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GEOMETRIC ASPECTS OF PRIMARY LATTICES

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The incidence structure derived from a primary lattice with a homogeneous basis of three n-cycles is a Hjelmslev plane of level n. A desarguesian Hjelmslev plane $H(R)$ is of level n if and only if R is completely primary and uniserial of rank n.

Introduction. The classical correspondence between vector spaces, projective spaces and complemented modular lattices was extended to finitely generated modules over completely primary and uniserial rings and primary lattices by Baer [5], Inaba [7] and, recently, by Jónsson and Monk [8]. In these extensions, however, an analogue to the classical projective space is missing. It is shown in the present paper, that the appropriate concept is that of a Hjelmslev space as defined by Klingenberg [9], [10] and by Lück [11]. To be correct, this is only shown for the case of a plane geometry, namely Hjelmslev planes of level n, corresponding to primary lattices with homogeneous basis of three n-cycles, and to free modules R^3. Also, we have the complete correspondence only in the desarguesian case. The restriction to this case is justified, as the author believes, by the fact it is well known to be typical for higher dimensional spaces in the classical theory.

In the non desarguesian case, there is a coordinatization theory for Hjelmslev planes of level n given by Drake [6], but this does not seem to lead to a construction of a lattice from the plane. Every primary lattice with a homogeneous basis of three n-cycles, however, leads to a Hjelmslev plane of level n (Theorem 2.13). Planes of level 1 (ordinary projective planes) and of level 2 (uniform Hjelmslev planes) can be shown to be obtainable from lattices. For uniform planes, this was done by the author in [2]. A combination of Theorem 2.13 with results of [4] shows that a desarguesian Hjelmslev plane $\mathscr{H}(\mathscr{R})$ is of level n if and only if $\mathscr{R}$ is completely primary and uniserial of rank n.

0. Definitions.

0.1. Let $\mathscr{H} = (\mathfrak{p}, \mathfrak{G}, I)$ be an incidence structure consisting of a set $\mathfrak{p}$ of points, a set $\mathfrak{G}$ of lines and an incidence relation $I \subseteqq \mathfrak{p} \times \mathfrak{G}$. We say that two points p, q of $\mathscr{H}$ are neighbors, $p \sim q$, if there are two different lines G, H such that $p, q\ IG, H$. Neighborhood for lines is defined dually. A mapping $\varphi: \mathscr{H} \to \mathscr{H}^*$ is a morphism of incidence structures, if it maps points on points, lines on lines and

pIG implies $\varphi pI\varphi G$.

An incidence structure $\mathscr{H}$ is called a projective Hjelmslev plane, short H-plane, if it satisfies the axioms [9, Def. 0]:

(i) For all points p, q of $\mathscr{H}$ there exists a line G of $\mathscr{H}$ such that p, qIG.

(ii) For all lines G, H of $\mathscr{H}$ there exists a point p of $\mathscr{H}$ such that pIG, H.

(iii) There exists an ordinary projective plane $\mathscr{P}$ and an epimorphism $\alpha: H \rightarrow \mathscr{P}$ such that $\alpha p = \alpha q$ is equivalent to $p \sim q$, and $\alpha G = \alpha H$ is equivalent to $G \sim H$.

Using (iii), we see that neighborhood is an equivalence relation and the factor structure $\mathscr{H}/\sim\, = \mathscr{H}'$ is a projective plane isomorphic to $\mathscr{P}$. We call $\mathscr{H}'$ the canonical epimorphic image and the projection $\varphi: \mathscr{H} \rightarrow \mathscr{H}'$ the canonical epimorphism of $\mathscr{H}$. In [9] it is shown that this set of axioms is equivalent to the ones used in [1] to define H-planes.

0.2. We deal with modular lattices with universal bounds N and U. The lattice operations are denoted by $\vee$, $\wedge$ and we make the convention that $\wedge$ shall bind closer than $\vee$, that is $a \vee b \wedge c = a \vee (b \wedge c)$. $L(a, b)$ is the interval of elements x such that $a \leqq x \leqq b$. We use $a \dot{\vee} b$ to denote independent join, i.e. to indicate $a \wedge b = N$. A cycle $a \in \mathscr{L}$ is an element such that $L(N, a)$ is a chain. A cycle of dimension k is a k-cycle.

