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ZEROS OF SUMS OF SERIES WITH HADAMARD GAPS

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If f is a function of the complex variable z in the unit disc and the power series expansion for f about zero can be expressed as a finite sum of series with Hadamard gaps, then $f(z)$ assumes every finite value infinitely often provided the coefficients in the power series expansion of f do not tend to zero and the average value of $(\log^+1/|f(re^{i\theta})|)^p$ does not grow too rapidly as $r \to 1^-$ for some $p > 1$.

1. Introduction and statement of results. Let f be a function analytic in $|z| < 1$ for which

$$f(z) = c_0 + \sum_{k=1}^{\infty} c_k z^{n_k} \tag{1}$$

where $\{n_k\}$ is a sequence of positive integers for which

$$\frac{n_{k+1}}{n_k} \geqq q > 1\,, \qquad (k \geqq 1)\,. \tag{2}$$

The series in (1) is said to have Hadamard gaps.

If q is greater than about 100, G. and M. Weiss [9] proved $f(z)$ assumes every finite value infinitely often provided

$$\sum_{k=0}^{\infty} |c_k| = \infty\,. \tag{3}$$

If $q > 1$, W. H. J. Fuchs [2] showed $f(z)$ assumes every finite value infinitely often provided

$$\limsup_{k\to\infty} |c_k| > 0\,. \tag{4}$$

In [3] Fuchs has extended his result to show that f assumes every finite value infinitely often in each sector

$$S = \{z \mid \alpha < \arg z < \beta \quad \text{and} \quad |z| < 1\}$$

where α and β are fixed real numbers.

The original result of Fuchs may also be extended as follows:

TEOREM 1. *Let $\{n_k\}$ be a sequence of positive integers for which (2) holds. Let l be a fixed positive integer, and let $n_k^{(i)}$ for $i=1, 2, \cdots, l$ be integers for which*

$$n_{k-1} < n_k^{(l)} < n_k^{(l-1)} < \cdots < n_k^{(1)} < n_k\,.$$

Suppose f is a function analytic in $|z| < 1$ for which

$$(5)\qquad \begin{aligned} f(z) &= a_0 + \sum_{k=1}^{\infty} (a_{n_k^{(l)}} z^{n_k^{(l)}} + a_{n_k^{(l-1)}} z^{n_k^{(l-1)}} + \cdots + a_{n_k^{(1)}} z^{n_k^{(1)}} + a_{n_k} z^{n_k}) , \\ &= \sum_{k=0}^{\infty} c_k z^k . \end{aligned}$$

Suppose (4) *holds and for some $p > 1$ there exists a constant C with $0 < C < +\infty$ such that*

$$(6)\qquad \frac{1}{2\pi}\int_0^{2\pi} (\log^+ 1/|f(re^{i\theta})|)^p d\theta \leqq C\Big(\frac{1}{2\pi}\int_0^{2\pi} (\log^+ 1/|f(re^{i\theta})|) d\theta\Big)^p$$

for a sequence of values of r approaching one. Then $f(z)$ assumes every finite value infinitely often in $|z| < 1$.

Two immediate corollaries of Theorem 1 are:

COROLLARY 1. *Assume the hypothesis of Theorem* 1 *with $n_k^{(\alpha)} = n_k - \alpha$ for $k = 1, 2, 3, \cdots$ and $0 < \alpha \leqq l$. Then $f(z)$ assumes every finite value infinitely often in $|z| < 1$.*

COROLLARY 2. *Let f be a function analytic in $|z| < 1$ for which*

$$f(z) = \sum_{k=0}^{\infty} c_k z^k = f_0(z) + f_1(z) + \cdots + f_t(z)$$

where for each $i, f_i(z)$ has a power series expansion about zero with Hadamard gaps. If (4) *holds and for some $p > 1$ there exists a constant C with $0 < C < +\infty$ such that* (6) *holds for a sequence of values of r approaching one, then $f(z)$ assumes every finite value infinitely often in $|z| < 1$.*

Corollary 1 is a special case of Theorem 1 and extends a result of C. Pommerenke [6] who showed that functions of the type of Corollary 1 without the assumption (6) must assume every value at least once. (G. Schmeisser [7] has recently extended the method in [6] to show the Pommerenke-type series assume every value infinitely often). Corollary 2 follows from Theorem 1 by noticing that $f(z)$ can be rewritten, if necessary, to be in the form (5).

