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UNICOHERENT COMPACTIFICATIONS

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In this paper we give necessary and sufficient conditions for the Freudenthal compactification of a rimcompact, locally connected and connected Hausdorff space to be unicoherent. We give several necessary and sufficient conditions for a locally connected generalized continuum to have a unicoherent compactification and show that if such a space X has a unicoherent compactification, then γX is the smallest unicoherent compactification of X in the usual ordering of compactifications.

A connected topological space X is said to be *unicoherent* if, $H \cdot K$ is connected whenever $X = H + K$ where H and K are closed connected sets. A continuum is a compact connected metric space and a generalized continuum is a locally compact, connected, separable metric space. By a mapping we will always mean a continuous function. If B is a subset of a space X, the closure of B in X will be denoted by $\mathrm{cl}_X B$ and the boundary of B in X will be denoted by $\mathrm{Fr}_X B$. An open set (respectively, a closed set) of a space X will be called a γ-open (respectively, γ-closed) subset of X provided it has a compact boundary in X. A space is rimcompact (or semicompact) provided every point has arbitrarily small neighborhoods with compact boundaries. All compactifications considered here are Hausdorff.

In [7] K. Morita showed that for any rimcompact Hausdorff space X there exists a topologically unique compactification γX of X satisfying:

(a) For every point x of γX and every open set R of γX containing x there exists an open set V of γX containing x such that $V \subset R$ and $\mathrm{Fr}_{\gamma X} V \subset X$.

(b) Any two disjoint γ-closed subsets of X have disjoint closures in γX.

Furthermore if C is any compactification of X satisfying (a), there exists a mapping h of γX onto C such that $h|X$ is the identity map. The compactification γX of X is called the Freudenthal compactification of X after H. Freudenthal who first defined it [4].

DEFINITION. We say that a connected space X is γ-unicoherent if whenever $X = H + K$, where H and K are γ-closed and connected sets, $H \cdot K$ is connected.

THEOREM 1. *If X is a locally connected, connected, rimcompact Hausdorff space, then γX, the Freudenthal compactification of X, is* *unicoherent iff X is γ-unicoherent.*

Proof. Suppose that X is γ-unicoherent and γX is not unicoherent. Then $\gamma X = H + K$ where H and K are closed and connected sets and $H \cdot K$ is not connected. Let $H \cdot K = A + B$ be a separation of $H \cdot K$ and let U and V be open subsets of γX containing A and B respectively such that $\mathrm{cl}_{\gamma X} U \cdot \mathrm{cl}_{\gamma X} V = \Phi$ and $(\mathrm{Fr}_{\gamma X} V + \mathrm{Fr}_{\gamma X} U) \subset X$. By Propositions (2.8) and (4.1) of [1], γX is locally connected so if C denotes the component of $U + V + H$ that contains H and D denotes the component of $U + V + K$ that contains K, C and D are open connected subsets of γX such that $(Fr_{\gamma X} C + \mathrm{Fr}_{\gamma X} D) \subset X$. By Lemma 5 of [6], $C \cdot X$ and $D \cdot X$ are connected so that $L = \mathrm{cl}_X (C \cdot X)$ and $M = \mathrm{cl}_X (D \cdot X)$ are γ-closed and connected subsets of X. Furthermore $X = L + M$ and $L \cdot M$ is not connected. This contradicts our hypothesis that X is γ-unicoherent and thus γX must be unicoherent.

