BI-INVARIANT PSEUDO-LOCAL OPERATORS ON LIE GROUPS

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Let G be a connected Lie group whose Lie algebra is not the semi-direct sum of a compact algebra and a solvable algebra. It is shown that any bi-invariant pseudo-local operator on G is the sum of an invariant differential operator and an operator with smooth kernel.

1. **Introduction.** We consider a class of operators on Lie groups, satisfying a weak local property. Roughly, a *pseudo-local operator* on a manifold M is a continuous linear operator, P, on the space of compactly supported functions on M, which extends to an operator P' on the space of compactly supported distributions on M, such that P' preserves singular support. It has been shown by Kohn and Nirenberg [3] that any pseudo-differential operator is pseudo-local. Stekaer [6] has proved that any bi-invariant pseudo-local operator on a complex semisimple Lie group is the sum of an invariant differential operator and an operator with smooth kernel. The proof of this theorem reduces to verifying that every smooth, invariant function on the Lie algebra of G minus the origin can be extended smoothly over the origin. Our main result is the verification of this hypothesis for a large class of Lie groups, proving the above theorem for these groups. For a given Lie group, this theorem implies that the class of bi-invariant differential operators on that group can be substantially extended only by considering operators which do not satisfy local properties.

After the original version of this paper had been submitted, the author learned that these results have been extended by Anders Melin [8] to include any Lie algebra which is not the direct sum of a compact algebra and an abelian one.[1] Independently, the author had extended the results to include the nilpotent case.

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2. **Definitions and notation.** Let G be a Lie group and $C^\infty(G)$, (resp. $C_0^\infty(G)$), the space of smooth functions (resp. smooth functions with compact support) on G. The dual of $C^\infty(G)$, which is the space of compactly supported distributions on G, will be denoted $\mathscr{E}'(G)$, while the dual of $C^\infty(G)$, the space of distributions on G, will be denoted $\mathscr{D}'(X)$.

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1. The author is indebted to Sigurdur Helgason for informing her of Melin's work.

For $u \in \mathscr{E}'(X)$ we define the singular support of u, denoted sing supp u, as $\{x \in X \mid u \notin C^\infty(U)$ for any neighborhood U of $x\}$.

A continuous linear operator $P: C_0^\infty(X) \to C^\infty(X)$ is called a *pseudo-local operator* if it extends to a continuous operator $P': \mathscr{E}'(X) \to \mathscr{D}'(X)$ such that P' preserves singular support; that is,

$$\text{sing supp } P'u \subseteqq \text{sing supp } u \quad \text{for } u \in \mathscr{E}'(X).$$

We now assume that there is a Lie group G which operates differentiably on X. That is, there is a differentiable map z

$$z: G \times X \to X$$

such that $z(ab, x) = z(a, z(b, x))$ for all $a, b \in G$, and all $x \in X$. If $f \in C_0^\infty(X)$ we define $_af$, the *left translate of* f *by* $a \in G$ as

$$_af(X) = f(z(a, x)) \quad \text{for } x \in X.$$

If $X = G$, then the *right translate of* f *by* $a \in G$ is defined by

$$f_a(b) = f(ba) \quad \text{for } a \in G.$$

We call the pseudo-local operator P *left invariant* (resp. *right invariant*) if

$$P(_af) = _a(Pf) \quad f \in C_0^\infty(X) \quad (\text{resp. } P(f_a) = (Pf)_a).$$

If $G = X$ is a Lie group, P is called *bi-invariant* if it is both left and right invariant.

Let $\mathfrak{g}$ be the Lie algebra of G, and let 1 denote the identity in G. G acts a group of automorphisms on $\mathfrak{g}$ via the adjoint representation, Ad. For any $a \in G$, $x \in \mathfrak{g}$, we write $a.x$ for $Ad(a)x$. A function f on $\mathfrak{g}$ or $\mathfrak{g} - \{0\}$ is called *invariant* if it is constant on the orbits of G on $\mathfrak{g}$. A function on G or $G - \{1\}$ is called *invariant* if it is constant on the conjugacy classes of G. If $\mathscr{F}$ is a family of functions, $\mathscr{F}^G$ will denote the subset of invariant functions.

3. Pseudo-local operators on Lie groups. Our main result is the following.

THEOREM. *Let G be a connected Lie group and $\mathfrak{g}$ its Lie algebra. If $\mathfrak{g}$ is not the semi-direct sum of a compact algebra and a solvable one, then every bi-invariant pseudo-local operator on G is the sum of an invariant differential operator and an operator with smooth kernel.*

This theorem has been proved by Stekaer-Hansen in the special case where $\mathfrak{g}$ is complex reductive, non-abelian, using the following reduction to a criterion involvling invariant functions on $\mathfrak{g}$.