For functions of the form (5) for which

$$\limsup_{r\to 1^-} \frac{\log M(r)}{-\log(1-r)} > 2(2+l)$$

where $M(r)$ denotes the maximum modulus of $f(z)$ on $|z| = r$, we remark in [8] that

$$\limsup_{r\to 1-} \frac{n(r)}{-\log(1-r)} > 0 ,$$

where $n(r)$ denotes the number of zeros of f in $|z| \leqq r$. It seems probable that functions of the type of Theorem 1 also assume every finite value infinitely often in each sector

$$\{z \mid \alpha < \arg z < \beta \quad \text{and} \quad |z| < 1\}$$

where α and β are fixed real numbers. It has been shown by P. Erdos and A. Renyi [1] that if $\{n_k\}$ is an increasing sequence of natural numbers satisfying

$$\liminf_{(k-j)\to\infty} (n_k - n_j)^{1/(k-j)} = 1 ,$$

then, for any sequence $\{\omega_k\}$ of natural numbers for which

$$\lim_{k\to\infty} \omega_k = +\infty$$

there exists a sequence $\{m_k\}$ of natural numbers such that

$$0 \leqq m_k - n_k < \omega_k$$

and a function g, analytic in $|z| < 1$ with the power series expansion

$$g(z) = \sum_{k=0}^{\infty} b_k z^{m_k} ,$$

where the b_k are positive, such that $g(z)$ is unbounded in $|z| < 1$, but bounded in the domain $|z| < 1, |\arg z| > \varepsilon$, for any $\varepsilon > 0$.

If f is an analytic function in $|z| < 1$, D. Gaier and W. Meyer-Konig [5] have defined the radius R_φ defined by $z = re^{i\varphi}, 0 \leqq r < 1$, *singular* for f if $f(z)$ is unbounded in any sector $|z| < 1, \varphi - \varepsilon < \arg z < \varphi + \varepsilon$ with $\varepsilon > 0$. They showed that if f is unbounded in $|z| < 1$ and the power series expansion for f about zero has Hadamard gaps, then every radius is singular for f. We have

THEOREM 2. *Suppose f is a function which is analytic in $|z| < 1$ and has the power series expansion* (5). *Suppose*

$$\limsup_{r\to 1^-} \{\max |c_k| r^k\} = \infty .$$

Then each radius $R_\varphi (0 \leqq \varphi < 2\pi)$ is singular for f.

In section two the necessary lemmas are stated and the theorems are proved, while section three contains the proof of the essential lemma which enables us to use the idea of G. H. Hardy and J. E. Littlewood of accentuating the dominance of the largest term in the series (5) by repeated differentiation (c.f. Fuchs [2]).

2. Proofs of the Theorems. We need three lemmas:

LEMMA 1 (Fuchs [2]). *Let g be a function analytic in $|z| < R$. If, for some positive integer p*

$$|g^{(p)}(z)| \leqq M \qquad (|z| < R),$$

and

$$|g^{(p)}(0)| \geqq A > 0,$$

then $g(z)$ assumes in $|z| < R$ every value w lying in the disc

$$|w - g(0)| < KR^p A^{p+1} M^{-p}$$

where K is a positive number depending only on p.

LEMMA 2 (Gaier [4]). *Let E be a closed subset of $\{z \mid |z| = 1\}$ and assume that E has measure $2\pi\gamma$ where $0 < \gamma < 1$. If p is a polynomial with N terms, then*

$$\max_{|z|=1} |p(z)| \leqq C_N(\gamma) \cdot \max_{z \in E} |p(z)|$$

where

$$\log C_N(\gamma) = \left(\frac{\gamma}{1-\gamma}\gamma^{-N} - \frac{1}{1-\gamma}\right) \log 3 . \tag{7}$$

