exist disjoint sets $U^*, V^* \in \sigma^*$ separating $\varphi(x)$ and $\sim B^*$. In which case, $\varphi^{-1}U^* - S$ and $\varphi^{-1}V^* \cup S$ are σ-open sets separating x and F. Note that σ is principal if σ^* is.

Next, if $A \in \sigma \wedge \tau$, then $\varphi A \in \tau^*$ and $A = \varphi^{-1}B^*$ for some $B^* \in \sigma^*$. Hence $\varphi: (E, \sigma \wedge \tau) \to (E^*, \sigma^* \wedge \tau^*)$ is open. If $\psi: (E, \sigma \wedge \tau) \to Y$ is any continuous function to a regular T_0 space Y, then $\psi(\mathrm{cl}_{\sigma\wedge\tau}(x)) = \psi(x)$ for any $x \in E$. Hence $\psi\varphi^{-1}: (E^*, \sigma^* \wedge \tau^*) \to Y$ is a welldefined continuous function. Since $\sigma^* \wedge^r \tau^* = 0$ then $\psi\varphi^{-1}$ must be constant, which implies that ψ is constant and hence $\sigma \wedge^r \tau = 0$.

Finally $\sigma \vee \tau = 1$. For $x \notin S$ we have $U^* \in \tau^*$ and $V^* \in \sigma^*$ such that $\{\varphi x\} = U^* \cap V^*$ which implies that

$$\{x\} = (\varphi^{-1}U^*) \cap (\varphi^{-1}V^* - S) \in \tau \vee \sigma .$$

Principal complementation and connectivity. In order for a regular topology τ and a principal regular topology σ to have a least upper bound of 1, it is necessary and sufficient that the minimal open sets of σ be discrete in τ. That they have a greatest lower bound of 0 is characterized in terms of continuous functions. Now a function is continuous on $(E, \sigma \wedge \tau)$ iff it is continuous on both (E, σ) and (E, τ). Relative to continuity on principal regular spaces, we have the following:

LEMMA 3.1. *Let σ be a principal regular topology on E. A function $f\colon (E, \sigma) \to (Y, \rho)$, where ρ is a T_1 topology, is continuous iff f is constant on each minimal σ-open set.*

THEOREM 3.2. *If (E, τ) is a regular T_0 space with a disjoint open cover $\{E_\alpha\}_\alpha$ of E and if, for each α, the topology $\tau_\alpha = \tau \mid E_\alpha$ has a principal complement σ_α in the lattice of regular topologies on E_α then τ has a principal complement in the lattice of regular topologies on E.*

Proof. For each α let B^α be some one minimal open set in σ_α. The set $\bigcup_\alpha B^\alpha$ and, for all α, all minimal open sets B_x in σ_α, different from B^α, define a minimal open base for a principal regular topology σ on E such that $\sigma \mid E_\alpha = \sigma_\alpha$.

Let f be any function on E to a regular T_0 space which is continuous relative to the topology $\sigma \wedge \tau$. Then for any α, $f_\alpha = f \mid E_\alpha$ is continuous relative to the topology $(\sigma \wedge \tau) \mid E_\alpha$. But $(\sigma \wedge \tau) \mid E_\alpha \leqq \sigma_\alpha \wedge \tau_\alpha$ so f_α is constant on E_α. Since f was continuous relative to σ then f must be constant on $\bigcup_\alpha B^\alpha$. Hence f is constant on all of E.

Lastly $\sigma \vee \tau = 1$: if x is any point of $E = \bigcup_\alpha B_\alpha$ then $\sigma_\alpha \vee \tau_\alpha = 1$ implies that there are sets $U \in \sigma$ and $V \in \tau$ such that $\{x\} = (U \cap E_\alpha) \cap (V \cap E_\alpha) = U \cap (V \cap E_\alpha) \in \sigma \vee \tau$.

The complementation problem for locally connected regular spaces is then reduced to the complementation problem for connected spaces. Further, the proof of the previous theorem suggests several lines of development.

THEOREM 3.3. *Let (E, τ) be a regular T_0 space whose set $\mathscr{E}$ of components satisfy the following conditions:*

(i) $\mathscr{E}$ is countable.

(ii) For each $C \in \mathscr{E}$ the restriction $\tau \mid C$ has a principal regular complement.

(iii) Either $\mathscr{E}$ has finitely many singletons or infinitely many nonsingletons.