Definition [8, Def. 4.2 and Def. 6.1]: A lattice $\mathscr{L}$ is said to be primary, if:

(i) $\mathscr{L}$ is modular of finite dimension.

(ii) Every element of $\mathscr{L}$ is the join of cycles and the meet of dual cycles.

(iii) Every interval in $\mathscr{L}$ that is not a chain contains at least three atoms.

Furthermore, we make the assumption

(iv) There are three independent n-cycles a_1, a_3, a_3 such that $U = a_1 \dot{\vee} a_2 \dot{\vee} a_3$ for the greatest element U of $\mathscr{L}$. This means that $\mathscr{L}$ is of type $(0, \cdots, 0, 3)$ in the sense of [8, Def. 4.10]. By [8, Lemma 6.4] it follows, that the a_i are pairwise perspective. Hence they form a homogeneous basis of order three of $\mathscr{L}$ (for a definition of that concept, see [1, Def 1]). Since the dual $\overline{\mathscr{L}}$ of a primary lattice $\mathscr{L}$ is again primary [8, Cor. 6.2], and the type of $\overline{\mathscr{L}}$ is equal to the type of $\mathscr{L}$ [8, Cor. 4.11], we may use duality in deriving results from (i)—(iv).

For the rest of this paper, $\mathscr{L}$ will always denote a lattice satisfying (i)—(iv), i.e. a primary lattice with a homogeneous basis of n-cycles a_1, a_2, a_3. For $\{i, j, k\} = \{1, 2, 3\}$ we put $A_i = a_j \dot{\vee} a_k$. Since the geometric dimension of $\mathscr{L}$ [8, Def. 5.1] is three, $\mathscr{L}$ may be non-arguesian.

1. The H-plane $\mathscr{H}(\mathscr{L})$.

1.1. Points and lines in $\mathscr{L}$. Let $\mathfrak{q}$ be the set on n-cycles of $\mathscr{L}$, and

$$\mathfrak{p} = \{p \in \mathscr{L} \mid \text{there is } i \in \{1, 2, 3\} \text{ such that } p \dot{\vee} A_i = U\} .$$

Every $p \in \mathfrak{p}$ is perspective to some a_i, hence is n-cycle. For an n-cycle q, assume $q \cap A_i \neq N \neq q \cap A_k$. Then we have $q \wedge A_i \wedge A_k = q \wedge a_j \neq N$ since q is a cycle, and by the same reason $q \wedge A_j = N$. Therefore $L(q, q \vee A_j)$ has dimension n and $q \dot{\vee} A_j = U$. Hence we have $\mathfrak{p} = \mathfrak{q}$.

By duality, we get: The set of dual cycles of $\mathscr{L}$ of codimension n is equal to the set

$$G = \{G \in \mathscr{L} \mid \text{ there is } i \in \{1, 2, 3\} \text{ such that } G \dot{\vee} a_i = U\} .$$

We call $\mathfrak{p}$ the set of points of $\mathscr{L}$ and $\mathfrak{G}$ the set of lines of $\mathscr{L}$.

1.2. Geometric elements. Every Element of $\mathscr{L}$ which is the join of independent points is said to be geometric [8, Def. 5.1]. By definition, a_1, a_2, a_3 and A_1, A_2, A_3 are geometric. From [8, Thm. 5.2] we derive (FC) (a) For every $b \in \{a_1, a_2, a_3, A_1, A_2, A_3\}$ and every

$$x \in \mathscr{L} \text{ with } x \wedge b = N, \text{ there exists } y \geqq x \text{ such that } y \dot{\vee} b = U.$$

Since the dual (b) of (a) is true as well, $\mathscr{L}$ satisfies the condition (FC) of [1, p. 77].