PROPOSITION 1 (*Stetkaer*). *Let G be a connected Lie group with Lie algebra* $\mathfrak{g}$. *If the restriction map* $r: (C^\infty(\mathfrak{g}))^G \to (C^\infty(\mathfrak{g} - \{0\}))^G$ *is surjective, then every bi-invariant pseudo-local operator on G is the sum of an invariant differential operator and an operator with smooth kernel.*

For the proof of this proposition see Stetkaer [6].

We shall refer to the condition on $\mathfrak{g}$ in the proposition as *Stetkaer's hypothesis.*

Stetkaer's verification of this hypothesis for the case where $\mathfrak{g}$ is complex reductive uses a result of Kostant ([4] Theorem 7). Kostant's theorem implies the existence of a hyperplane $\mathfrak{h} \subset \mathfrak{g} - \{0\}$ and a smooth map $t: \mathfrak{g} \to \mathfrak{h}$ satisfying the following conditions.

(i)   There is a dense subset $\mathfrak{r} \subset \mathfrak{g}$ such that for any $x \in \mathfrak{r}$, there exists a unique $x' \in \mathfrak{h}$ with $a.x = x'$ for some $a \in G$.

(ii)   For any $x \in \mathfrak{r}$, $t(x) = x'$. In particular, if $y \in \mathfrak{h} \cap \mathfrak{r}$, then $t(y) = y$. Conditions (i) and (ii) above show that any invariant function f on $\mathfrak{g} - \{0\}$ is completely determined by its values on $\mathfrak{h}$. Since $\mathfrak{h} \subset \mathfrak{g} - \{0\}$, the function $f \circ t$ is defined and smooth on all of $\mathfrak{g}$. Therefore $f \circ t$ is the desired extension of f since in agrees with f on $\mathfrak{g} - \{0\}$.

Since Kostant's result does not extend even to real reductive Lie groups, we shall use a substantially different approach in our proof.

4.   Proof of the main theorem.   We shall verify Stetkaer's hypothesis in the case where $\mathfrak{g}$ is not the semi-direct sum of a compact Lie algebra and a solvable Lie algebra. If f is an invariant function which is smooth on $\mathfrak{g} - \{0\}$, it will be shown by explicit computation that all partial derivatives of f can be extended continuously over 0. We shall define a one parameter subgroup $\{a_t\}_{t \in R}$ of G and show that for a suitable basis of $\mathfrak{g}$ the transformation of the partial derivatives with respect to this basis can be easily computed (Lemma 3). Invariance of f under the action of this one-parameter group is sufficient to prove the theorem, since the action of $\{a_t\}_{t \in R}$ pushes "most" small elements in $\mathfrak{g} - \{0\}$ to the unit sphere.

Let $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$ be a Levi decomposition of $\mathfrak{g}$, with $\mathfrak{g}_1$ semisimple, and $\mathfrak{g}_2$ solvable. By assumption, $\mathfrak{g}_1$ is not compact. From the structure theory of semisimple algebras, it is well known that $\mathfrak{g}_1$ contains a subalgebra $\mathfrak{u}$, where $\mathfrak{u}$ is isomorphic to $\mathfrak{sl}(2, R)$, the Lie algebra of the real special linear group. (For the proof of this, as well as the details of the representation theory of $\mathfrak{u}$, to be used later, see

Serre (5), Chapitre IV and VI) or Helgason ([1] Chapter VII].) For any $x \in \mathfrak{g}$ let $ad\ x$ be the endomorphism defined by $ad\ x(y) = [x, y]$ for all $y \in \mathfrak{g}$; i.e. ad is the adjoint representation. If $\mathfrak{u}$ is any such subalgebra, let $x \in \mathfrak{u}$ be the inverse image of the element $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ under a fixed isomorphism of $\mathfrak{u}$ with $\mathfrak{sl}(2, \boldsymbol{R})$. From the representation theory of three-dimensional simple Lie algebras there is a vector sum decomposition

$$\mathfrak{g} = \mathfrak{g}^{(0)} + \sum_{j=1}^{k} \mathfrak{g}^{(r_j)} + \sum_{j=1}^{k} \mathfrak{g}^{(-r_j)} \, ,$$

where $\mathfrak{g}^{(r_j)} = \{y \in \mathfrak{g} \,|\, [x, y] = r_j y\}$, $\mathfrak{g}^{(-r_j)} = \{y \in \mathfrak{g} \,|\, [x, y] = -r_j y\}$, $\mathfrak{g}^{(0)} = \{y \in \mathfrak{g} \,|\, [x, y] = 0\}$ where the r_j are all positive integers.

