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THE EQUATION $y'(t) = F(t, y(g(t)))$

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A functional differential equation, in general, is a relationship in which the rate of change of the state of the system at time t depends on the state of the system at values of time, perhaps other than the present.

In this paper, sufficient conditions are given for g so that the initial value problem $y'(t) = F(t, y(g(t)))$, $y(p) = q$, may be solved uniquely; where F is both continuous into the Banach space B, and is Lipschitzean in the second position.

1. DEFINITIONS. If p is a real number and $I = \{I_1, I_2, \cdots\}$ is a collection of intervals so that $p \in I_1$ and $I_n \subseteqq I_{n+1}$ for each positive integer n, then I is said to be a nest of intervals about p. Let $I_0 = \{p\}$ and $a_0 = b_0 = p$. Also, let $[a_n, b_n] = I_n$ for each nonnegative integer n. Let I^* denote the union of all elements of I.

In general B denotes a Banach space; and if D is a real number set, let $C[D, B]$ denote the set of continuous functions from D into B. Whenever D is an interval, $C[D, B]$ is taken to be a Banach space with supremum norm $|\cdot|$.

If g is a continuous function from I^* into I^* so that $g(I_n) \subseteqq I_n$ for each positive integer n, then g is said to be an I-function. If g is an I-function then for each positive integer n, define the following:

$$A_n = \{x \in [a_n, a_{n-1}]: g(x) \notin I_{n-1}\} ,$$
$$B_n = \{x \in [b_{n-1}, b_n]: g(x) \notin I_{n-1}\}, \text{ and}$$
$$E_n(s) = [p, g(s)] \cap (A_n \cup B_n), \text{ for each } s \in I_n .$$

Let $\int_D h(s)ds$ denote the Lebesgue integral of h over the subset D of the domain of the Lebesgue integrable function h.

Let F denote a continuous function from $I^* \times B$ into B so that $\|F(x, y) - F(x, z)\| \leqq M(x)\cdot\|y - z\|$ for all $x \in I^*$ and $y, z \in B$, where M is Lebesgue integrable on each I_n. Furthermore, if f is a continuous nonnegative valued function from I^* to the reals, and m is a positive integer, let $\int_p^x (M, f, g, m)$ denote

$$\left|\int_p^x M(s_1)\left|\int_p^{(g_1)} M(s_2)\right| \cdots \left|\int_p^{g(s_{m-1})} M(s_m)f(s_m)ds_m\right| \cdots |ds_2|ds_1\right| .$$

If D is either A_n or B_n, let $\int (M, f, D, m)$ denote

$$\int_D M(s_1)\int_{E_n(s_1)} M(s_2) \cdots \int_{E_n(s_{m-1})} M(s_m)f(s_m)ds_m \cdots ds_2ds_1 .$$

If D is a subset of the domain of the function h, let $h|_D$ denote the restriction of h to D. Also, let $f\circ g$ denote the composition of f with g, whenever applicable; $f\circ g(x) = f(g(x))$.

2. Main results.

THEOREM A. *Suppose I is a nest of intervals about $p, q \in B$, g is an I-function, k is a sequence of positive integers, and for each positive integer n, $\alpha_n = \int(M, 1, A_n, k(n)) < 1$ and $\beta_n = \int(M, 1, B_n, k(n)) < 1$. Then there is a unique function $y \in C[I^*, B]$ so that $y'(t) = F(t, y(g(t)))$ and $y(p) = q$, for all $t \in I^*$. [We say then that the initial value problem (IVP) has unique solution.]*

Proof. Since, $I_0 = \{p\}$, then certainly $y_0 = \{(p, q)\}$ is the unique function in $C[I_0, B]$ so that for all $t \in I_0$, $y_0(t) = q + \int_p^t F(s, y_0(g(s)))ds$.

Next, suppose n is a nonnegative integer so that there is a unique function $y_n \in C[I_n, B]$ so that, for each $t \in I_n$, $y_n(t) = q + \int_p^t F(s, y_n(g(s)))ds$. The following is the construction of y_{n+1}. Let $D = \{f \in C[I_{n+1}, B]: f|_{I_n} = y_n\}$ and let $m = k(n+1)$. Then, if $f \in D$ and $t \in I_{n+1}$, let T be so that $Tf(t) = q + \int_p^t F(s, f(g(s)))ds$. Then, certainly T is from D into D.