Let G be a line of $\mathscr{L}$, say $G \dot{\vee} a_i = U$, and $r = G \wedge A_k$ and $s = G \wedge A_j$. We claim that r and s are points such that $G = r \dot{\vee} s$. Obviously we have $a_i \wedge (r \vee s) = N$. Then, $a_i \vee r = a_i \vee G \wedge A_k = (a_i \vee G) \wedge A_k = A_k$, so that r and a_j are perspective with center a_i. Hence r and s are points. From $a_i \vee (r \vee s) = A_k \vee A_j = U$ and $r \vee s \leqq G$ we get $r \vee s = G$ by the indivisibility of complements.

In particular, every line of $\mathscr{L}$ is geometric.

Since the independent join of three points is U, and it is easy to see that the independent join of two points is always a line (by (FC) and [1, Lemma 8]), points and lines make up all geometric elements of $\mathscr{L}$ except for N and U.

1.3. For a line G and a point $p \leqq G$, the interval $L(p, G)$ is a chain. Proof: Consider two points r, s such that $r \dot{\vee} s = G$. For at least one of them, say r, we have $r \wedge p = N$. Then $r \dot{\vee} p = G$ and we have $L(p, G) \cong L(N, r)$, the assertion.

1.4. Neighbors of p on G. Again let p be a point, G a line and

$p \leqq G$. We use $\ll$ to denote the covering relation in $\mathscr{L}$. Let $N = z_0 \ll z_1 \ll \cdots \ll z_n = p$ be the chain of elements less than or equal to p, and let $p = y_0 \leqq \cdots \leqq y_n = G$ be the chain of elements between p and G.

LEMMA. *For every $i \in \{0, 1 \cdots, n\}$ there exists a point $c_i \leqq G$ such that $y_i = p \vee c_i$ and $z_{n-i} = p \wedge c_i$.*

Proof. For every i, p is a maximal cycle contained in y_i [8, Cor. 4.7]. By [8, Thm. 4.8] p has a relative complement x_i in $L(N, y_i)$ and by [8, Lemma 6.4] there exists a cycle c_i such that $y_i = p \dot{\vee} x_i = p \vee c_i = x_i \dot{\vee} c_i$. Since c_i and p are perspective, c_i is an n-cycle, hence a point. Counting the relative dimensions shows $p \wedge c_i = z_{n-i}$.

1.5. Let G and H be two lines and p a point such that $p \leqq G \wedge H$. By the last lemma, there is a point $q \leqq G$ such that $p \vee q = G \wedge H$. This and the dual statement yield

(S) (a) For points p, q of $\mathscr{L}$ and a line G with $p \vee q \leqq G$ there exists a line H such that $p \vee q = G \wedge H$.

(b) For lines G, H of $\mathscr{L}$ and a point $p \leqq G \wedge H$ there exists a point q such that $p \vee q = G \wedge H$.

1.6. In [1, p. 77/78] it was defined: A modular lattice with a homogeneous basis of order three consisting of cycles is called an H-lattice, if it satisfies (FC) and (S). By 1.2 and 1.5, $\mathscr{L}$ is an H-lattice. From an H-lattice an incidence structure $(\mathfrak{p}, \mathfrak{G}, I)$ is derived by defining $\mathfrak{p}$ and $\mathfrak{G}$ as in 1.1 and incidence by the ordering of the lattice. Using Theorem 1 of [1], we can now state:

PROPOSITION. *$\mathscr{L}$ is an H-lattice and the incidence structure $\mathscr{H} = \mathscr{H}(\mathscr{L}) = (\mathfrak{p}, \mathfrak{G}, I)$ derived from $\mathscr{L}$ is a projective H-plane. Two points p, q of $\mathscr{L}$ are neighbors in $\mathscr{H}$ if and only if $p \wedge q > N$, two lines G, H are neighbors if and only if $G \vee H < U$.*

More information about $\mathscr{H}$ will be given in the next section.

2. $\mathscr{H}(\mathscr{L})$ is of level n.