$$\text{Let } \mathfrak{g}^+ = \sum_{j=1}^{k} \mathfrak{g}^{(r_j)} \text{ and } \mathfrak{g}^- = \sum_{j=1}^{k} \mathfrak{g}^{(-r_j)} \, .$$

We make the convention that $r_{-j} = -r_j$. Let $x_{0_1}, x_{0_2}, \cdots, x_{0_{p(0)}}$ be a basis for $\mathfrak{g}^{(0)}$, and for each j, positive and negative, let $x_{j_1} x_{j_2}, \cdots x_{j_{p(j)}}$ be a basis for $\mathfrak{g}^{(r_j)}$. Give $\mathfrak{g}$ the metric for which the above basis is orthonormal. We write $|y|$ for the length of an element $y \in \mathfrak{g}$. Any $y \in \mathfrak{g}$ has a unique decomposition $y = y_+ + y_0 + y_-$, with $y_+ \in \mathfrak{g}^+$, $y_0 \in \mathfrak{g}^{(0)}$ and $y_- \in \mathfrak{g}^-$. Then $|y|^2 = |y_+|^2 + |y_0|^2 + |y_-|^2$.

Let $\boldsymbol{D}$ be the family of all partial derivatives for the given basis. We write D_{n_j} for the partial derivative

$$\frac{\partial^{n_{j_1}}}{\partial x_{j_1}} \frac{\partial^{n_{j_2}}}{\partial x_{j_2}} \cdots \frac{\partial^{n_{j_m}}}{\partial x_{j_m}}$$

where $n_j = (n_{j_1}, n_{j_2}, \cdots, n_{j_m})$, with $n_{j_i} \geqq 0$ for all i, for all j, positive, negative or zero. Any $D \in \boldsymbol{D}$ can be written $D_{n_{-k}} \cdots D_{n_k}$. The *order* of D_{n_j}, $O(D_{n_j})$, is defined by

$$O(D_{n_j}) = \sum_{i=1}^{m} n_{j_i} \, .$$

Then the order of D is defined by

$$O(D) = \sum_{j=-k}^{k} O(D_{n_j}) \, .$$

The *height of* D, $h(D)$ is defined by

$$h(D) = \sum_{j=-k}^{h} r_j\, O(D_{n_j}) \, .$$

For any real t, let $a_t = \exp tx$, where $\exp \colon \mathfrak{g} \to G$ is the exponential map. Then $\{a_t\}_{t \in \boldsymbol{R}}$ is a one-parameter subgroup; we shall need only

the invariance of f under $\{a_t\}$.

Then if $y \in \mathfrak{g}^{(r_j)}$, $a_t \cdot y = e^{tr_j} y$, where $a_t \cdot y$ denotes the adjoint action of a_t on the element $y \in \mathfrak{g}$. The following lemma shows how the partial derivatives transform under the action of a_t.

LEMMA 2. $Df(a_t \cdot y) = e^{-th(D)} Df(y)$ for $y \in \mathfrak{g}$, $y \neq 0$, for any $D \in \mathbf{D}$.

Proof. We prove the formula by induction on the order of D. Suppose first that $D = \partial/\partial x_{j_s}$. Then

$$\frac{\partial f}{\partial x_{j_s}}(a_t \cdot y) = \lim_{\varepsilon \to 0} \frac{f(a_t \cdot y + \varepsilon x_{j_s}) - f(a_t \cdot y)}{\varepsilon},$$

$$= \lim_{\varepsilon \to 0} \frac{f(y + \varepsilon a_t^{-1} \cdot x_{j_s}) - f(y)}{\varepsilon},$$

by the invariance of f,

$$= \lim_{\varepsilon \to 0} \frac{f(y + \varepsilon e^{-tr_j} x_{j_s}) - f(y)}{\varepsilon}$$

$$= e^{-tr_j} \frac{\partial f}{\partial x_{j_s}}(y) = e^{-th(D)} Df(y).$$

Now assume the lemma is true whenever $O(\bar{D}) < k$. If $O(D) = k$, then $D = (\partial/\partial x_{j_l})\bar{D}$, where $O(\bar{D}) = k - 1$.

$$Df(a_t \cdot y) = \lim_{\varepsilon \to 0} \frac{\bar{D}f(a_t \cdot y + \varepsilon x_{j_l}) - \bar{D}f(a_t y)}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} e^{-th(\bar{D})} \frac{(\bar{D}f(y + \varepsilon a_t^{-1} \cdot x_{j_l}) - \bar{D}f(y))}{\varepsilon}$$

$$= e^{-th(\bar{D})} \lim_{\varepsilon \to 0} \frac{\bar{D}f(y + \varepsilon e^{-tr_j} x_{j_l}) - \bar{D}f(y)}{\varepsilon}$$

$$= e^{-th(\bar{D})} e^{-tr_j} \frac{\partial}{\partial x_{j_l}} Df(y)$$

$$= e^{-th(D)} D,$$

which proves Lemma 2.