LEMMA 3. *Assume the hypothesis of Theorem 1. Let p, ν, and α be positive integers where $0 \leqq \alpha \leqq l$. For $k = 1, 2, 3, \cdots$ let $n_k^{(0)} = n_k$. Define*

$$S_0 = \exp\{-p/n_\nu^{(\alpha)}\}, \quad S_1 = \exp\left\{-\frac{P}{2n_\nu^{(\alpha)}}\left(1 + \frac{\log q}{q-1}\right)\right\},$$

and

$$W_{k,(\beta)} = n_k^{(\beta)}(n_k^{(\beta)} - 1)(n_k^{(\beta)} - 2) \cdots (n_k^{(\beta)} - p + 1) S^{n_k^{(\beta)}}$$

where $S_0 < S < S_1$; $0 \leqq \beta \leqq l$; and $k = 1, 2, 3 \cdots$. Then for a fixed γ with $0 < \gamma < 1$ there exists an integer p_0 depending on q and l such that for $p > p_0$ and $\nu > \nu_0(p)$

$$\sum_{k \neq \nu-1, \nu, \nu+1} (W_{k,(l)} + W_{k,(l-1)} + \cdots + W_{k,(0)}) < \frac{1}{4}(C_{3l+3}(\gamma))^{-1} W_{\nu,(\alpha)},$$

where $C_{3l+3}(\gamma)$ is defined by (7).

Proof of Theorem 1. We note that it suffices to show $f(z)$ assumes zero infinitely often in $|z| < 1$. So suppose $f(z)$ is zero only a finite number of times in $|z| < 1$ and denote these zeros by $z_1, z_2, \cdots, z_j$. Then $N(r, 1/f) = O(1)$, and it follows from the first fundamental theorem of Nevanlinna theory that

$$(8) \qquad m(r, f) = m(r, 1/f) + O(1) .$$

For $0 < r < 1$,

$$|f(re^{i\theta})| > \lambda_q\Big(\sum_{k=0}^{\infty} |c_k|^2 r^{2k}\Big)^{1/2} > 0$$

on a set of θ of measure not less than $\mu_q > 0$ [10, p. 216[1]], so

$$(9) \qquad m(r, f) \to \infty$$

as r approaches one by condition (4).

For a value r' for which (6) holds, let $\mathscr{E}(r')$ denote the set of θ in $[0, 2\pi]$ at which

$$(10) \qquad \log^+ \frac{1}{|f(r'e^{i\theta})|} > \frac{1}{2} m(r', 1/f) .$$

Denote the measure of $\mathscr{E}(r')$ by $|\mathscr{E}(r')|$. Then using Hölder's inequality and (6)

$$\begin{aligned}\pi m(r', 1/f) &\leqq \int_{\mathscr{E}(r')} \log^+ \frac{1}{|f(r'e^{i\theta})|} d\theta \\ &\leqq \Big(\int_{\mathscr{E}(r')} \Big(\log^+ \frac{1}{|f(r'e^{i\theta})|}\Big)^p d\theta\Big)^{1/p} |\mathscr{E}(r')|^{1/q} \\ &\leqq (2\pi C)^{1/p} m(r', 1/f) |\mathscr{E}(r')|^{1/q} .\end{aligned}$$

Thus,

$$(\pi/(2\pi C)^{1/p})^q \leqq |\mathscr{E}(r')| .$$

Define γ by

$$2\pi\gamma = (\pi/(2\pi C)^{1/p})^q .$$

Let $\rho = \max_{1 \leqq k \leqq j} |z_k|$, and let

$$U = \limsup_{k \to \infty} |c_k| .$$

If $U < \infty$, let N be the least integer such that

$$|c_k| < \frac{3}{2} U$$

for $k > N$. If $U = \infty$, let $N = 0$.

Define

$$\mu(r) = \sup_{k > N} |c_k| r^k , \qquad (0 \leqq r < 1) .$$

Let $V = V(r)$ be the largest integer such that

[1] Theorem 8.20 on page 215 easily extends to finite sums of series with Hadamard gaps, and so Theorem 8.25 on page 216 does also.

$$|c_V|r^V > \frac{1}{2}\mu(r) \ .$$

If $U = \infty$, we see

$$|c_V|r^V > 1 \ , \qquad (r > r_0) \ ,$$

and also $V(r) \to \infty$ as $r \to 1^-$. If $U < \infty$, we see

$$|c_V|r^V > \frac{1}{3}U$$

and again $V(r) \to \infty$ as $r \to 1^-$ since there are infinitely many integers k with

$$|c_k| > \frac{3}{4}U > \frac{1}{2}\mu(r) \ , \qquad (r < 1) \ .$$

Using the notation of Lemmas 1 and 3, we choose $p \geqq \max(N, p_0)$ and choose r so close to one that

$$\rho < rS_0 = r\exp\{-p/n_\nu^{(\alpha)}\}$$

where $n_\nu^{(\alpha)} = V(r)$, $n_{\nu-1}^{(l)} > 2p$, and $\nu > \nu_0(p)$. We may assume $r(S_0S_1)^{1/2}$ is a value r' for which (6) holds. Let

$$T(z) = \sum_{k=\nu-1}^{\nu+1}(a_{n_k^{(l)}}z^{n_k^{(l)}} + \cdots + a_{n_k}z^{n_k}) \ .$$

