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THE FIXED POINT PROPERTY FOR ARCWISE CONNECTED SPACES: A CORRECTION

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Several years ago the second author stated a fixed point theorem for a class of arcwise connected spaces which includes the dendroids as well as certain nonunicoherent continua. Subsequently the first author detected a flaw in the proof. The present collaboration has produced a correct proof. Since the theorem has not been subsumed in the literature of the intervening years and since other authors have alluded to it, it seems desirable to publish the new proof.

For recent references to the theorem, see [1], [4] and [7]. The original, erroneous argument can be found in [5]. (The error (p. 1277) occurs in the assertion that $S' = \bigcup \{S'_\gamma\}$ is connected, and hence that $\mathscr{N}$ has a maximal member.)

In the present exposition a few changes have been made in terminology. In what follows an *arc* is a compact connected Hausdorff space with exactly two non-cutpoints. A space X is *arcwise connected* if for each two elements x and y of X with $x \neq y$, there exists an arc $[x, y]$ contained in X. It is convenient to write $[x, x] = \{x\}$ and $[x, y) = (y, x] = [x, y] - \{y\}$. A *circle* is the union of two arcs which meet only in their endpoints. We write $\square$ to denote the empty set. If $e \in X$ then an *e-ray* is the union of a maximal nest of arcs $[e, x]$. If R is an e-ray then

$$K_R = \bigcap \{\overline{R - [e, x)}: [e, x] \subset R\} ,$$

where the bar denotes closure. If X is not compact then it may be that K_R is empty, but in the compact case this cannot occur.

THEOREM. *If X is an arcwise connected Hausdorff space which contains no circle, if $e \in X$ and if $f: X \to X$ is continuous, then f has a fixed point or there exists an e-ray R such that $f(K_R) \subset K_R$.*

COROLLARY. *If X is an arcwise connected Hausdorff space which contains no circle and if there exists $e \in X$ such that K_R has the fixed point property for each e-ray R, then X has the fixed point property.*

Before embarking on the proof of the theorem, some subsidiary results will be helpful.

LEMMA 1. *If X is a Hausdorff space, A is an arc and $f: A \to$*

X is continuous, then $f(A)$ is arcwise connected.

Since A is locally connected and compact it follows that $f(A)$ is locally connected. In contrast to the case where A is separable, the arcwise connectivity of $f(A)$ is not immediate [3]. A proof of Lemma 1 can be found in the thesis of J. K. Harris [2]; it is a modification of an argument first used by J. L. Kelley (see, for example, [6; p. 39].) We give a sketch of that argument.

If x and y are elements of $f(A)$, then there exists a closed subset F of A which is minimal with respect to $\{x, y\} \subset f(F)$ and $f(a) = f(b)$ whenever a and b are the endpoints of a complementary interval of $A - F$. It follows from this minimality that $f(F)$ is connected and that x and y are the only non-cutpoints of $f(F)$. Therefore $f(F)$ is an arc, and so $f(A)$ is arcwise connected.

For the remainder of this paper X is an arcwise connected Hausdorff space which contains no circle and $e \in X$. In particular, if x and y are distinct elements of X then the arc $[x, y]$ is unique. Consequently the relation $x \leqq y$ if and only if $x \in [e, y]$ is a partial order. As usual, if $x \leqq y$ and $x \neq y$ we write $x < y$.

Of course each arc in X has a natural order which does not necessarily agree with the partial order $\leqq$. If a and b are elements of X and if p precedes q in the natural order on $[a, b]$, we write $[a, p, q, b]$.

LEMMA 2. *If a, b and c are elements of X such that $a < b$ and $a \nleq c$, then $a \in [b, c]$.*

Proof. If $b \leqq c$ then by transitivity the hypothesis that $a \nleqq c$ is contradicted. Therefore, by the uniqueness of arcs there exists $d \neq b$ such that $[e, b] \cap [e, c] = [e, d]$. Moreover,

$$a \in [e, b] - [e, d] \subset [d, b] \subset [d, b] \cup [d, c] = [b, c] .$$

LEMMA 3. *Let $f: X \rightarrow X$ be continuous and suppose x and t are elements of X such that $x < t < f(x)$, $t < f(t)$ and $f(x) \nleqq f(t)$. Then there exists $y \in (x, t]$ such that $f(y) \in [f(x), f(t)]$ and $f(y) \leqq f(x)$.*

Proof. By the uniqueness of arcs there exists $z \in X$ such that $[z, f(x)] = [e, f(x)] \cap [f(t), f(x)] \subset [f(t), f(x)]$, and therefore by Lemma 1, $[z, f(x)] \subset f([x, t])$. Because $f(x) \nleqq f(t)$ and $z \leqq f(t)$ it follows that $z \neq f(x)$. Consequently there exists $y \in (x, t]$ such that $z = f(y)$.

LEMMA 4. *If $f: X \rightarrow X$ is continuous and if p and q are elements of X such that $[f(p), p, q, f(q)]$, then there exists $x \in [p, q]$ such that*

$x = f(x)$.