DEFINITION 2.1. (cf. [3] and [6]) Let $\mathscr{H}$ and $\mathscr{H}^*$ be H-planes with canonical epimorphisms $\varphi: \mathscr{H} \to \mathscr{H}'$ and $\kappa: \mathscr{H}^* \to (\mathscr{H}^*)'$ onto ordinary projective planes. Let $\psi: \mathscr{H} \to \mathscr{H}^*$ be an epimorphism and $\lambda: (\mathscr{H}^*)' \to \mathscr{H}'$ an isomorphism. If $\varphi = \lambda\kappa\psi$ we say $\mathscr{H}$ has a refined neighbor property defined by $\psi: \mathscr{H} \to \mathscr{H}^*$. We define $p \equiv q$ by $\psi p = \psi q$ and $G \equiv H$ by $\psi G = \psi H$. Then $\equiv$ is called a refined neighbor relation in $\mathscr{H}$.

We say $\equiv$ is minimal provided the following conditions hold:

(M) Let p, q be points on G and p on H.

(a) If $p \equiv q$ and $G \sim H$, then q is on H.

(b) If $p \sim q$ and $G \equiv H$, then q is on H.

(c) There exist distinct points a and b and distinct lines A and B such that $a \equiv b$ and $A \equiv B$.

DEFINITION 2.2. The ordinary projective planes make up the class of projective H-planes of height 1. Suppose $\mathscr{H}$ is an H-plane with a minimal neighborhood defined by ψ: $\mathscr{H} \to \mathscr{H}^*$, where $\mathscr{H}^*$ is of height $n-1$. Then one calls $\mathscr{H}$ an H-plane of height n.—It is suitable to denote an H-plane of height n by $\mathscr{H}_n$ and by $\mathscr{H}_{n-1}$ the plane and by $\psi_{n-1}, \varphi_{n-1}, \lambda_{n-1}$ the maps which define the minimal neighborhood in $\mathscr{H}_n$. Proceeding thus we obtain, for every H-plane of height n, the following commutative diagram

$$\begin{array}{ccccccc}
\mathscr{H}_n & \xrightarrow[\psi_{n-1}]{} & \mathscr{H}_{n-1} & \xrightarrow[\psi_{n-2}]{} & \cdots & \xrightarrow[\psi_1]{} & \mathscr{H}_1 \\
\Big\downarrow{\scriptstyle \varphi_n} & & \Big\downarrow{\scriptstyle \varphi_{n-1}} & & & & \Big\downarrow{\scriptstyle \varphi_1} \\
\mathscr{H}_n' & \xleftarrow[\lambda_{n-1}]{} & \mathscr{H}_{n-1}' & \xleftarrow[\lambda_{n-2}]{} & \cdots & \xleftarrow[\lambda_1]{} & \mathscr{H}_1' .
\end{array}$$

We set $\mu_k = \psi_k \cdots \psi_{n-2}\psi_{n-1}$ and take μ_n to be the identity on $\mathscr{H}_n$. We denote by $(\sim k)$ the refined neighborhood defined by μ_k: $\mathscr{H}_n \to \mathscr{H}_k$ in $\mathscr{H}_n$.

DEFINITION. 2.3. If $\mathscr{H}_n$ is an H-plane of height n, then the H-planes $\mathscr{H}_i$ in the defining sequence of $\mathscr{H}_n$ are of height i. The notion of $(\sim k)$-neighborhood is defined in $\mathscr{H}_i$ as in $\mathscr{H}_n$. A k-segment in $\mathscr{H}_i$ is the nonempty intersection of a line with a class of $(\sim k)$-neighbor points. An H-plane $\mathscr{H}_n$ of height n is called of level n, if the following axiom of reciprocal segments holds in every plane $\mathscr{H}_i$ of the defining sequence of $\mathscr{H}_n$:

(RS) (a) For all lines G, H of $\mathscr{H}_i$, the set of common points of G and H is a k-segment, for some $k \in \{1, 2, \cdots, i\}$.

(b) $G(\sim k)H$ if and only if the set of common points of G and H contains an $(i-k)$-segment.

REMARK. For the change of (N) [3, p. 175] to (RS), see [4].