By Lemma 3

$$\begin{aligned}\sum_{k\neq\nu-1,\nu,\nu+1}\left(\sum_{i=0}^{l}|a_{n_k^{(i)}}|r^{n_k^{(i)}}W_{k,(i)}\right) &\leqq \mu(r)\sum_{k\neq\nu-1,\nu,\nu+1}(W_{k,(l)} + \cdots + W_{k,(0)}) \ , \\ &\leqq 2|a_{n_\nu^{(\alpha)}}|r^{n_\nu^{(\alpha)}}\cdot\frac{1}{4}(C_{3l+3}(\gamma))^{-1}W_{\nu,(\alpha)} \ .\end{aligned}$$

Hence,

$$f^{(p)}(rSe^{i\theta}) = T^{(p)}(rSe^{i\theta}) + E(rSe^{i\theta})$$

where

$$|E(rSe^{i\theta})| < \frac{1}{2}(C_{3l+3}(\gamma))^{-1}n_\nu^{(\alpha)}(n_\nu^{(\alpha)}-1)\cdots(n_\nu^{(\alpha)}-p+1)|a_{n_\nu^{(\alpha)}}|(rS)^{n_\nu^{(\alpha)}-p} \ .$$

Consequently,

$$|f^{(p)}(rSe^{i\theta})| \leqq C_1(p, l)(n_\nu^{(\alpha)})^p|a_{n_\nu^{(\alpha)}}|(rS_1)^{n_\nu^{(\alpha)}} \ ,$$

and using Lemma 2 on the polynomial $T^{(p)}(rSe^{i\theta})$ where rS is a value r' for which (6) is valid we find

$$|f^{(p)}(rSe^{i\theta})| \geqq C_2(p, l, \gamma)(n_\nu^{(\alpha)})^p|a_{n_\nu^{(\alpha)}}|(rS_0)^{n_\nu^{(\alpha)}}$$

for values of θ in $\mathscr{E}(rs)$.

Therefore if $r' = r(S_0S_1)^{1/2}$ is a value for which (6) holds, then we may apply Lemma 1 to

$$g(\zeta) = f(r(S_0S_1)^{1/2}e^{\zeta+i\theta})$$

with

$$R = \frac{1}{4}\left(1 + \frac{\log q}{q-1}\right)\frac{p}{n_\nu^{(\alpha)}}$$

where p and ν satisfy the hypotheses of Lemma 3. Then

$$\begin{aligned} R^p A^{p+1} M^{-p} &> C_3(p, q, l, \gamma)\,|a_{n_\nu^{(\alpha)}}|\,(rS_0)^{n_\nu^{(\alpha)}}(S_0/S_1)^{pn_\nu^{(\alpha)}}\,, \\ &> C_4(p, q, l, \gamma)\mu(r)\,, \\ &> C_5(p, q, l, \gamma, U)\,. \end{aligned}$$

Thus $f(z)$ takes every value w in the disc

$$|w - f(r(S_0S_1)^{1/2}e^{i\theta})| < C_6(p, q, l, \gamma, U)\,.$$

But by (10) we note that

$$|f(r(S_0S_1)^{1/2}e^{i\theta})| < \exp\left(-\frac{1}{2}m(r(S_0S_1)^{1/2}), 1/f)\right),$$

and because of (8) and (9) we conclude that when r is near enough to one

$$|f(r(S_0S_1)^{1/2}e^{i\theta})| < C_6(p, q, l, \gamma, U)\,.$$

Thus $f(z)$ will assume the value zero at points arbitrarily near $|z| = 1$ which contradicts our earlier assumption and proves the theorem.