Now suppose that γX is unicoherent and X is not γ-unicoherent. Then $X = H + K$ where H and K are γ-closed and connected subsets of X and $H \cdot K$ is not connected. Let $H \cdot K = A + B$ be a separation of $H \cdot K$ and let H', K', A' and B' denote the closures of H, K, A and B, respectively, in γX. Since the boundary of $H \cdot K$ in X is a subset of the union of the boundaries of H and K in X, $H \cdot K$ and hence A and B are γ-closed subsets of X. Then by property (b) of Morita's characterization of γX, A' and B' are disjoint closed subsets of γX. We now argue that $H' \cdot K'$ is a subset of $A' + B'$. Suppose to the contrary that there exists a point x in $H' \cdot K'$ that does not belong to $A' + B'$. Let U be any open subsets of γX containing x such that U does not intersect $A' + B'$ and such that $\mathrm{Fr}_{\gamma X} \subset X$. Let Q be the component of U that contains x and note that $\mathrm{Fr}_{\gamma X} Q$ is a subset of X and Q is an open subset of γX. Then since X is dense in γX and x is a limit point of H' and K', $Q \cdot H$ and $Q \cdot K$ are nonempty sets. But by Lemma 5 of [6], $Q \cdot X$ is connected and since Q misses $H \cdot K$, $Q \cdot X$ must lie entirely in H or K. Of course this implies that either $Q \cdot H$ or $Q \cdot K$ is empty and this is a contradiction. Thus $H' \cdot K' = A' + B'$ and this contradicts the unicoherence of γX. Therefore X is γ-unicoherent.

We need the following notation and definitions. Let S^1 denote the unit circle in the complex plane, let $I_1 = \{z = e^{i\theta}: 0 \leqq \theta \leqq \Pi\}$ and let $I_2 = \{z = e^{i\theta}: \Pi \leqq \theta \leqq 2\Pi\}$. For any space W let $\mathscr{A}(W)$ denote the set of mappings of W into S^1 and let $\mathscr{A}_j(W)$ be the set of all mappings of W into I_j, $j = 1, 2$. For each $f \in \mathscr{A}_j(W)$, $j = 1, 2$, let $B_j(f)$ denote the set of all points $t \in I_j$ such that $\mathrm{Fr}\, f^{-1}(t)$ contains a compact set K that separates W into two disjoint open sets M and N where f maps M into the arc from 1 to t on I_j and f maps N into the arc from t to -1 on I_j. Finally let $E(W) = \{f \in \mathscr{A}(W): B_1(f \mid f^{-1}(I_1)) +$

$B_2(f \mid f^{-1}(I_2))$ is dense in $S^1\}$.

THEOREM 2. *Suppose that X is a locally connected, rimcompact Hausdorff space. A necessary and sufficient condition that γX be unicoherent is that every element of $E(X)$ be nullhomotopic.*

Proof of the necessity. Suppose that γX is unicoherent and let f be an element of $E(X)$. For $j = 1, 2$, there exists a point $t_j \in I_j$ such that $\mathrm{Fr}_X f^{-1}(t_j)$ contains a compact set K_j that separates $f^{-1}(I_j)$ into two disjoint open sets M_j and N_j where f maps M_j into the arc from 1 to t_j on I_j and f maps N_j into the arc from t_j to -1 on I_j. Then if we let M denote $K_1 + K_2 + M_1 + M_2$ and let N denote $K_1 + K_2$ $N_1 + N_2$, $X = M + N$ and the boundaries (relative to X) of M and N are subsets of $K = K_1 + K_2$. We assert that the boundaries of $M_0 = \mathrm{cl}_{\gamma X}M$ and $N_0 = \mathrm{cl}_{\gamma X}N$ relative to γX are also subsets of K. In order to see this suppose that x is an element of the boundary of M_0 and $x \notin K$. Then since γX is locally connected, there exists an open connected set R of γX containing x such that $R \cdot K = \Phi$ and $\mathrm{Fr}_{\gamma X}R \subset X$. Then $R \cdot M \neq \Phi$ and $R \cdot (X \backslash M) \neq \Phi$ since X is dense in γX. Furthermore $R \cdot X$ is connected by Lemma 5 of [6] and so $R \cdot X$ is a connected subset of X that meets M and $X \backslash M$. This implies that R meets K and this contradicts our selection of x. Hence the boundaries of M_0 and N_0 in γX are subsets of K. Also by Theorem 3 of [7], M_0 and N_0 are topologically equivalent to γM and γN respectively. Then by Lemma 1 of [3], $f \mid M$ has a continuous extension f_M to M_0 and $f \mid N$ has a continuous extension f_N to N_0. Then since $N_0 \cdot M_0 \subset K$, the function h of γX into S^1 defined by $h \mid M_0 = f_M$ and $h \mid N_0 = f_N$ is continuous. By Lemma (7.4) of [9, p. 228], h is exponentially representable on γX, i.e. there exists a real valued function θ on γX such that $h(x) = e^{i\theta(x)}$ for all $x \in X$. It is evident that this implies that $f = h \mid X$ is exponentially representable an X and by Theorem (6.2) of [9, p. 226], f is nullhomotopic.