Then τ has a principal regular complement.

Proof. Without loss of generality, by (i) the collection of components forms a sequence $\{E_n\}_n$ such that, by (iii) each singleton is followed by a nonsingleton. For each n, let $\tau_n = \tau \mid E_n$ and σ_n its principal regular complement.

Now for any nonsingleton E_n there must be at least two distinct minimal open sets in σ_n; otherwise $\tau_n = 1$. But 1 is not connected unless $|E_n| = 1$.

For each n, choose A^n and B^n minimal open sets in σ_n such that $B^n \neq A^n$ if $|E_n| > 1$. Then the sets

(i) $B^n \cup A^{n+1}$ for all n such that $|E_n| \neq 1$ and $|E_{n+1}| \neq 1$

(ii) $B^n \cup E_{n+1} \cup A^{n+2}$ for all n such that $|E_{n+1}| = 1$

(iii) $B^{n-1} \cup E_n \cup A^{n+1}$ for all n such that $|E_n| = 1$

(iv) B_x for all minimal σ_n open sets with $B_x \neq A^n, B^n, n = 1, \cdots$

define a base of minimal open sets for a principal regular topology σ on E such that $\sigma_n = \sigma \mid E_n$ for each n.

Let f be any function on E to a regular T_0 space which is continuous relative to the topology $\sigma \wedge \tau$. Then $f_n = f \mid E_n$ is continuous relative to the topology $\sigma_n \wedge \tau_n$ for each n. Hence f_n is constant on E_n and since f is constant on each set in σ then f is constant on all of E.

For each x not in some B^n or A^n there are sets $U \in \tau$ and $B_x \in \sigma_n$ such that $\{x\} = (U \cap E_n) \cap B_x = U \cap B_x \in \sigma \vee \tau$. For any $x \in B^n$ there is a neighborhood $U \in \tau$ of x such that $U \cap B^n = \{x\}$ and, since components are closed and $x \notin E_{n\pm 1}, E_{n+2}$, such that $U \cap E_{n\pm 1} = \varnothing$ and $U \cap E_{n+2} = \varnothing$. Hence

$$\begin{aligned} \{x\} &= U \cap (B^n \cup A^{n+1}) \in \tau \vee \sigma \text{ if } |E_n|, |E_{n+1}| \neq 1 ; \\ &= U \cap (B^n \cup E_{n+1} \cup A^{n+2}) \text{ if } |E_{n+1}| = 1 ; \\ &= U \cap (B^{n-1} \cup E_n \cup A^{n+1}) \text{ if } |B_n| = |E_n| = 1 . \end{aligned}$$

Similarly for any $x \in A^n$. Thus $\sigma \vee \tau = 1$.

THEOREM 3.4. *Let (E, τ) be a regular space and D a dense subset. If $\tau \mid D$ has a complement σ^* in the lattice of regular topologies on D, then τ has a complement in the lattice of regular topologies on E.*

Proof. Define σ to be the topology on E with the base $\sigma^* \cup \{\{y\}: y \notin D\}$. Then σ is regular, $\sigma \mid D = \sigma^*$; σ is principal iff σ^* is principal. Now clearly $(\sigma \wedge \tau) \mid D \leqq \sigma \mid D \wedge \tau \mid D$ so $(\sigma \bigwedge^r \tau) \mid D \leqq \sigma \mid D \bigwedge^r \tau \mid D = 0$. In which case, for any nonemply $U \in \sigma \bigwedge^r \tau$ we have $U \supset D$ since $U \cap D = \varnothing$ is impossible. Hence $\sigma \bigwedge^r \tau = 0$. Obviously $\sigma \vee \tau = 1$.

It is now clear that the complementation problem can be reduced to considering spaces without isolated points, because in the following result $(W, \tau \mid W)$ has no isolated points.

COROLLARY 3.5. *Let (E, τ) be a regular T_0 space, I the set of isolated points, $W = \mathrm{int}_\tau (E - I)$ the interior of $E - I$. If $(W, \tau / W)$ has a principal regular complement then there is a principal regular complement for τ.*

Classes with complements. In this section our task is to construct principal regular complements for various classes of regular T_0 topologies. The first result provides the basic construction used in the following theorem to handle the class of supra-*DN* spaces. The definition of this class is a generalization of the *DN* spaces of B. A. Anderson [1, p. 989] and was suggested by Harold Bell as a means of extending methods developed for the *DN* spaces. The question remains open whether this class exhausts the regular T_0 spaces. Subsequent results show an approach to a different class of spaces and to arbitrary products of such spaces.