LEMMA 1. *If $f, h \in D$ and $t \in I_{n+1}$, then*

$\|T^m f(t) - T^m h(t)\| \leqq \int_p^t (M, \|f\circ g - h\circ g\|, g, m)$, *for each positive integer m.*

Proof of Lemma 1. (by induction on m) If $m = 1$,

$$\begin{aligned}\|Tf(t) - Th(t)\| &= \left\|\int_p^t [F(s, f(g(s))) - F(s, h(g(s)))]ds\right\| \\ &\leqq \left|\int_p^t \|F(s, f(g(s))) - F(s, h(g(s)))\|ds\right| \\ &\leqq \left|\int_p^t M(s)\cdot\|f(g(s)) - h(g(s))\|ds\right| = \int_p^t (M, \|f\circ g - h\circ g\|, g, 1)\,.\end{aligned}$$

Now, suppose the lemma holds for $m = r$. Then,

$$\begin{aligned}&\|T^{r+1}f(t) - T^{r+1}h(t)\| \\ &\quad= \left\|\int_p^t [F(s, T^r f(g(s))) - F(s, T^r h(g(s)))]ds\right\| \\ &\quad\leqq \left|\int_p^t \|F(s, T^r f(g(s))) - F(s, T^r h(g(s)))\|ds\right| \\ &\quad\leqq \left|\int_p^t M(s)\cdot\|T^r f(g(s)) - T^r h(g(s))\|ds\right| \\ &\quad\leqq \left|\int_p^t M(s_1)\cdot\int_p^{g(s_1)} (M, \|f\circ g - h\circ g\|, g, r)ds_1\right|\,,\end{aligned}$$

by the induction hypothesis, but this equals $\int_p^t (M, \|f\circ g - h\circ g\|, g, r+1)$.

Lemma 2. *If N is a bounded, measurable function from I_{n+1} to the reals so that $N(s) = 0$ whenever s is in $I_{n+1}\backslash(A_{n+1} \cup B_{n+1})$, then*

$$\int_p^{a_{n+1}} (M, N, g, m) = \int (M, N, A_{n+1}, m) ,$$

and

$$\int_p^{b_{n+1}} (M, N, g, m) = \int (M, N, B_{n+1}, m) .$$

Proof of Lemma 2. (by induction on m) If $m = 1$, $\int_p^{a_{n+1}} (M, N, g, 1) = \left|\int_p^{a_{n+1}} M(s)N(s)ds\right| = \int_{A_{n+1}} M(s)N(s)ds = \int (M, N, A_{n+1}, 1)$, because N is 0 at each point of $[p, a_{n+1}]\backslash A_{n+1}$. Suppose the lemma is true for $m = r$. Then, $\int_p^{a_{n+1}} (M, N, g, r+1) = \int_p^{a_{n+1}} (M, U, g, r)$, where $U(s) = \left|\int_p^{g(s)} M(t)N(t)dt\right|$, for all $s \in I_{n+1}$. If $s \in I_{n+1}\backslash(A_{n+1} \cup B_{n+1})$, $g(s) \in I_n$. Thus, N is 0 on $[p, g(s)]$, and so $U(s) = 0$. Whence, U satisfies the conditions for N in the lemma. So, by the induction hypothesis, $\int_p^{a_{n+1}} (M, U, g, r) = \int (M, U, A_{n+1}, r) = \int (M, N, A_{n+1}, r+1)$, because $U(s) = \int_{E_{n+1}(s)} M(t)N(t)dt$. The proof of the second equality in the lemma is similar. Thus, Lemma 2 is proven.

Now, the two lemmas are applied. By Lemma 1, $\|T^m f(t) - T^m h(t)\| \leqq \int_p^t (M, \|f\circ g - h\circ g\|, g, m)$, for all $t \in I_m$, $\leqq \max\left\{\int_p^{a_{n+1}} (M, \|f\circ g - h\circ g\|, g, m), \int_p^{b_{n+1}} (M, \|f\circ g - h\circ g\|, g, m)\right\}$ which by Lemma 2 is $= \max\left\{\int (M, \|f\circ g - h\circ g\|, A_{n+1}, m), \int (M, \|f\circ g - h\circ g\|, B_{n+1}, m)\right\}$, because $\|f(g(s)) - h(g(s))\| = 0$ for all $s \in I_{n+1}\backslash(A_{n+1} \cup B_{n+1})$. Thus, $|T^m f - T^m h| \leqq \max\left\{\int (M, \|f\circ g - h\circ g\|, A_{n+1}, m), \int (M, \|f\circ g - h\circ g\|, B_{n+1}, m)\right\} \leqq \max\left\{\int (M, 1, A_{n+1}, m), \int (M, 1, B_{n+1}, m)\right\}\cdot|f - h|$. Thus, T^m is a contraction map from the complete metric space D into D. Thus T^m has a unique fixed point y_{n+1}. It is a known result that this implies that y_{n+1} is the unique fixed point of T. $[(Ty_{n+1}) = T(T^m(Ty_{n+1}) = T^m(Ty_{n+1})$, but only y_{n+1} is so that $y_{n+1} = T^m y_{n+1}$. So $Ty_{n+1} = y_{n+1}$, and uniqueness is clear.]

