PACIFIC JOURNAL OF MATHEMATICS
Vol. 43, No. 2, 1972

SESQUILINEAR FORMS IN INFINITE DIMENSIONS

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This paper is concerned with sesquilinear forms defined on vector spaces of arbitrary dimension. Motivation is taken from classical Hilbert space theory, as the orthogonality relation induced by the form is used to replace the topology. First, an algebraic version of the Frechet-Riesz Representation Theorem is proved for linear functionals having an orthogonally closed kernel. Next, the notion of adjoint is formulated, following von Neumann, in the language of linear relations. It is proved that the adjoint of an arbitrary relation is a single valued linear relation precisely when the domain of that relation is orthogonally dense. Finally, an algebraic version of a continuous linear operator is introduced and the relationship with the notion of adjoint and linear functional is studied. The main result here is that an operator is orthogonally continuous precisely when it has an everywhere defined adjoint. These general results of pure algebra imply standard topological results in the context of a Hilbert space.

There are two directions in which to generalize away from the concept of a Hilbert space. One is the familiar topological generalization via Banach spaces, linear topological spaces. The other direction is algebraic via inner product spaces, sesquilinear forms. The finite dimensional theory of sesquilinear forms is well worked out. However, the infinite dimensional case seems fraught with pathology. Kaplansky and others have initiated a study of the infinite dimensional case [6], [7], [8]. Gross and Fischer [4] have used topological methods. In this paper, we propose an algebraic approach to infinite dimensions motivated by the "happy accidents" in Hilbert space theory that correlate algebraic and topological conditions. In particular, we prove an algebraic version of the Frechet-Riesz Representation Theorem, von Neumann's theorem on the single valuedness of the adjoint relation, and discuss continuity, all in the algebraic context of a vector space over a division ring with no "natural" topology present.

2. Quadratic spaces. We shall follow the terminology of Bourbaki [2] on sesquilinear forms.

By a *quadratic space* we mean a triple (k, E, Φ) where E is a left vector space over the division ring k and Φ is a nondegenerate orthosymmetric θ-sesquilinear form on E with respect to the involutive anti-automorphism θ of k. Given vectors x and y in E, we say x is *orthogonal* to y and write $x \perp y$ when $\Phi(x, y) = 0$. For any subset M of E, we define the *orthogonal of M* by

$$M^{\perp} = \{x \text{ in } E \mid x \perp m \text{ for all } m \text{ in } M\} .$$

It is clear that $M^{\perp}$ is always a subspace of E. A vector x of E is called *isotropic* if $x \perp x$ and is *anisotropic* otherwise.

The two main differences between general quadratic spaces and Hilbert space is first in the general nature of the scalars and second, in the possible existence of nonzero isotropic vectors. The role of isotropic vectors is important in physical theories and indeed a good example to hold in mind is the geometry of space-time with the Minkowski metric. Here, of course, $k = \boldsymbol{R}$, $E = \boldsymbol{R}^4$, and

$$\Phi((\alpha_1, \alpha_2, \alpha_3, \alpha_4), (\beta_1, \beta_2, \beta_3, \beta_4)) = \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 - \alpha_4\beta_4 .$$

The first "happy accident" to note is that in Hilbert space, a subspace M is metrically closed precisely when $M = M^{\perp\perp}$. Thus we are led to consider the closure operator $M \mapsto M^{\perp\perp}$ on the lattice of all subspaces of E, Lat (k, E), as an algebraic substitute for the topology. Let $P_c(E, \Phi) = \{M \text{ in Lat } (k, E) \mid M = M^{\perp\perp}\}$. The geometry of $P_c(E, \Phi)$, which is of interest in the study of the logical foundations of quantum mechanics, has been considered in [9].

In a Hilbert space H we have that each closed space M yields an orthogonal direct sum decomposition $H = M \oplus M^{\perp}$. This is not true for a general quadratic space. A subspace F of E is said to be a *splitting subspace* provided $E = F + F^{\perp}$. Let $P_s(E, \Phi)$ be the collection of all splitting subspaces of E. It is easy to see that each splitting subspace is $\perp$-closed. We shall show later that the converse need not hold.

It is well known that the lattice of closed subspaces of a Hilbert space is an orthomodular lattice. We have shown elsewhere [10] that orthomodularity actually residues in $P_s(E, \Phi)$ in general and $P_s(E, \Phi)$ is an orthomodular poset which need not be a lattice. Thus the orthomodularity of the lattice of closed subspaces of Hilbert space arises from the "happy accident" that $P_s(H) = P_c(H)$.

3. Linear functionals.

The next "happy accident" we note is that a linear functional on a Hilbert space is continuous exactly when it has a closed kernel. This motivates our next definition.

Let (E, Φ) be a quadratic space. Let f be a linear functional on E. Call f *orthocontinuous* if $\ker(f) = \ker(f)^{\perp\perp}$. Let E' denote the set of all orthocontinuous linear functionals on E and call it the *orthodual of* E. Let E^* denote the algebraic dual space of E.