Proof of Theorem 2. Suppose there is some radius R_φ which is not singular for f, and so there exists an $\varepsilon > 0$ such that $|f(z)|$ is bounded in the sector $\mathscr{S} = \{z \mid \varphi - \varepsilon < \arg z < \varphi + \varepsilon, |z| < 1\}$. Then for each complex number a, $f(z) - a$ is also bounded in $\mathscr{S}$. Thus taking $2\pi\gamma = 2\varepsilon$, the argument of Theorem 1 shows f assumes a infinitely often in $\mathscr{S}$. Since a is arbitrary, $|f|$ is unbounded in $\mathscr{S}$, and therefore R_φ is singular for f.

3. *Proof of Lemma* 3. If $n_k^{(\beta)} < p$, then $W_{k,(\beta)} = 0$. Turning to $p \leqq n_k^{(\beta)} < n_\nu^{(\alpha)}$, we first observe that for fixed β with $0 \leqq \beta \leqq l$,

$$\frac{n_{k+2}^{(\beta)}}{n_k^{(\beta)}} \geqq q > 1\,, \qquad (k = 1, 2, 3, \cdots)\,.$$

Assume $W_{k+2,(\beta)} \neq 0$ and $n_{k+2}^{(\beta)} \leqq n_\nu^{(\alpha)}$. Then

$$
(11)\quad \begin{aligned}
\frac{W_{k,(\beta)}}{W_{k+2,(\beta)}} &\leqq \left(\frac{n_k^{(\beta)}}{n_{k+2}^{(\beta)}}\right)^p (S_0)^{-n_{k+2}^{(\beta)}+n_k^{(\beta)}} \leqq \left(\frac{n_k^{(\beta)}}{n_{k+2}^{(\beta)}}\right)^p \exp\left\{p\left(1-\frac{n_k^{(\beta)}}{n_{k+2}^{(\beta)}}\right)\right\}, \\
&< \sup_{0<t<1/q} \exp\{p(1-t+\log t)\} \\
&\leqq \exp\{p(1-q)-\log q\}.
\end{aligned}
$$

Hence the right-hand side of (11) is less than $A^{-1}=(1+16(l+1)C_{3l+3}(\gamma))^{-1}$ provided

$$p > \frac{\log A}{\log q - 1 + q^{-1}} = p_1 .$$

Proceeding in a similar manner, we may also show

$$\frac{W_{\nu-3,(\beta)}}{W_{\nu,(\alpha)}} \leqq \frac{1}{A} \quad\text{and}\quad \frac{W_{\nu-2,(\beta)}}{W_{\nu,(\alpha)}} \leqq \frac{1}{A}$$

when $p > p_1$. Consequently for $W_{k,(\beta)} \neq 0$ where $k \leqq \nu-3$ and $k+2n=\nu-1$, we see

$$(12)\quad \frac{W_{k,(\beta)}}{W_{k+2,(\beta)}} \cdot \frac{W_{k+2,(\beta)}}{W_{k+4,(\beta)}} \cdots \frac{W_{\nu-5,(\beta)}}{W_{\nu-3,(\beta)}} \frac{W_{\nu-3,(\beta)}}{W_{\nu,(\alpha)}} \leqq \left(\frac{1}{A}\right)^n, \quad (0 \leqq \beta \leqq l),$$

and for $W_{k,(\beta)} \neq 0$ where $k \leqq \nu - 2$ and $k + 2n = \nu$ we see

$$(13)\quad \frac{W_{k,(\beta)}}{W_{k+2,(\beta)}} \cdot \frac{W_{k+2,(\beta)}}{W_{k+4,(\beta)}} \cdots \frac{W_{\nu-4,(\beta)}}{W_{\nu-2,(\beta)}} \cdot \frac{W_{\nu-2,(\beta)}}{W_{\nu,(\alpha)}} \leqq \left(\frac{1}{A}\right)^n, \quad (0 \leqq \beta \leqq l).$$

Using (12) and (13) provided $p > p_1$, we get

$$
(14)\quad \begin{aligned}
\sum_{k<\nu-1} (W_{k,(l)} + W_{k,(l-1)} + \cdots + W_{k,(0)}) &< \frac{2(l+1)}{A} \sum_{n=0}^{\infty} \left(\frac{1}{A}\right)^n W_{\nu,(\alpha)}, \\
&\leqq \frac{2(l+1)}{A-1} W_{\nu,(\alpha)} .
\end{aligned}
$$