THEOREM 4.1. *Let (E, τ) be a regular T_0 space, $\xi > |E|$, and $\{S_n: 0 \leqq n < \eta \leqq \xi\}$ a wellordered family of disjoint discrete nonempty subsets of E whose union is dense in E. Suppose that for such $n > 0$, any open set containing $\mathrm{cl}_\tau (\bigcup_{\gamma < n} S_\gamma)$ meets S_n. Then τ has a principal regular complement σ. Moreover there is some point $x \in E$ such that $\mathrm{cl}_{\sigma \wedge \tau} (x) = E$.*

Proof. Define σ to be the principal regular topology with the base of minimal open sets $\{S_n: n \geqq 0\} \cup \{\{x\}: x \notin \bigcup_{n \geqq 0} S_n\}$. Then for any $x \in E$ we have $\{x\} \in \sigma \vee \tau$.

On the other hand, for each S_n let x_n be any point in $\mathrm{cl}_\tau (S_n)$. Suppose there is an ordinal n such that

$$\mathrm{cl}_{\sigma \wedge^r \tau} (x_n) \neq \mathrm{cl}_{\sigma \wedge^r \tau} (x_0) .$$

Let m be the least such ordinal. Then there are disjoint sets U^*, $V^* \in \sigma \bigwedge^r \tau$ such that $\mathrm{cl}_{\sigma \wedge^r \tau} (x_0) \subset U^*$ and $\mathrm{cl}_{\sigma \wedge^r \tau} (x_m) \subset V^*$. Also, for every $\gamma < m$, $\mathrm{cl}_{\sigma \wedge^r \tau} (x_0) = \mathrm{cl}_{\sigma \wedge^r \tau} (x_\gamma)$. But then $\mathrm{cl}_{\sigma \wedge^r \tau} (x_0)$ is a τ-closed set containing all the sets $\mathrm{cl}_{\sigma \wedge^r \tau} (x_\gamma) \supset S_\gamma$ for $\gamma < m$. By the regularity, every $U \in \sigma \wedge^r \tau$ such that $x_0 \in U$ must contain $\mathrm{cl}_{\sigma \wedge^r \tau} (x_0) \supset \mathrm{cl}_\tau (\bigcup_{\gamma < m} S_\gamma)$. So U^* meets $S_m \subset \mathrm{cl}_{\sigma \wedge^r \tau} (x_m) \subset V^*$ which is a contradiction. Hence $\mathrm{cl}_{\sigma \wedge^r \tau} (x_0) = E$ and $\sigma \bigwedge^r \tau = 0$.

DEFINITION. A space (E, τ) is said to be supra-*DN* if, for any open set U such that $\mathrm{cl}_\tau (U) - U \neq \emptyset$ there is a discrete set $S \subset U$ such that $\mathrm{cl}_\tau (S) - U \neq \emptyset$.

Note that any first countable space is supra-*DN*.

THEOREM 4.2. *If (E, τ) is a regular T_0 supra-DN space without* *isolated points then τ has a principal regular complement.*

Proof. Let x_1 be any point of E and $U_1 = E - \{x_1\} \in \tau$. Then there is a discrete set $S_1 \subset U_1$ such that $\{x_1\} = \mathrm{cl}_\tau(S_1) - U_1$. For the induction, consider any ordinal n between 1 and ξ, where $\xi > |E|$; suppose that for each $\beta < n$ the set $S_\beta \subset E - \mathrm{cl}_\tau(\bigcup_{\gamma<\beta} S_\gamma)$ is defined, nonclosed, discrete, and either $\mathrm{cl}_\tau(\bigcup_{\gamma<\beta} S_\gamma) \in \tau$ or any open set containing $\mathrm{cl}_\tau(\bigcup_{\gamma<\beta} S_\gamma)$ meets S_β. Now for any subset $A \subset E$, either the boundary of $E - \mathrm{cl}_\tau(A)$ is nonempty or $\mathrm{cl}_\tau(A)$ is open. Hence if $\mathrm{cl}_\tau(\bigcup_{\gamma<n} S_\gamma)$ is not open then the boundary of $U_n = E - \mathrm{cl}_\tau(\bigcup_{\gamma<n} S_\gamma) \in \tau$ contains some point x_n and U_n contains a discrete set S_n such that $x_n \in \mathrm{cl}_\tau(S_n) - U_n$. So any open set containing $\mathrm{cl}_\tau(\bigcup_{\gamma<n} S_\gamma)$ contains the boundary of U_n and hence, as a neighborhood of x_n, meets S_n. If, on the other hand, $\mathrm{cl}_\tau(\bigcup_{\gamma<n} S_\gamma) \in \tau$, let x_n be any point of $V_n = E - \mathrm{cl}_\tau(\bigcup_{\gamma<n} S_\gamma)$ and $U_n = V_n - \{x_n\} \in \tau$. Then there is a discrete set $S_n \subset U_n$ such that $\{x_n\} = \mathrm{cl}_\tau(S_n) - U_n$.