3.1. *Frechet-Riesz Representation Theorem.* Let (E, Φ) be a quadratic space. Then the induced map $d: E \rightarrow E^*$ defined by

$$d(y)(x) = \Phi(x, y)$$

is a θ-linear monomorphism and $im(d) = E'$. Moreover, the image under d of all anisotropic vectors consists precisely of all those linear functionals whose kernel is a splitting subspace of E.

Proof. For y in E we have $\ker d(y) = (ky)^\perp$ which is a closed subspace of E. Thus $\operatorname{im}(d) \subseteq E'$. Next let f be in E'. If f is the zero functional then $f = d(0)$ and f is in $\operatorname{im}(d)$. So assume f is not identically zero. Then $\ker(f)$ is a hyperplane in E. Thus there is line kw with $E = \ker(f) \oplus kw$. Now pick a nonzero vector z in $\ker(f)^\perp$. Then

$$(0) = E^\perp = (\ker(f) \oplus kw)^\perp = \ker(f)^\perp \cap (kw)^\perp$$

so that $\Phi(w, z) \neq 0$. Let $y = ((\Phi(w, z))^{-1} f(w))^{\theta^{-1}} z$. Note y is in kz which is contained in $\ker(f)^\perp$. Thus $\Phi(w, y) = f(w)$.

Now let x be any vector in E. Then there is a unique x_1 in $\ker(f)$ and x_2 in kw such that $x = x_1 + x_2$. Then $f(x) = f(x_2)$ and $\Phi(x, y) = \Phi(x_2, y)$. But $x_2 = \lambda w$ so $f(x) = f(x_2) = \lambda f(w) = \lambda \Phi(w, y) = \Phi(\lambda w, y) = \Phi(x_2, y) = \Phi(x, y)$. Thus $f = d(y)$ and hence $\operatorname{im}(d) = E'$.

The fact that d is a monomorphism follows from the non-degeneracy of Φ.

If y is anisotropic, then y does not belong to $ky^\perp$ so $\ker d(y) = (ky)^\perp$ and $(ky)^\perp \oplus ky = E$. On the other hand if $\ker(f) \oplus \ker(f)^\perp = E$, then $\ker(f)$ is closed so there is a y with $f = d(y)$. Since $(ky)^\perp \oplus ky = E$, y is clearly anisotropic.

Note that the theorem above implies the usual Frechet-Riesz Representation theorem for real, complex, and quaternionic Hilbert spaces.

The corollaries below follow readily.

COROLLARY 3.2. *If Φ admits nonzero isotropic vectors, then there are closed subspaces of E that are not splitting.*

COROLLARY 3.3. *The orthodual of E is a total subspace of E^*.*

COROLLARY 3.4. *Let M be a closed subspace of E with x a vector not in M. Then there is an orthocontinuous linear functional f such that $f(x) \neq 0$, but $M \subseteq \ker(f)$.*

4. Adjoint. Let (E, Φ) be a quadratic space. We shall imitate the von Neumann formulation of the notion of adjoint. Let T be a relation on E with graph $G(T)$. We say T is a *closed relation* if $G(T) = G(T)^{\perp\perp}$ where $\perp$ is taken relative to $\Phi \oplus \Phi$ $E \oplus E$. Note a closed relation is necessarily a linear relation i.e. T or $G(T)$ if you prefer, is a subspace of $E \oplus E$. The closure $\bar{T}$ of the relation T is defined by $G(\bar{T}) = G(T)^{\perp\perp}$. Clearly $\bar{T}$ extends T. We also note that if T is a closed linear relation, then $\ker(T)$ is a closed linear subspace of E.

Now define $U: E \times E \to E \times E$ by $U(x, y) = (-y, x)$. Then U is an everywhere defined linear bijection with $U^{-1}(y, x) = (x, -y)$. Also note that $\Phi \oplus \Phi(Uz, w) = \Phi \oplus \Phi(z, U^{-1}w)$ and for $M \subseteqq E \times E$, we have $U(M^{\perp}) = U(M)^{\perp}$. For T any relation on E, define T^* a relation on E by $G(T^*) = U(G(T))^{\perp}$. Call T^* the *adjoint* of T. Note then that *every* linear operator has an adjoint. The question is whether or not the adjoint is single valued.

The usual definition of adjoint is given by demanding the existence of a linear operator T^* for a given linear operator T, such that the identity $\Phi(Tx, y) = \Phi(x, T^*y)$ holds for all x and y. It is interesting to note this formal identity persists. For if T is a relation on E with (x, z) in $G(T)$ and (y, w) in $G(T^*)$, then

$$\Phi(z, y) = \Phi(x, w) .$$

If we formally write $z = Tx$ and $w = T^*y$, we recover the previous equation.

It was brought to our attention that the next theorem was previously obtained by R. Arens [1] p. 16, Prop. 3.32. The Hilbert space origin of the idea goes back to J. von Neumann [12].