2.4. If the cycles a_i of $\mathscr{L}$ are of dimension 1, then $\mathscr{H}(\mathscr{L})$ is an ordinary projective plane (an H-plane such that two points p, q are neighbors if and only if $p = q$), hence an H-plane of level 1. If the a_i are bicycles, that is of dimension 2, then by 1.5 every point of $\mathscr{H}(\mathscr{L})$ has at least one proper neighbor and by [2, Satz 3], $\mathscr{H}(\mathscr{L})$ is a uniform H-plane, that is of level 2 [3, p. 179].

We are going to apply induction to show that $\mathscr{H}(\mathscr{L})$ is of level n if the a_i are n-cycles. We may assume $n > 2$. First we have to show that $\mathscr{H}(\mathscr{L})$ is of height n.

2.5. Let a_i cover b_i and $B = b_1 \vee b_2 \vee b_3$. Then b_1, b_3, b_3 form a homogeneous basis of $\mathscr{L}^* = L(N, B)$ (cf. [8, Cor. 4.13]). By [8, Cor. 4.4] $\mathscr{L}^*$ satisfies (i) and (ii) of Def. 0.2. Moreover, every interval of $\mathscr{L}^*$ is an interval of $\mathscr{L}$, so $\mathscr{L}^*$ satisfies (iii) as well. Hence $\mathscr{L}^*$ is a primary lattice with the homogeneous basis b_1, b_2, b_3 of three $(n-1)$-cycles. Let the derived H-plane be $\mathscr{H}^* = \mathscr{H}(\mathscr{L}^*) = (\mathfrak{p}^*, \mathfrak{G}^*, I)$.

Let p be a point of $\mathscr{H} = \mathscr{H}(\mathscr{L})$ and G be a line of $\mathscr{H}$. We define

$$\psi \colon \mathscr{H} \to \mathscr{H}^*$$

by

$$\psi p = p \wedge B \text{ and } \psi G = G \wedge B \,.$$

In the following paragraphs, we will show that ψ is an epimorphism.

If $p \leqq G$, then $p \wedge B \leqq G \wedge B$, so the fact that ψ preserves incidence is trivial.

2.6. Let p be a point of $\mathscr{H}$, say $p \dot{\vee} A_i = U$, and let $B_i = b_j \vee b_k$. Then $(p \wedge B) \vee B_i = B$, and ψ maps $\mathfrak{p}$ into $\mathfrak{p}^*$. We want to show that it is onto. Let p^* be a point of $\mathscr{H}^*$, say $p^* \vee B_i = B$. Then $p^* \wedge A_i = N$, and by [8, Thm. 5.2], p^* is contained in some complement p of A_i. It follows $p \in \mathfrak{p}$ and $\psi p = p^*$.

2.7. Let G be a line of $\mathscr{H}$, say $G \dot{\vee} a_i = U$, and $G \wedge A_j = s$ and $G \wedge A_k = r$ as in 1.3. We have $b_i \vee r \geqq b_j$ and $b_i \vee s \geqq b_k$, hence $b_i \vee (G \wedge B) = (b_i \vee G) \wedge B = B$. Since $G \wedge B \wedge b_i = N$, ψ maps $\mathfrak{G}$ into $\mathfrak{G}^*$. Again we have to show that it is onto. Let G^* be a line of $\mathscr{H}^*$ and $G^* = r^* \dot{\vee} s^*$ for two points of $\mathscr{H}^*$. There exist points r, s of $\mathscr{H}$ such that $r \wedge B = r^*$ and $s \wedge B = s^*$. For $G = r \vee s$ we have $\psi G = G^*$.

2.8. Since $p \sim q$ in $\mathscr{H}$ means $p \wedge q > N$ in $\mathscr{L}$, we have $p \sim q$ in $\mathscr{H}$ if and only if $\psi p \sim \psi q$ in $\mathscr{H}^*$.

We want to show that the same is true for lines. Assume $G \sim H$ in $\mathscr{H}$. We know that this means $G \wedge H > p$ for some common point p of G and H. Let x be a cycle $\leqq G$ such that $G \wedge H = p \dot{\vee} x$. We may assume $G \neq H$, hence the dimension of x is at most $n-1$. Therefore $x \leqq G \wedge B$ and $x \leqq H \wedge B$, and we have

$$\begin{aligned} G \wedge H \wedge B &= (p \dot{\vee} x) \wedge B \\ &= p \wedge B \vee x \\ &= \psi p \vee x \,, \end{aligned}$$

and from $x > N$ we deduce $\psi G \sim \psi H$.