Proof of the sufficiency. Suppose that every element of $E(X)$ is nullhomotopic and suppose that γX is not unicoherent. Then by the proof of Theorem 1 there exists closed and connected sets H and K of γX such that $H \cdot K$ is not connected, $\mathrm{Fr}\, H$ and $\mathrm{Fr}\, K$ are subsets of X and $L = H \cdot X$ and $M = K \cdot X$ are connected. Let $H \cdot K = A + B$ be a separation of $H \cdot K$. We note that L and M are γ-closed subsets of and thus by Theorem 3 of [7], γL is homeomorphic to H and γM is homeomorphic to K. It then follows from Lemma 2 of [3] that there exists a mapping f of H into I_1 such that $f(A) = 1$, $f(B) = -1$ and $B_1(f \mid H \cdot X)$ is dense in I_1. Similarly there exists a mapping g of K into I_2 such that $g(A) = 1$, $g(B) = -1$ and $B_2(g \mid K \cdot X)$ is dense in I_2. Then if we define $h: \gamma X \to S^1$ by $h|H = f$ and $h|K = g$ we have that h is continuous and $k = h|X$ is an element of $E(X)$. Then by our hypothesis and Proposition 6.2 of [9, p. 226], k is exponentially representable, i.e. there exists a real-valued mapping θ on X such that for each $x \in X$, $k(x) = e^{i\theta(x)}$. But then $\theta(A) \subset \{0, \pm 2\Pi, \pm 4\Pi, \cdots\}$ and $\theta(B) \subset \{\pm \Pi, \pm 3\Pi, \cdots\}$ and so if $a \in \theta(A)$ and $b \in \theta(B)$, the interval $[a, b]$ lies in $\theta(A) \cdot \theta(B)$ since L and M are connected. This is a contradiction since then $k(L) \cdot k(M)$ would then contain a semicircle whereas it consists of the points -1 and 1. Hence γX is unicoherent.

DEFINITION. A connected space X is said to be weakly unicoherent if whenever $X = H + K$ where H and K are closed and connected sets and K is compact, $H \cdot K$ is connected.

THEOREM 3. *Let X be a locally connected generalized continuum. A necessary and sufficient condition for γX to be unicoherent is that X be weakly-unicoherent.*

Proof of the necessity. Suppose that γX is unicoherent. Since X is locally compact, X is open in γX and $X^* = \gamma X \backslash X$ is closed. Then by Theorem (2.3) of [2], $X = \gamma X \backslash X^*$ is weakly-unicoherent.