Thus, $y_{n+1}(t) = Ty_{n+1}(t) = q + \int_p^t F(s, y_{n+1}(g(s)))ds$, for all $t \in I_{n+1}$, and is the unique such function. Hence, by inductive definition, for each positive integer i, there is a unique function $y_i \in C[I_i, B]$ so that for all $t \in I_i$, $y_i(t) = q + \int_p^t F(s, y_i(g(s)))ds$. Now, define $y \in$

$C[I^*, B]$ so that $y(t) = y_n(t)$, whenever $t \in I_n$. Since $m \leqq n$ implies $y_n|_{I_m} = y_m$, y is well-defined, and $y(t) = q + \int_p^t F(s, y(g(s)))ds$, for all $t \in I^*$. Now, suppose $z(t) = q + \int_p^t F(s, z(g(s)))ds$, for all $t \in I^*$, and $z \in C[I^*, B]$. Then, if n is a positive integer, and $t \in I_n$, $z|_{I_n}(t) = q + \int_p^t F(s, z|_{I_n}(g(s)))ds$. So, $z|_{I_n} = y_n = y|_{I_n}$ for each positive integer n. Thus, $z = y$.

COROLLARY 1. *Let M be the constant 1 function, and let $k(n) = 2$, for all n. Suppose for each n, $\int_{A_n} \min\{|g(x) - a_{n-1}|, |g(x) - b_{n-1}|\}dx < 1$, and $\int_{B_n} \min\{|g(x) - a_{n-1}|, |g(x) - b_{n-1}|\}dx < 1$. Then, the IVP has a unique solution.* [See Figure 1. All the shaded area between each pair of vertical dashed lines is less than one.]