Now let $G \not\sim H$ in $\mathscr{H}$, then $G \wedge H = p$ for a unique point p. There are points r, s of $\mathscr{H}$ such that $G = p \dot{\vee} r$ and $H = p \dot{\vee} s$. From this we derive $\psi p \vee \psi r \vee \psi s \leqq \psi G \vee \psi H$, and since ψp, ψr, ψs are three independent $(n-1)$-cycles, it follows $G \wedge B \vee H \wedge B = B$, hence $\psi G \not\sim \psi H$.

Thus we have arrived at: $G \sim H$ in $\mathscr{H}$ if and only if $\psi G \sim \psi H$ in $\mathscr{H}^*$.

2.9. By 2.5 – 2.8 we know:
$\psi\colon \mathscr{H} \to \mathscr{H}^*$ is an epimorphism and

$$\begin{aligned} &p \sim q \text{ if and only if } \psi p \sim \psi q \,, \\ &G \sim H \text{ if and only if } \psi G \sim \psi H \,. \end{aligned}$$

Now, for $n - 1 > 1$, we may repeat the procedure and, changing notation to $\mathscr{H} = \mathscr{H}_n$, $\mathscr{H}^* = \mathscr{H}_{n-1}$ and $\psi = \psi_{n-1}$, get a sequence

$$\mathscr{H}_n \xrightarrow[\psi_{n-1}]{} \mathscr{H}_{n-1} \xrightarrow[\psi_{n-2}]{} \cdots \xrightarrow[\psi_1]{} \mathscr{H}_1 \,,$$

where the final incidence structure $\mathscr{H}_1$ is an ordinary projective plane. The mapping

$$\mu_1 = \psi_1 \cdots \psi_{n-1}\colon \mathscr{H}_n \to \mathscr{H}_1$$

is an epimorphisms such that

$$(*) \qquad \begin{aligned} &p \sim q \text{ in } \mathscr{H} \text{ if and only if } \mu_1 p = \mu_1 q \text{ in } \mathscr{H}_1 \text{ and} \\ &G \sim H \text{ in } \mathscr{H} \text{ if and only if } \mu_1 G = \mu_1 H \text{ in } \mathscr{H}_1. \end{aligned}$$

Now the canonical epimorphism $\varphi_n\colon \mathscr{H}_n \to \mathscr{H}_n'$ is universal with the property (*), hence we have a unique isomorphism $\phi\colon \mathscr{H}_n' \to \mathscr{H}_1$ such that $\mu_1 = \theta\varphi_n$. By the same reasoning for $\mathscr{H}_{n-1}$ and $\nu_1\colon \mathscr{H}_{n-1} \to \mathscr{H}_1$ we get the following commutative diagram

$$\begin{tikzcd} & \mathscr{H}_n \arrow[r, "\psi_{n-1}"'] \arrow[dl, "\varphi_n"'] \arrow[d, "\mu_1"] & \mathscr{H}_{n-1} \arrow[d, "\nu_1"'] \arrow[dr, "\psi_{n-1}"] & \\ \mathscr{H}_n' \arrow[r, "\theta"'] & \mathscr{H}_1 & \mathscr{H}_1 & \mathscr{H}_{n-1}' \arrow[ll, "\eta"] \end{tikzcd}$$

If we put $\lambda_{n-1} = \theta^{-1}\eta$, we have $\varphi_n = \lambda_{n-1}\varphi_{n-1}\psi_{n-1}$ and $\psi_{n-1}\colon \mathscr{H}_n \to \mathscr{H}_{n-1}$ defines a refined neighborhood in $\mathscr{H}_n$. Clearly, the same is true for all $\psi_i: \mathscr{H}_{i+1} \to \mathscr{H}_i$ $(1 \leqq i < n)$. Thus we arrive at a commutative diagram as required in Definition 2.2 We did not yet show that the refined neighborhood defined by $\psi_{n-1}: \mathscr{H}_n \to \mathscr{H}_{n-1}$ is minimal. Without knowing this, we define μ_i and $(\sim i)$ as in 2.2.