LEMMA 3. Let $y \in \mathfrak{g} - \{0\}$ and $\delta > 0$ such that $|y| < \delta$. Then for $D \in \mathbf{D}$ and any $\varepsilon > 0$, there exists $y' \in \mathfrak{g}$, $|y'| < \delta$, such that

(1) $y'_+ \neq 0$ and $y'_- \neq 0$

and

(2) $|Df(y) - Df(y')| < \varepsilon$.

Proof. Since Df is continuous at y, there exists a neighborhood

V of y such that $z \in V$ implies $|Df(y) - Df(z)| < \varepsilon$. The intersection of V with the ball of radius δ around the origin is again a neighborhood, V', of y. Since $|y|^2 = |y_+|^2 + |y_0|^2 + |y_-|^2$, if either y_+ or y_- is 0, we may choose $z'_+ \in \mathfrak{g}^+$ and $z'_- \in \mathfrak{g}^-$ sufficiently small so that $y' = y + z'_+ + z'_-$ is still in V' which proves the lemma.

We now define a compact neighborhood U of width $1/2$ around the unit sphere, i.e.,

$$U = \{ y \in g \,|\, 1/2 \leq |y| \leq 3/2 \} \,.$$

Since U is compact, for any $D \in \boldsymbol{D}$, $\varepsilon > 0$, there exists $\delta_D > 0$ such that

$$|y_1 - y_2| < \delta_D \Longrightarrow |Df(y_1) - Df(y_2)| < \varepsilon \text{ for any } y_1, y_2 \in U \,.$$

LEMMA 4. *Let $D \in \boldsymbol{D}$ with $h(D) \neq 0$. Then for any $\varepsilon > 0$, there exists a neighborhood S_D of 0 in $\mathfrak{g}$ such that if $y \in S_D - \{0\}$, then $|Df(y)| < \varepsilon$.*

Proof. Let $M = \max_{z \in U} |Df(z)|$. Then for any $t \in \boldsymbol{R}$, $z \in U$ $|Df(a_t \cdot z)| \leq |e^{-th(D)} M|$, by Lemma 2. We shall assume, to minimize notation, that $h(D) > 0$. The proof for $h(D) < 0$ is similar. Choose t_1 satisfying $|e^{-t_1 h(D)} M| < \varepsilon$. Then $|e^{-th(D)} M| < \varepsilon$ for all $t > t_1$. Now let $r = \max_{j=1,2\ldots k} r_j$, and let $\delta_D = \min (1/2, e^{-t_1}/r)$. We define S_D as the sphere of radius δ_D, and we shall show that this satisfies our condition. For suppose $y = y_+ + y_0 + y_- \in S_D$. By Lemma 3 it suffices to assume that $y_+ \neq 0$ and $y_- \neq 0$. Since $|y|^2 = |y_+|^2 + |y_0|^2 + |y_-|^2$, we have $|y_-| < \delta_D$. Since $y_- \neq 0$, there exists t such that $|a_{-t} \cdot y| = 1$. Then

$$|a_{-t} \cdot y - a_{-t} \cdot y_-| = |a_{-t} \cdot (y_0 + y_+)| \leq |y_0 + y_+| < \delta_D \leq \frac{1}{2} \,.$$

So that $a_{-t} \cdot y \in U$. But

$$1 = |a_{-t} \cdot y_-| \leq |r| \, |e^t| \, |y_-| \,,$$

so that

$$e^t \geq 1/r |y_-| > 1/r(e^{-t_1}/r) = e^{t_1} \,,$$

which proves that $t > t_1$. Therefore, since $a_{-t} \cdot y \in U$, it follows from the definition of t_1 that $|Df(a_t \cdot (a_{-t} \cdot y))| < \varepsilon$. Since $a_t \cdot (a_{-t} \cdot y) = y$, this proves Lemma 4.