Consequently $\mathrm{cl}_\tau(\bigcup_{1\leqq n} S_n) = E$ and $S_0 = \{x_n\colon \mathrm{cl}_\tau(\bigcup_{\gamma<n} S_\gamma) \in \tau\}$ is discrete. Lastly, if $\mathrm{cl}_\tau(\bigcup_{1\leqq\gamma<n} S_\gamma) \in \tau$ then any τ-open set containing $\mathrm{cl}_\tau(\bigcup_{0\leqq\gamma<n} S_\gamma) \supset S_0$, and hence containing x_n, meets S_n. Otherwise $\mathrm{cl}_\tau(\bigcup_{1\leqq\gamma<n} S_\gamma) \notin \tau$ and any $U \in \tau$ such that $U \supset \mathrm{cl}_\tau(\bigcup_{0\leqq\gamma<n} S_\gamma)$ must meet S_n. The conclusion then follows by the previous theorem.

DEFINITION. A space (E, τ) is said to be Bolzano-Weierstrass compact if every infinite subset of E has a limit point in E.

DEFINITION. A space (E, τ) is said to be locally-B.W.-compact if each point in the space has a fundamental system of neighborhoods each of which is Bolzano-Weierstrass compact.

THEOREM 4.3. *If (E, τ) is a separable, regular T_0 locally-B.W.-compact space without isolated points, then τ has a principal regular complement.*

Proof. Let $Q = \{q_1, q_2, \cdots\}$ be a countable dense subset of E. Let V_1 be a B.W. compact neighborhood of $x_1 = q_1$. Since $\tau | Q$ is T_2 without isolated points, there is a countably infinite discrete $S_1 \subset \mathrm{int}_\tau(V_1) \cap Q$ with $x_1 \in S_1$. For every $x \in S_1$, the T_2 regularity of E and the discreteness of the countable set S_1 imply that there is an open set V_x such that $x \in V_x \subset \mathrm{cl}_\tau V_x \subset V_1$, $\mathrm{cl}_\tau V_x \cap \mathrm{cl}_\tau S_1 = \{x\}$, and if $x, y \in S_1$ and $x \neq y$, then $\mathrm{cl}_\tau V_x \cap \mathrm{cl}_\tau V_y = \varnothing$. Hence, for each $x \in S_1$, an infinite discrete set S_x may be chosen so that $x \in S_x \subset V_x \cap Q$.

The points of S_1 may be denoted by x_{1n} for $n = 1, 2, \cdots$, with $x_{11} = x_1$. The corresponding discrete sets may be denoted by S_{1n}. For each n, let $y_{1n} \in \mathrm{cl}_\tau(S_{1n}) - S_{1n} \subset \mathrm{cl}_\tau V_{x_{1n}}$.

For each $k > 1$ let $Q_k = Q - \mathrm{cl}_\tau (\bigcup_{p<k} \bigcup_{n=1}^{\infty} S_{pn}) \in \tau \mid Q$. If $Q_k \neq \varnothing$, let x_k be the least element in the order on Q_k.

V_k a B.W.-compact neighborhood of x_k in $\sim\mathrm{cl}_\tau (\bigcup_{p<k} \bigcup_{n=1}^{\infty} S_{pn})$

S_k a countably infinite discrete set in $V_k \cap Q_k$ with $x_k \in S_k$

x_{kn} $n = 1, 2, \cdots$ the points of S_k in the induced order

S_{kn} the corresponding countably infinite discrete sets chosen from the intersection of Q and a neighborhood, of x_{kn}, whose closure is in V_k with $x_{k1} = x_k \in S_{k1}$ and satisfying $\mathrm{cl}_\tau S_{kn} \cap \mathrm{cl}_\tau S_{kp} = \varnothing$ for $n \neq p$, and

$y_{kn} \in \mathrm{cl}_\tau (S_{kn}) - S_{kn}$.