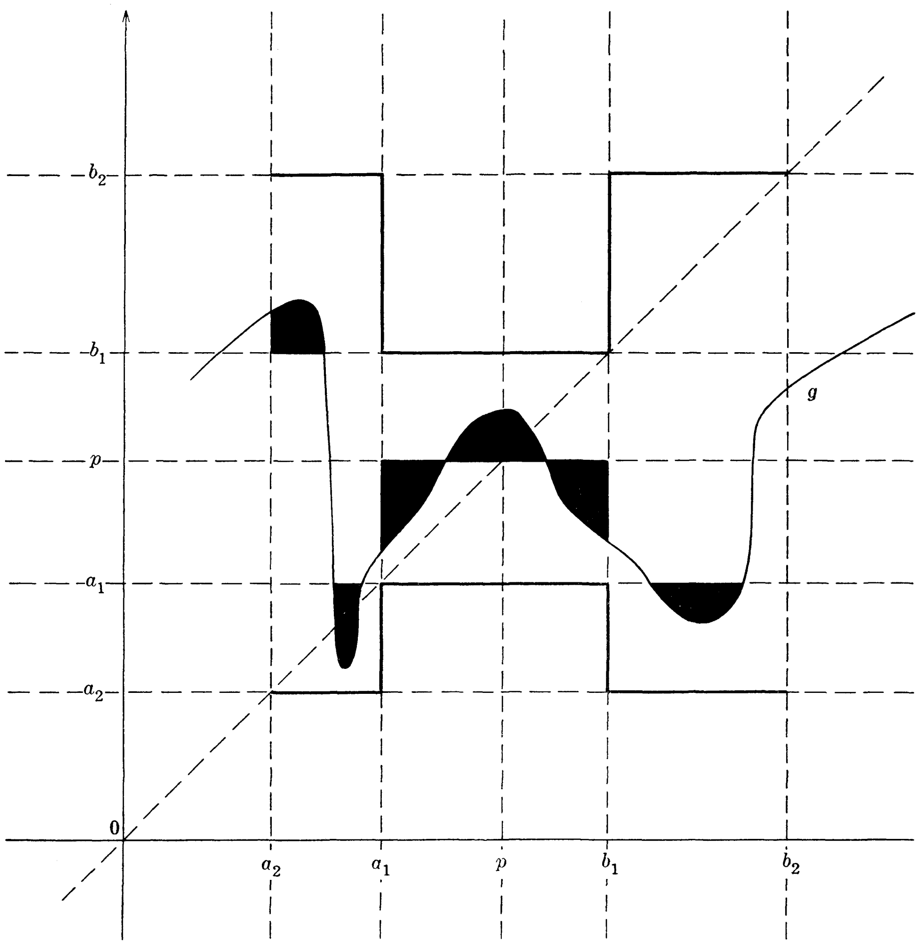

FIGURE 1

Proof. $\alpha_n = \int_{A_n} M(s_1) \int_{E_n(s_1)} M(s_2) ds_2 ds_1 = \int_{A_n} \int_{E_n(s_1)} ds_2 ds_1$. Now, $s_1 \in A_n$ implies

$$E_n(s_1) = \begin{cases} A_n \cap [p, g(s_1)], \text{ if } g(s_1) \in A_n, \text{ and} \\ B_n \cap [p, g(s_1)], \text{ if } g(s_1) \in B_n \,. \end{cases}$$

Thus, $E_n(s_1) \subseteqq [g(s_1), a_{n-1}]$ if $g(s_1) \in A_n$, and in this case, $|g(s_1) - a_{n-1}| \leqq |g(s_1) - b_{n-1}|$. Also, $E_n(s_1) \subseteqq [b_{n-1}, g(s_1)]$ if $g(s_1) \in B_n$, and in this case, $|g(s_1) - b_{n-1}| \leqq |g(s_1) - a_{n-1}|$. Thus, $E_n(s_1)$, which is certainly measurable, must have measure $\leqq \min\{|g(s_1) - a_{n-1}|, |g(s_1) - b_{n-1}|\}$. Hence, $\int_{A_n} \int_{E_n(s_1)} ds_2 ds_1 \leqq \int_{A_n} \min\{|g(s_1) - a_{n-1}|, |g(s_1) - b_{n-1}|\} ds_1$, because $\int_{E_n(s_1)} ds_2$ is the measure of $E_n(s_1)$. Thus, $\alpha_n < 1$, and similarly $\beta_n < 1$, for each positive integer n. Apply Theorem A.

COROLLARY 2. *Suppose $k(n) = 1$ for each n. Then, if $\int_{A_n} M < 1$ and $\int_{B_n} M < 1$, for each n, the IVP has unique solution.*

Proof. Immediate.

COROLLARY 3. *Suppose M is the constant 1 function and $k(n) = 1$ for each n. Then if $\max\{b_n - b_{n-1}, a_{n-1} - a_n\} < 1$, for each n, the IVP has unique solution.*

Proof. $A_n \subseteqq [a_{n-1}, a_n]$ and $B_n \subseteqq [b_{n-1}, b_n]$ implies $\int_{A_n} 1 \leqq \int_{a_n}^{a_{n+1}} 1 = a_{n-1} - a_n$ and $\int_{B_n} 1 \leqq \int_{b_{n-1}}^{b_n} 1 = b_n - b_{n-1}$. Apply Corollary 2.

The following example illustrates the advantage of allowing $k(n)$ to assume integral values other than 1.

EXAMPLE. Let F be so that $M = 1$ in the *IVP*—$y(p) = q$, $y'(t) = F(t, y(g(t)))$, where

$$g(x) = \begin{cases} 2x \quad , \text{ if } x \in [0, p], \text{ and} \\ 4p - 2x, \text{ if } x \in [p, 2p] \,. \end{cases}$$

then it is straightforward to show that if J is a subinterval of $[0, 2p]$ and $g(J) \subseteqq J$, then $J = [0, 2p]$. Thus, if I is a nest of intervals about any point of $[0, 2p]$ and $I^* = [0, 2p]$, then $I_n = [0, 2p]$ for each positive integer n, if g is to be an I-function. Thus, in order to apply Corollary 3, it seems necessary to require $p < 1$, in order to solve the IVP. However, if Theorem A is applied with $k(n) = m$ for all n, then Theorem B, which follows, shows that the condition $p < 2^{(m-1)/m}$ gives the best apparent bound for the size of p in order to solve the IVP. Now, since m is arbitrary, clearly, p may be any positive number less than 2.