2.10. In order to prove the axioms (M) and (RS) of Definitions 2.1 and 2.3, it is useful to have an alternative description $p(\sim i)q$ and $G(\sim i)H$ in $\mathscr{H} = \mathscr{H}(\mathscr{L})$.

(i) Let $N = p_0 \ll p_1 \ll \cdots p_{n-1} \ll p_n = p$ and $N = q_0 \ll \cdots \ll q_n = q$ be the chains of elements below the points p and q. We have $\psi_{n-1}p = p \wedge B = p_{n-1}$, hence $\psi_{n-1}p = \psi_{n-1}q$ if and only if $p_{n-1} = q_{n-1}$. Repeating the argument we obtain $\mu_i p = p_i$, which yields $\mu_i p = \mu_i q$ if and only if $p_i = q_i$.

(ii) Let G, H be lines of $\mathscr{H}(\mathscr{L})$ and p, r, s be points such that $G = p \dot{\vee} r$ and $H = p \dot{\vee} s$. Let r_i, s_i be defined like p_i in (i), and $p = x_0 \ll x_1 \ll \cdots \ll x_n = G$. If $\mu_i G = \mu_i H$, then

$$r_i = \mu_i r \leqq \mu_i G = \mu_i H$$

and

$$s_i = \mu_i s \leqq \mu_i H = \mu_i G \;.$$

Hence $p \dot{\vee} r_i = x_i \leqq G \wedge H$ and from Lemma 1.4 we get
(+) There exists a point q such that $p_{n-i} = q_{n-i}$ and

$$p \vee q = x_i \leqq G \wedge H \;.$$

Conversely, assume (+). There exists a cycle r_i such that $p \vee q = p \dot{\vee} r_i = x_i$ and points r, s such that $r_i \leqq r \leqq G$ and $r_i \leqq s \leqq H$ [8, Thm. 4.8]. From this we derive $G = p \dot{\vee} r$ and $H = p \dot{\vee} s$ and

$$\mu_i G = p_i \dot{\vee} r_i = \mu_i H \;.$$

Letting $G = g_0 \ll g_1 \ll \cdots \ll g_n = U$ and $H = h_0 \ll \cdots \ll h_n = U$ we may equivalently say

$$\mu_i G = \mu_i H \text{ if and only if } g_{n-i} = h_{n-i} \;.$$

Or, using $p = y_0 \ll \cdots \ll y_n = H$:

$$\mu_i G = \mu_i H \text{ if and only if } x_i = y_i \;.$$

2.11. We are now ready to verify that $\psi_{n-1}: \mathscr{H}_n \to \mathscr{H}_{n-1}$ defines a minimal neighborhood in $\mathscr{H}_n$.

(Ma) From $p \wedge B = q \wedge B$ it follows that p and q cover $p \wedge q$. Hence $p \vee q$ covers p and q. Now if $G \vee H < U$, then $G \wedge H > p$ and since $L(p, G)$ is a chain, we have

$$p \ll p \vee q \leqq G \wedge H ,$$

hence $q \leqq H$.

(Mb) Let x_i and y_i be as in 2.10. By 2.10 (ii) we know $x_{n-1} = y_{n-1}$. Now if $p \sim q$, then $p \vee q < G$, hence $p \vee q \leqq x_{n-1} = y_{n-1}$ which implies $q \leqq H$.

(Mc) Taking $i = 1$ in 1.5 we get points with the desired property. By duality, we have lines $G \neq H$ such that $G \vee H$ is a cocycle of codimension $n - 1$, hence $\psi_{n-1}G = \psi_{n-1}H$.

2.12. The axiom of reciprocal segments. By 2.10 (i) an i-segment is a set of points on a line G such that $p_i = q_i$ for any two points p, q of the set.