LEMMA 5. *For any $D \in \boldsymbol{D}$ with $h(D) = 0$ and any $\varepsilon > 0$, there exists a neighborhood S_D of 0 in $\mathfrak{g}$ such that if $y, y' \in S_D - \{0\}$, then $|Df(y) - Df(y')| < \varepsilon$.*

Proof of Lemma 5.   Choose δ, $0 < \delta \leqq 1$, such that for any z, $z' \in U$,

$$|z - z'| < \delta \text{ implies } |Df(z) - Df(z')| < \frac{\varepsilon}{5},$$

and let S_D be the ball of radius $\delta/2$ around 0.   Now let y, $y' \in S_D$ be arbitrary.   We will show that $|Df(y) - Df(y')| < \varepsilon$.

We write

$$y = y_+ + y_0 + y_-$$

and

$$y' = y'_+ + y'_0 + y'_-$$

as before.   By Lemma 3 we may assume that $y_+, y_-, y'_+ y'_-$ are all nonzero.   We show first

(3) $|Df(y_0 + y_+) - Df(y)| < \dfrac{\varepsilon}{5}$ and $|Df(y'_0 + y'_+) - Df(y')| < \dfrac{\varepsilon}{5}$.

For this, choose $t > 0$ such that $|a_t \cdot y_+| = 1$.   As in the proof of Lemma 4, $a_t \cdot y$ and $a_t \cdot (y_0 + y_+) \in U$.   Then $|a_t \cdot y - a_t \cdot (y_0 + y_t)| = |a_t \cdot y_-| < |y_-| < \delta$.   By the choice of δ, we have $|Df(a_t \cdot y) - Df(a_t \cdot (y_0 + y_+))| < \varepsilon/5$.   Since $h(D) = 0$, Df is invariant under a_t, so that the first inequality of (3) holds.   The proof of the second is the same.

By continuity of Df at $y_0 + y_+$ and $y'_0 + y'_+$, we may choose $q_- \in$ $\mathfrak{g}^-$ with $|q_-|$ sufficiently small so that

(4) $|Df(\bar{y}) - Df(y_0 + y_+)| < \varepsilon/5$ and $|Df(\bar{y}') - Df(y'_0 + y'_+)| < \varepsilon/5$,

where $\bar{y} = q_- + y_0 + y_+$ and $\bar{y}' = q_- + y'_0 + y'_+$.   Now choose $s > 0$ such that $|a_{-s} \cdot q_-| = 1$, which is possible since $q_- \neq 0$.   Then $a_{-s} \cdot \bar{y} \in U$ and $a_{-s} \cdot \bar{y}' \in U$.   We shall show

(5) $|Df(\bar{y}) - Df(\bar{y}')| < \dfrac{\varepsilon}{5}$.

Indeed,

$$
\begin{aligned}
|a_{-s} \cdot \bar{y} - a_{-s} \cdot \bar{y}'| &= |a_{-s} \cdot (y_0 + y_+) - a_{-s} \cdot (y'_0 + y'_+)| \\
&\leqq |a_{-s} \cdot (y_0 + y_+)| + |a_{-s}(y'_0 + y'_+)| \\
&\leqq |y_0 + y_+| + |y'_0 + y'_+| \\
&< \frac{\delta}{2} + \frac{\delta}{2} = \delta ,
\end{aligned}
$$

which proves $|Df(a_{-s} \cdot \bar{y}) - Df(a_{-s} \cdot \bar{y}')| < \varepsilon/5$.   Then (5) follows immediately since Df is invariant under a_{-s}.

To complete the proof of Lemma 5, note that

$$|Df(y) - Df(y')| \leqq |Df(y) + Df(y_0 + y_+)| + |Df(y') - Df(y_0' + y_+')|$$
$$+ |Df(y_0 + y_+) - Df(\bar{y})| + |Df(y_0' + y_+')$$
$$- Df(\bar{y}')| + |Df(\bar{y}) - D(\bar{y}')|$$
$$< \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} \text{ by (3), (4) and (5)} = \varepsilon .$$

We may now complete the proof of the theorem. Lemmas 4 and 5 show that for any $D \in \boldsymbol{D}$, the function Df can be extended continuously over **0**.

We shall assume the following, which can be proved using elementary calculus.

(6) *If $h(x)$ is a function on $\boldsymbol{R}$ such that dh/dx exists and is continuous off* **0**, *then h is differentiable if the function dh/dx can be extended continuously over* **0**.

By (6) and induction, it follows that Df exists and is continuous on all of $\mathfrak{g}$ for any $D \in \boldsymbol{D}$.

This finishes the proof of the theorem.

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