THEOREM B. *If g is as in the above example, and for each positive integer n, $F_n(x) = \int_p^x (1, 1, g, n+1)$, then*

(1) *F_n is symmetric about p. That is, for each n, $F_n(x) = F_n(2p - x)$, for all $x \in [0, p]$; and*

(2) *$F_n(x) + F_n(p - x) = p^{n+1}/2^n$, for each n, and for all $x \in [0, p/2]$.*

Proof. (induction on n) Suppose $n = 1$. Then, if $x \in [0, 2p]$, $F_1(x) = \left|\int_p^x |g(s) - p|\, ds\right|$, which is

$$F_1(x) = \begin{cases} p^2/2 - px + x^2, & \text{if } x \in [0, p/2]\ , \\ px - x^2, & \text{if } x \in [p/2, p]\ , \\ -2p^2 + 3px - x^2, & \text{if } x \in [p, 3p/2], \text{ and} \\ 5p^2/2 - 3px + x^2, & \text{if } x \in [3p/2, 2p]\ . \end{cases}$$

It is straightforward to show that F_1 satisfies the conditions (1) and (2) of the theorem. Now, suppose the theorem is true for the positive integer k. Then, for each $x \in [0, 2p]$, $F_{k+1}(x) = \left|\int_p^x F_k(g(s))ds\right|$. If $x \in [0, p]$, $F_{k+1}(2p - x) = \left|\int_p^{2p-x} F_k(g(s))ds\right|$. Thus, if $x \leqq s \leqq p$, $g(s) = 2s = 4p - 2(2p - s) = g(2p - s)$. So, $F_{k+1}(x) = \int_p^x F_k(g(s))ds = \int_{2p-x}^p F_k(g(2p - s))(-1)ds$, by change of variable, but this is $\int_p^{2p-x} F_k(g(2p - s))ds = \int_p^{2p-x} F_k(g(s))ds = F_{k+1}(2p - x)$. Thus, F_{k+1} is symmetric about p.

Now, suppose $x \in [0, p/2]$. Then,

$$\begin{aligned} &F_{k+1}(x) + F_{k+1}(p - x) \\ &\quad = \int_x^p F_k(g(s))ds + \int_{p-x}^p F_k(g(s))ds \\ &\quad = \int_x^p F_x(2s)ds + \int_{p-x}^p F_k(2s)ds, \text{ because } g(s) = 2s \\ &\quad = \int_x^p F_k(2s)ds + \int_{p-x}^p F_k(2p - 2s)ds, \text{ because } g(z) = g(2p - z) \\ &\quad = \int_x^p F_k(2s)ds - \int_{2x}^0 (1/2)F_k(s)ds, \text{ by change of variable} \end{aligned}$$

$$= \int_x^p F_k(2s)ds + (1/2)\int_0^{2x} F_k(s)ds$$
$$= \int_x^p F_k(2s)ds + \int_0^x F_k(2s)ds, \text{ by change of variable}$$
$$= \int_0^p F_k(2s)ds$$
$$= (1/2)\int_0^{2p} F_k(s)ds, \text{ by change of variable}$$
$$= \int_0^p F_k(s)ds, \text{ because } F_k \text{ is symmetric about } p$$
$$= \int_0^{p/2} F_k(s)ds + \int_{p/2}^p F_k(s)ds$$
$$= \int_0^{p/2} F_k(s)ds - \int_0^{p/2} F_k(p - s)(-1)ds, \text{ by change of variable}$$
$$= \int_0^{p/2} \{F_k(s) + F_k(p - s)\}ds$$
$$= \int_0^{p/2} \{p^{k+1}/2^k\}ds, \text{ by the induction hypothesis}$$
$$= p^{k+2}/2^{k+1}.$$

By Theorem B, $F_n(0) + F_n(p - 0) = p^{n+1}/2^n$. But, $F_n(p) = 0$, by definition of F_n, and thus $F_n(0) = p^{n+1}/2^n$. Also, $F_n(2p) = F_n(2p - 0) = F_n(0) = p^{n+1}/2^n$. Thus, if $p^{n+1}/2^n < 1$, then $\alpha_{n+1} \leqq F_n(0) = p^{n+1}/2^n < 1$, and $\beta_{n+1} \leqq F_n(2p) = p^{n+1}/2^n < 1$. Apply Theorem A.

3. Applications.

The following is a generalization of a theorem by Anderson [1].

Let F be a continuous real-valued function with domain D of the plane $R \times R$ so that the partial derivative F_2 is continuous on D and $(0, b) \in D$. Let h' and k be so that if $|x| \leqq h'$ and $|y - b| \leqq k$, then $(x, y) \in D$. Let $K = \sup\{|F(x, y)|: |x| \leqq h' \text{ and } |y - b| \leqq k\}$, $h = \min\{k', k/K\}$, and $M = \sup\{|F_2(x, y)|: |x| \leqq h \text{ and } |y - b| \leqq k\}$.