Clearly $\mathrm{cl}_\tau (\bigcup_{p=1}^{\infty} \bigcup_{n=1}^{\infty} S_{pn}) \supset \mathrm{cl}_\tau (Q) = E$.

Define a principal regular complement σ for τ with a base of minimal open sets consisting of

$$\begin{aligned} &U_1 = S_{11}\,, \\ &U_k = S_{1k} \cup \{y_{1(k-1)}\} \cup S_{k1} && \text{for } k > 1, \\ &U_{pk} = S_{pk} \cup \{y_{p(k-1)}\} && \text{for } p, k > 1, \\ &\{y\} && \text{for all } y \notin (\textstyle\bigcup_k U_k) \cup (\bigcup_{p,k} U_{pk}). \end{aligned}$$

The minimal open sets are discrete in (E, τ) because S_{k1} was chosen in a closed neighborhood outside $\mathrm{cl}_\tau (\bigcup_{p<k} \bigcup_{m=1}^{\infty} S_{pm})$ which contains $\mathrm{cl}_\tau (S_{1k})$, and because $y_{n(k-1)} \in \mathrm{cl}_\tau (S_{n(k-1)})$ and $\mathrm{cl}_\tau (S_{n(k-1)}) \cap \mathrm{cl}_\tau (S_{nk}) = \varnothing$.

Lastly, if $U \in \tau \wedge \sigma$, $U \neq \varnothing$, then $U \cap (\bigcup_{p,k} S_{pk}) \neq \varnothing$. Let $\bar\alpha$ be the least ordinal for which there is a β such that $U \cap S_{\bar\alpha\beta} \neq \varnothing$ and $\bar\beta$ the least such β. Suppose $\bar\alpha \neq 1$. Then $\bar\beta \neq 1$ and $y_{\bar\alpha(\bar\beta-1)} \in U_{\bar\alpha\bar\beta} \subset U \in \sigma$. But $y_{\bar\alpha(\bar\beta-1)}$ is a τ-limit point of $S_{\bar\alpha(\bar\beta-1)}$ so $U \in \tau$ meets $S_{\bar\alpha(\bar\beta-1)}$ which contradicts the minimality of $\bar\beta$. Hence $\bar\alpha = 1$. Similarly $\bar\beta = 1$ and $S_{11} = U_1 \subset U$ for every $U \in \tau \wedge \sigma$ and $\sigma \bigwedge^r \tau = 0$.

Note that local compactness and countable compactness imply local-B.W.-compactness.

THEOREM 4.4. *For each $i \in \theta$ let (E_i, τ_i) be a regular T_0 space for which there exists a principal regular topology σ_i on E_i such that*

(a) $\sigma_i \vee \tau_i = 1$.

(b) *There is a subset $W_i \subset E_i$ such that $U \in \sigma_i \wedge \tau_i$ and $U \neq \varnothing$ imply that $U \supset W_i$.*

(c) *If $U \in \tau_i$ satisfies $U \supset W_i$ then there are σ_i-isolated points in U.*

(d) *The set of σ_i-nonisolated points is dense in (E_i, τ_i).*

If $E = \prod_{i\in\theta} E_i$ and $\tau = \prod_{i\in\theta} \tau_i$ then (E, τ) has a principal regular complement.

Proof. Well order θ; let $(x_i)_i \in E$. If x_i is isolated in σ_i for every $i \in \theta$, then let $B(x_i)_i = \{(x_i)_i\}$. Otherwise, there is a least element $\bar\imath \in \theta$ such that $x_{\bar\imath}$ is not $\sigma_{\bar\imath}$-isolated; let $B(x_i)_i = B_{\bar\imath} \times (x_i)_{i\neq\bar\imath}$ where $B_{\bar\imath}$

is the minimal σ_i-open set containing x_i. The collection $\{B(x_i)_i: (x_i)_i \in E\}$ forms a base of minimal open sets for a principal regular topology σ on E.