(RSa) Let $p \leqq G \wedge H$ and p_i, x_i as before. Assume $G \wedge H = x_{n-i}$. Then for every point $q \leqq H$ we have that

$p \wedge q \geqq p_i$ implies $q \leqq G$, and

$p \wedge q < p_i$ implies $q \nleqq G$, since otherwise $G \wedge H > x_{n-i}$.

Hence the set of points incident with both G and H is an i-segment.

(RSb) By 2.10 (ii), $\mu_i G = \mu_i H$ if and only if G and H have (at least) an i-segment in common.

THEOREM 2.13. *The H-plane $\mathscr{H}(\mathscr{L})$ derived from a primary lattice L with a homogeneous basis of three n-cycles is an H-plane of level n.*

Proof. By 2.9, $\psi_{n-1}: \mathscr{H}_n \rightarrow \mathscr{H}_{n-1}$ defines a refined neighborhood in $\mathscr{H}_n$ which is minimal by 2.11. By 2.12, the axiom (RS) of reciprocal segments holds in $\mathscr{H}_n$. Since $\mathscr{H}_{n-1}$ is derived from a primary lattice with a homogeneous basis of $(n-1)$-cycles, we may assume that $\mathscr{H}_{n-1}$ is of level $n - 1$. But then $\mathscr{H}_n$ is of level n.

3. Desarguesian H-planes of level n.

DEFINITION 3.1. [8, Def. 6.6]. A ring $\mathscr{R}$ (associative with unit) is said to be completely primary and uniserial if there is a two-sided ideal $\mathscr{N}$ of $\mathscr{R}$ such that every left or right ideal of $\mathscr{R}$ is of the form $\mathscr{N}^k$ (where $\mathscr{N}^0 = \mathscr{R}$). The rank of such a ring is the smallest integer k such that $\mathscr{N}^k = (0)$.

It is a simple exercise to verify that a completely primary and uniserial ring is an H-ring in the sense of [9 Def. 9].

DEFINITION 3.2. Let $\mathscr{R}$ be a completely primary and uniserial ring of rank n. The lattice $\mathscr{L}(\mathscr{H}^3)$ of all submodules of the ($\mathscr{R}$-

left) module $\mathscr{R}^3$ is primary [8, Thm. 6.7] and has the homogeneous basis $a_1 = \mathscr{R}(1, 0, 0)$, $a_2 = \mathscr{R}(0, 1, 0)$, $a_3 = \mathscr{R}(0, 0, 1)$ of n-cycles. Let $\mathscr{H}(\mathscr{R}) = \mathscr{H}(\mathscr{L}(\mathscr{R}^3))$ be the H-plane derived from $\mathscr{L}(\mathscr{R}^3)$. It is easy to check that this plane is essentially the same as defined by Klingenberg [9 Def. 10] via homogeneous coordinates. An H-plane $\mathscr{H}$ is called desarguesian if there exists an H-ring $\mathscr{R}$ such that $\mathscr{H}$ is isomorphic to $\mathscr{H}(\mathscr{R})$, the latter defined as in [9].

THEOREM 3.3. *If $\mathscr{R}$ is a completely primary and uniserial ring of rank n, then the H-plane $\mathscr{H}(\mathscr{R})$ is of level n.*

Proof. Theorem 2.13 and Definition 3.2.

3.4. In [4] it is shown: If $\mathscr{H} = \mathscr{H}(\mathscr{R})$ is a desarguesian H-plane of level n, then $\mathscr{R}$ is a completely primary and uniserial ring of rank n. We combine this with 3.3:

COROLLARY. *A desarguesian H-plane $\mathscr{H}(\mathscr{R})$ is of level n if and only if $\mathscr{R}$ is completely primary and uniserial of rank n.*

3.5. Since the lattice $\mathscr{L}(\mathscr{R}^3)$ defined in 3.2 is arguesian, we have a correspondence between completely primary and uniserial rings of rank n, arguesian primary lattices with a homogeneous basis of three n-cycles and desarguesian H-plane of level n as in the classical theory of projective spaces. With the appropriate definitions, it should be not too hard to verify the analogues correspondences for finite dimensional H-spaces. The coordinatization theorems relevant for this can be found in [7] and [8] for lattices and in [10] and [11] for Hjelmslev spaces.

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