THEOREM C. *Suppose there are intervals* $I_1 \subseteqq I_2 \subseteqq \cdots \subseteqq I_m = [-h, h]$ *so that max* $\{b_n - b_{n-1}, a_{n-1} - a_n\} \cdot M < 1$ *for each integer in* $[1, m]$, *and so that* $0 \in I_1$. *Let* $I_j = I_m$ *for each* $j \geqq m$. *Then, if* g *is an I-function, there is a unique function* y *so that* $y(0) = b$ *and* $y'(t) = F(t, y(g(t)))$, *for all* $t \in [-h, h]$.

Proof. Let $E = \{(x, y): |x| \leqq h, |y - b| \leqq k\}$, and let G be an extension of $F|_E$ so that

$$G(x, y) = \begin{cases} F(x, b - k), \text{ if } y \leqq b - k, \text{ and} \\ F(x, b + k), \text{ if } y \geqq b + k\,. \end{cases}$$

By continuity of F_2 and the mean value theorem, it follows that F is Lipschitzean in the second position with constant M. It follows, also, that G has the same Lipschitz constant M. Then, by Corollary 2, there is a unique function $y \in C[I^*, B] = C[[-h, h], R]$ so that $y'(t) = G(t, y(g(t)))$, $y(0) = b$, for all $t \in [-h, h]$. Equivalently, $y(t) = b + \int_0^t G(s, y(g(s)))ds$, for all $|t| \leqq h$. Thus, $|y(t) - b| = \left|\int_0^t G(s, y(g(s)))ds\right| \leqq h \cdot \sup\{|G(s, y(g(s)))|: |s| \leqq h\}$, and since the range of G is a subset of the range of $F|_E$, we have that this is $\leqq h \cdot \sup\{|F(x, v)|: |x| \leqq h, |v - b| \leqq k\} = h \cdot K \leqq k$, by definition of h. Thus, $G(x, y(g(x))) = F(x, y(g(x)))$, for all $|x| \leqq h$. So, $y'(t) = F(t, y(g(t)))$, $y(0) = b$, for all $t \in [-h, h]$.

The following is a generalization of a theorem by Kuller [3].

THEOREM D. *Suppose only that g is a continuous function with connected, real domain E so that g is not the identity, but gog is the identity. Then, if $M = 1$ and $q \in B$, there is a segment Q about the unique fixed point p' of g so that if $p \in Q \cap E$, the IVP has unique solution.*

Proof. Kuller proves that g has a unique fixed point p' and that g is strictly decreasing. Let $0 < k < 1/2$. Let $\beta_0 = p$ and let β be a nondecreasing sequence of reals so that $\beta_i - \beta_{i-1} < k$, for each positive integer i, and so that β converges to the right boundary of E, which may be $+\infty$. Then, for each positive integer i, let $\{\alpha_{i1}, \alpha_{i2}, \cdots, \alpha_{in_i}\}$ be so that $g(\beta_i) = \alpha_{in_i} \geqq \cdots \geqq \alpha_{i2} \geqq \alpha_{i1} = g(\beta_{i-1})$ and also so that $\alpha_{ij} - \alpha_{i,j+1} < k$, for all j. Then, $\{[\alpha_{ij}, g(\alpha_{ij})]: i \geqq 1$ and $1 \leqq j \leqq n_i\}$ is a monotonic collection of intervals, each containing p. Let $I_1 = [\alpha_{11}, g(\alpha_{11})]$. Suppose I_m has been defined to be $[\alpha_{ij}, g(\alpha_{ij})]$. Then, let $g(\alpha_{ij})$

$$I_{m+1} = \begin{cases} [\alpha_{i,j+1}, g(\alpha_{i,j+1})], \text{ if } j < n_i, \text{ and} \\ [\alpha_{i+1,2}, g(\alpha_{i+1,2})], \text{ if } j = n_i . \end{cases}$$

Relabel I_n to be $[a_n, b_n]$. Then, $\max\{a_{n-1} - a_n, b_n - b_{n-1}\} < 1$, for each positive integer n. Let $Q = (a_1, b_1)$. Then apply Corollary 3.

Kuller required differentiability of g in order to solve $y' = y \circ g$, $y(p') = q$, where p' is the unique fixed point of g.

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