Proof of the sufficiency. Suppose that γX is not unicoherent. Then as in the proof of Theorem 1, γX has a representation $\gamma X = P + Q$ where P and Q are open connected subsets of γX, the boundaries of P and Q in γX are subsets of X, $\mathrm{cl}_{\gamma x} P \cdot \mathrm{cl}_{\gamma x} Q = A + B$ where A and B are disjoint nonempty closed sets and P has a nonempty intersection with both the boundary of A and the boundary of B. By Lemma 5 of [6], $P' = P \cdot X$ is a connected open subset of X and thus is arcwise connected. Furthermore since the boundaries of A and B are subsets of X there exists an arc $\alpha\beta$ in P' such that $\alpha\beta \cdot A = \alpha$ and $\alpha\beta \cdot B = \beta$. Let R be the component of $P' \backslash (A + B)$ that contains $\alpha\beta \backslash (\alpha + \beta)$ and let W be an open subset of γX containing A such that $B \cdot \mathrm{cl}\, W = \phi$ and the boundary of W is a subset of X. Then $H = R \cdot \mathrm{Fr}_{\gamma x} W$ is a nonempty compact subset of R and there exists a continuum K_0 of X such that $H \subset K_0 \subset R$. Let K be the union of K_0 together with all the components of $R \backslash K_0$ with boundary entirely in K_0, i.e. having no boundary points in $X \cdot (A + B)$. Then K separates R since $W \cdot R$ contains a subarc $\alpha b \backslash \alpha$ from some point $b \in \alpha\beta$ and $X \backslash \mathrm{cl}_x W$ contains a subarc $a\beta$ of $\alpha\beta$. But $X \backslash K$ is connected since $X \backslash K$ is the union of the closure of Q in X plus all of the components of $X \backslash (A \cdot B)$ except R plus all of the components of $R - K_0$ having a boundary point in $X \cdot (A + B)$. This contradicts Whyburn's characterization of weak-unicoherence in [8, p. 185].

COROLLARY 3.1. *Let X be a locally connected generalized continuum. Then X is weakly-unicoherent iff X is γ-unicoherent.*

This corollary follows immediately from Theorems 1 and 3.

REMARK. The authors have been unable to discover a direct proof of Corollary (3.1). In general the two types of unicoherency are not equivalent and in the absence of local compactness, Theorem 3 is not valid.

EXAMPLE. Let $Y = \{z \text{ complex } \mid 1/2 \leqq |z| \leqq 1\}$,

$$\begin{aligned} S &= \{z \mid |z| = 1\},\ A \text{ a countable dense subset of } S\,, \\ L_z &= Y \cdot \{\text{ray from origin thru } z\} \\ C_r &= \{z \mid |z| = r\},\ r \in [1/2, 1]; \\ Z &= \{C_r \cdot L_a \mid r \text{ is rational, } a \in A\}\,. \end{aligned}$$

The set Z is countable and dense in Y. Let $X = Y - Z$. The set X is evidently T_2, connected and locally connected (in fact, path connected and locally path connected), rim compact but not locally compact. Moreover:

1. X is weakly-unichoherent. To see this, note that any continuum $K \subset X$ has empty interior in X. If therefore $X = H + K$, H closed and connected and K compact and connected, then necessarily the open set $X - H$ is a subset of K, and thus empty. It follows that $H \cdot K = K$, which is connected.

2. X is not γ-unicoherent. For let $p, q \in S - A$ be two distinct points. Then L_p and L_q are compact and disjoint subsets of X. Assume $0 \leqq ARGp < ARGq$. Then

$$H = \{z \in X \mid ARGp \leqq ARGz \leqq ARG_q\} \text{ and}$$
$$K = \{z \in X \mid ARGq \leqq ARG_z \leqq ARGp + 2\pi\}$$

are closed, connected subsets of X such that $X = H + K$, $H \cdot K = L_p + L_q$ is compact but not connected.

3. γX is not unicoherent. To show this it is sufficient to show that γX is just the set Y. To this end we use the characterization of γX obtained by Morita [6]. We show that

(a) For any point $x \in \gamma X$ and open set R of γX containing x, there is an open set V of rX containing x such that $V \subset R$ and $\mathrm{Fr}_{\gamma X} V \subset X$.

(b) Any two disjoint γ-closed subsets of X have disjoint closures in γX.

That (a) holds is evident from the definition of X. To see that (b) holds, let A and B be disjoint γ-closed subsets of X and suppose that $p \in \mathrm{cl}_{\gamma X} A \cdot \mathrm{cl}_{\gamma X} B$. First of all we note that p cannot belong to X for then it would lie in $A \cdot B$ which is empty. In particular p does not lie in the compact set $(\mathrm{Fr}_X A + \mathrm{Fr}_X B)$. By our construction of X there exists an open subset V of Y containing p such that $V \cdot (\mathrm{Fr}_X A + \mathrm{Fr}_X B) = \Phi$ and $V \cdot X$ is connected. Since p belongs to the closure of A in Y, $V \cdot X \cdot A$ is not empty and since $V \cdot X$ misses $\mathrm{Fr}_X A$, $V \cdot X$ must lie entirely in A. But this is a contradiction since $V \cdot X$ must meet B. Therefore A and B have disjoint closures in Y.