Using hypothesis (a) for the first nonisolated coordinate, it is easily seen that $\sigma \vee \tau = 1$.

Next let A^1, $A^2 \in \sigma \wedge \tau$ be nonempty. Now A^1, $A^2 \in \tau$ implies that there are indices $i_1, i_2, \cdots, i_k \in \theta$ such that A^1 and A^2 contain rectangular neighborhoods. Hence there are points $(x_i)_i \in A^1$ and $(y_i)_i \in A^2$ such that $x_i = y_i$ for $i \neq i_1, \cdots, i_k$ and, by (d), x_i, y_i are σ_i-nonisolated for $i = i_1, \cdots, i_k$ only. Let $j = \min\{i_1, \cdots, i_k\}$ and $A_j^1 = \{z \in E_j: \{z\} \times (x_i)_{i\neq j} \in A^1\} \in \tau_j$, the inverse image of A^1 under the $(x_i)_{i\neq j}$-section; since x_i is σ_i-isolated for $i < j$ then for any σ_j-nonisolated point $x \in A_j^1$, $B_x \times (x_i)_{i\neq j} \subset A^1$. In which case, $B_x \subset A_j^1$ and hence $A_j^1 \in \sigma_j$. Similarly $A_j^2 = \{z \in E_j: \{z\} \times (y_i)_{i\neq j} \in A^2\} \in \sigma_j \wedge \tau_j$. Thus by (b), $W_j \subset A_j^1 \cap A_j^2 \in \sigma_j \wedge \tau_j$ and by (c), there is an isolated point $x_j' = y_j'$ in $A_j^1 \cap A_j^2$ which means that

$$(x_i)_{i\neq j} \times x_j' \in A^1 \text{ and } (y_i)_{i\neq j} \times y_j' \in A^2 .$$

Continuing this process and replacing $x_{i_1}, \cdots, x_{i_k}$ and $y_{i_1}, \cdots, y_{i_k}$ locates a point common to A^1 and A^2. The absence of disjoint sets in $\tau \wedge \sigma$ implies that $\tau \bigwedge^r \sigma = 0$.

In particular, the principal regular complement constructed in Theorem 4.3 satisfies conditions (a), (b) and (d) required of the factor spaces in Theorem 4.4; condition (c) can be accomodated without losing others.

REFERENCES

1. B. A. Anderson, *A class of topologies with T_1 complements*, Notices Amer. Math. Soc., **16** (1969), 69T-G141, 989.
2. B. A. Anderson and D. G. Stewart, *T_1 complements of T_1 topologies*, Proc. Amer. Math. Soc., **23** (1970), 77-81.
3. R. W. Bagley, *On the characterization of the lattice of topologies*, J. London Math. Soc., **30** (1955), 247-249.
4. N. Bourbaki, *Topologie Generale*, Hermann, Paris, 1961.
5. O. Frohlich, *Das Halbordnunssystem der topologischen Raume auf einer Mange*, Math. Ann., **156** (1964), 79-95.
6. H. Gaifman, *The lattice of all topologies*, Canad. J. Math., **18** (1966), 83-85.
7. J. Hartmanis, *On the lattice of topologies*, Canad. J. Math., **10** (1958), 547-553.
8. M. J. Norris, *A note on regular and completely regular topological spaces*, Proc. Amer. Math. Soc., **1** (1950), 754-755.
9. W. J. Pervin, *Foundations of General Topology*, Academic Press, New York, 1964.
10. A. K. Steiner, *The lattice of topologies: structure and complementation*, Trans. Amer. Math. Soc., **122** (1966), 379-398.
11. ———, *Complementation in the lattice of T_1 topologies*, Proc. Amer. Math. Soc., **17** (1967), 884-886.
12. A. K. Steiner and E. F. Steiner, *Topologies with T_1 complements*, Fund. Math., **61** (1967), 23-28.

13. ———, *A T_1 complement for the reals*, Proc. Amer. Math. Soc., **19** (1968) 177-179.
14. J. P. Thomas, *Associated regular spaces*, Canad. J. Math., **20** (1968), 1087-1092.
15. W. J. Thron, *Topological Structures*, Holt Rinehart Winston, New York, 1966.
16. A. C. M. van Rooij, *The lattice of all topologies is complemented*, Canad. J. Math., **20** (1968), 805-807.

Received July 15, 1971 and in revised form September 6, 1971.