Now for $k \geqq \nu > \nu_0(p)$ and x any integer with $0 < x \leqq p-1$, we have simultaneously

$$\frac{n_{k+2}^{(\beta)} - x}{n_k^{(\beta)} - x} < 2^{1/p}\left(\frac{n_{k+2}^{(\beta)}}{n_k^{(\beta)}}\right), \quad \frac{n_{\nu+2}^{(\beta)} - x}{n_\nu^{(\alpha)} - x} < 2^{1/p}\left(\frac{n_{\nu+2}^{(\beta)}}{n_\nu^{(\alpha)}}\right),$$

and

$$\frac{n_{\nu+3}^{(\beta)} - x}{n_\nu^{(\alpha)} - x} < 2^{1/p}\left(\frac{n_{\nu+3}^{(\beta)}}{n_\nu^{(\alpha)}}\right).$$

Then when $n_k^{(\beta)} \geqq n_\nu^{(\alpha)}$,

$$
\begin{aligned}
\frac{W_{k+2,(\beta)}}{W_{k,(\beta)}} &< 2\left(\frac{n_{k+2}^{(\beta)}}{n_k^{(\beta)}}\right)^p S^{n_{k+2}^{(\beta)}-n_k^{(\beta)}}, \\
&< 2t^p (S_1^{n_k^{(\beta)}})^{(t-1)},
\end{aligned}
$$

$$\leqq 2t^p\left\{\exp\left(-\frac{1}{2}\left(1+\frac{\log q}{q-1}\right)\right)\right\}^{p(t-1)} = \theta(t)$$

where $t = n_{k+2}^{(\beta)}/n_k^{(\beta)} \geqq q$. For $t \geqq q$, $\theta(t) \leqq \theta(q)$, so

$$\frac{W_{k+2,(\beta)}}{W_{k,(\beta)}} \leqq 2\exp\left\{-\frac{1}{2}p(q-1-\log q)\right\}, \tag{15}$$

when $n_k^{(\beta)} \geqq n_\nu^{(\alpha)}$. Hence the right-hand side of (15) is less than $1/A$ provided

$$p > \frac{2\log(2A)}{q-1-\log q} = p_2\,.$$

Proceeding in a similar manner we may also show

$$\frac{W_{\nu+2,(\beta)}}{W_{\nu,(\alpha)}} \leqq \frac{1}{A} \quad\text{and}\quad \frac{W_{\nu+3,(\beta)}}{W_{\nu,(\alpha)}} \leqq \frac{1}{A},$$

when $p > p_2$. Consequently for $k \geqq \nu + 2 \geqq \nu_0(p) + 2$ and $k = \nu + 2n$, we see

$$\frac{W_{k,(\beta)}}{W_{k-2,(\beta)}}\cdot\frac{W_{k-2,(\beta)}}{W_{k-4,(\beta)}}\cdots\frac{W_{\nu+4,(\beta)}}{W_{\nu+2,(\beta)}}\,\frac{W_{\nu+2,(\beta)}}{W_{\nu,(\alpha)}} \leqq \left(\frac{1}{A}\right)^n, \quad (0 \leqq \beta \leqq l)\,, \tag{16}$$

and for $k \geqq \nu + 3 > \nu_0(p) + 3$ and $k = \nu + 2n + 1$, we see

$$\frac{W_{k,(\beta)}}{W_{k-2,(\beta)}}\cdot\frac{W_{k-2,(\beta)}}{W_{k-4,(\beta)}}\cdots\frac{W_{\nu+5,(\beta)}}{W_{\nu+3,(\beta)}}\,\frac{W_{\nu+3,(\beta)}}{W_{\nu,(\alpha)}} \leqq \left(\frac{1}{A}\right)^n, \quad (0 \leqq \beta \leqq l)\,. \tag{17}$$

Using (16) and (17) provided $p > p_2$, we get

$$\begin{aligned}\sum_{k>\nu+1}(W_{k,(l)} + W_{k,(l-1)} + \cdots + W_{k,(0)}) &< \frac{2(l+1)}{A}\sum_{n=0}^{\infty}\left(\frac{1}{A}\right)^n W_{\nu,(\alpha)}\,,\\ &\leqq \frac{2(l+1)}{A-1}W_{\nu,(\alpha)}\,.\end{aligned} \tag{18}$$

Combining (14) and (18) we now have the lemma provided p_0 is the maximum of p_1 and p_2 (and remembering that $A = 1 + 16(l+1)C_{3l+3}(\gamma)$).

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