DEFINITION. A mapping $f: X \in Y$ is *monotone* provided for every $y \in Y$, $f^{-1}(y)$ is compact and connected.

THEOREM 4. *If X is a locally connected generalized continuum and Y is any unicoherent compactification of X, then there exists a monotone mapping g of Y onto γX such that $g \mid X$ is the identity.*

Proof. Let Z denote the quotient space of Y obtained from the decomposition whose only nondegenerate elements are the components of $Y \backslash X$ and let p denote the natural map of Y onto Z. Then since X is open in Y, Z is a Hausdorff compactification of X. Furthermore since point inverses of p are connected, it follows from Proposition (2.2.1) of [9], that Z is unicoherent. Also $Z \backslash X$ is totally disconnected and by the maximality of γX there exists a mapping h of γX onto Z such that $h \mid X$ is the identity and $h(\gamma X \backslash X) = Z \backslash X$. We assert that h is a homeomorphism. In order to prove this we need only show that h is one-to-one on $\gamma X \backslash X$. To this end let $x, y \in \gamma X$, $x \neq y$ and suppose that $h(x) = h(y)$. There exists a connected and open set R of γX containing x such that $y \notin \mathrm{cl}_\gamma R = K$ and $\mathrm{Fr}_\gamma R \subset X$. Then $Z = h(K) + h(\gamma X \backslash R)$ and $h(K) \cdot h(\gamma X \backslash R) = h(x) + h(\mathrm{Fr}\, R)$ is not connected. This contradicts the unicoherence of Z and hence h must be a homeomorphism. Then $g = h^{-1} \circ p$ is the desired monotone mapping.

COROLLARY 4.1. *Suppose that X is a locally connected generalized continuum. Then X has a unicoherent compactification if and only if γX is unicoherent.*

Proof. This result follows immediately from Theorem 4 and the fact that monotone images of compact unicoherent continua are unicoherent.

THEOREM 5. *Suppose that X is a locally connected generalized continuum. Then the following are equivalent*

(i) *X is weakly-unicoherent*
(ii) *γX is unicoherent*
(iii) *X is γ-unicoherent*
(iv) *X has a unicoherent compactification*
(v) *every mapping of X into S^1 with compact boundaries of point inverses is null-homotopic.*

Proof. The equivalence of (i)—(iv) has been established in Theorems (1) – (4). As an immediate consequence of Theorem (3.3) of [2], we have that (v) implies (i) and (ii) implies (v) follows from Theorem 1 of this paper.

DEFINITION. A connected space X is said to have the *complementation property* provided whenever K is a compact set in X, X/K has at most one component with a non-compact closure. See [2] for some characterizations of this property.

THEOREM 6. *Let X be a locally connected generalized continuum and let Y be any unicoherent, locally connected continuum. There exists a unicoherent compactification Z of X with $Z\backslash X$ homeomorphic to Y if and only if X is weakly-unicoherent and has the complementation property.*

Proof of the necessity. Suppose that Z is a unicoherent compactification of X and $Z\backslash X$ is homeomorphic to Y. Then by Theorem (4.2) of [2], X is weakly-unicoherent and has the complementation property.

Proof of the sufficiency. Suppose that X is weaklyunicoherent and has the complementation property. Then by Theorem (2.2) of [5] there exists a compactification Z of X with $Z\backslash X$ homeomorphic to Y and by Theorem (4.2) of [2], Z is unicoherent. This completes the proof.

REMARK. It appears to be difficult to establish results concerning the unicoherence of a compactification of an arbitrary completely regular space. We can show that the Freudenthal compactification of a rim-compact, locally connected γ-unicoherent space is the smallest unicoherent compactification of X with $\gamma X\backslash X$ zero-dimensional.

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