Proof. By a straightforward maximality argument there exists $[x, y] \subset [p, q]$ which is minimal relative to $[f(x), x, y, f(y)]$. If $f(x) \neq x$ then $x = f(x_1)$ where $x_1 \in (x, y]$ so that $[x_1, y]$ contradicts the minimality of $[x, y]$. Therefore $f(x) = x$.

A subset C of X is called a *chain* if it is simply ordered with respect to the partial order $\leqq$.

LEMMA 5. *If $x \in X$ such that $x \not\leqq f(x)$ and if there exists $t_1 \in X$ such that $t_1 \leqq f(t_1) \leqq x$, then f has a fixed point.*

Proof. Let T be a subset of X which is maximal with respect to $T \cup f(T) \subset [e, x]$ and $t \leqq f(t)$ for all $t \in T$. Since $T \subset [e, x]$, there is a least upper bound t_0 of T. We will show that $t_0 = f(t_0)$.

Suppose first that $t_0 \not\leqq f(t_0)$ and $f(t_0) \not\leqq t_0$. Then there exist disjoint open sets U and V such that $t_0 \in V$, $f(V) \subset U$ and $U \cap [e, t_0] = \square = V \cap [e, f(t_0)]$. If $t \in T$ is chosen so that $[t, t_0] \subset V$, then $[f(t), f(t_0)] \subset f([t, t_0]) \subset U$ since, by Lemma 1, $f([t, t_0])$ is arcwise connected. Since $t < f(t)$ and $t \not\leqq f(t_0)$, it follows from Lemma 2 that $t \in [f(t), f(t_0)] \subset U$, and this contradicts our assumption that U and V are disjoint. Therefore, either $f(t_0) \leqq t_0$ or $t_0 \leqq f(t_0)$.

If $f(t_0) < t_0$ then there exist disjoint open sets 0 and W such that $t_0 \in 0$ and $f(0) \subset W$. If $y \in T$ is chosen so that $[y, t_0] \subset 0$, then $[f(y), f(t_0)] \subset W$ and, since $f(t_0) < y \leqq f(y)$, it follows that $y \in W$. Again this is a contradiction and therefore $t_0 \leqq f(t_0)$.

If $t_0 < f(t_0)$ then there are disjoint open sets U' and V' such that $t_0 \in V'$, $f(V') \subset U'$ and $U' \cap [e, t_0] = \square$. If $s \in [t_0, x]$ is chosen so that $[t_0, s] \subset V'$, then $s < f(t_0)$ and hence $[f(t_0), f(s)] \subset U'$. By Lemma 3 there exists $z \in (t_0, s]$ such that $f(z) \in [f(t_0), f(s)]$ and $f(z) \leqq f(t_0)$. Since $z < f(z) \leqq f(t_0) \leqq x$, the maximality of the set T is contradicted. Therefore $t_0 = f(t_0)$.

Proof of the theorem. Let $\mathscr{S}$ denote the family of all subsets S of X such that $S \cup f(S)$ is a chain and $t \leqq f(t)$ for each $t \in S$. Clearly $\{e\} \in \mathscr{S}$, so by Zorn's Lemma $\mathscr{S}$ has a maximal member S_0.

Suppose $S_0 \cup f(S_0) \subset [e, x]$ for some $x \in X$. If $x \not\leqq f(x)$ then f must have a fixed point by Lemma 5. If $x \leqq f(x)$ for each x such that $S_0 \cup f(S_0) \subset [e, x]$ then by maximality both x and $f(x)$ are members of S_0 and hence $x = f(x)$.

Therefore we may assume that $S_0 \cup f(S_0)$ is cofinal in some ray R. It follows readily that S_0 is cofinal in R. We will show that if

$f(K_R) - K_R \neq \square$ then f has a fixed point. Choose $y \in K_R$ such that $f(y) \in X - K_R$; then there is a generalized sequence x_n (i.e., a function whose domain is some ordinal number) in R such that $x_n < x_{n+1}$ and $x_n \to y$. Since S_0 is cofinal in R, the sequence x_n can be so chosen that there exists $y_n \in S_0 \cap [x_n, x_{n+1}]$, for each n.

If there exists n_1 such that $x_{n_1} \notin [e, f(x_{n_1})]$ then $[f(y_{n_1}), y_{n_1}, x_{n_1}, f(x_{n_1})]$, so that by Lemma 4, f has a fixed point. Consequently we may assume $x_n \leqq f(x_n)$ for each n. Moreover, since $f(y) \notin K_R$ we may assume $f(x_n) \notin R$, for each n.

If there exists n_2 such that $f(x_{n_2}) \leqq f(f(x_{n_2}))$ then we may find m such that $y_m \notin [e, f(f(x_{n_2}))]$ and therefore $[f(y_m), y_m, f(x_{n_2}), f(f(x_{n_2}))]$. Again, f has a fixed point by Lemma 4. Hence we may assume that $x_n < f(x_n) \nleqq f(f(x_n))$. But then the hypotheses of Lemma 5 are satisfied.

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