THEOREM 4.1. *Let T be a relation on E. Then T^* is single valued if and only if* $(\mathrm{dom}(T))^{\perp\perp} = E$.

In view of [1], we omit the proof.

It is interesting to note that the single valuedness of T^* depends only on the nature of the domain of T and not whether T is single valued or even linear.

COROLLARY 4.2. (1) *Let T be a linear relation on E. Then T^* is single valued if and only if T has an orthogonally dense domain;*

(2) *T^* has dense domain if and only if T^{**} is single valued;*

(3) *The closure of a linear operator is single valued exactly when its adjoint has a dense domain.*

Following S. S. Holland Jr., (to whom we are indebted for several ideas of this section), we shall use the term *CDD operator* to mean a closed domain dense linear operator.

COROLLARY 4.3. *The adjoint* T^* *of a CDD operator* T *is CDD and* $T = T^{**}$.

THEOREM 4.4. *Let* T *be a CDD operator. Then* T^* *satisfies* $\Phi(Tx, y) = \Phi(x, T^*y)$ *for all* x *in* $\operatorname{dom}(T)$ *and* y *in* $\operatorname{dom}(T^*)$. *Also any linear operator* S *satisfying* $\Phi(Tx, y) = \Phi(x, Sy)$ *for all* x *in* $\operatorname{dom}(T)$ *and* y *in* $\operatorname{dom}(S)$ *is such that* $S \subseteqq T^*$. *If* $\operatorname{dom}(S) = \operatorname{dom}(T^*)$, *then* $S = T^*$.

Proof. Since T is domain dense, T^* is single valued and

$$\Phi(Tx, y) = \Phi(x, T^*y)$$

for all x in $\operatorname{dom}(T)$ and y in $\operatorname{dom}(T^*)$. If $\Phi(Tx, y) = \Phi(x, Sy)$ for all x in $\operatorname{dom}(T)$ then $\Phi \oplus \Phi((y, Sy), (-Tx, x)) = -\Phi(y, Tx) + \Phi(Sy, x) = 0$ for all x in $\operatorname{dom}(T)$ so that (y, Sy) is in $U(G(T))^\perp = G(T^*)$. Thus y is in $\operatorname{dom}(T^*)$ and $T^*y = Sy$. Thus $S \subseteqq T^*$.

In Hilbert space, a bounded linear has a topologically closed graph and conversely. We can prove that if T is a domain dense linear operator on E and T^* is domain dense then T has a $\perp$-closed graph. It would be more interesting to prove the following open question: *Algebraic Closed Graph Theorem* if T is an everywhere defined closed linear operator then T has an everywere defined adjoint. We conjecture this is not true in general but is true in the case that every closed subspace of our quadratic space is splitting.

5. **Orthocontinuity.** In Hilbert space, the continuous linear operators are of great interest. We shall show how to approach these algebraically.

Let (E, Φ) be a quadratic space with $T\colon E \to E$ linear. We say T is *orthocontinuous* if for all subspaces M of E we have

$$T(M^{\perp\perp}) \subseteqq T(M)^{\perp\perp} .$$

PROPOSITION 5.1. *Let* $T\colon E \to E$ *be linear. Then the following statements are equivalent*

(1) $M = M^{\perp\perp}$ *implies* $T^{-1}(M) = (T^{-1}(M))^{\perp\perp}$
(2) M *closed implies* $T^{-1}(M)$ *closed*
(3) $T(M^{\perp\perp}) \subseteqq T(M)^{\perp\perp}$
(4) $T^{-1}(N^{\perp\perp}) \supseteqq (T^{-1}(N))^{\perp\perp}$
(5) T *is orthocontinuous*

The proof is easy and is omitted.

LEMMA 5.2. *Let* $T: E \to E$ *be an everywhere defined linear operator. Suppose* $\mathrm{dom}(T^*) = E$. *Then for any* $M = M^{\perp\perp}$ *we have* $T^{-1}(M) = (T^*(M^\perp))^\perp$. *In particular then* $\ker(T) = \mathrm{im}(T^*)^\perp$.

Proof. Let $M = M^{\perp\perp}$. Then x is in $T^{-1}(M)$ if and only if Tx is in $M = M^{\perp\perp}$ if and only if $\Phi(Tx, y) = 0$ for all y in $M^\perp$ if and only if $\Phi(x, T^*y) = 0$ for all y in $M^\perp$ if and only if x is orthogonal to $T^*(M^\perp)$.

Next we make a connection between the domain of the adjoint and the orthodual.

THEOREM 5.3. *If T is an everywhere defined linear operator on E, then* $\mathrm{dom}(T^*)$ *comprises exactly those y in E for which the linear functional* $f_y(x) = \Phi(Tx, y)$ *is orthocontinuous.*

Proof. Since T is domain dense, T^* is single valued and

$$\Phi(Tx, y) = \Phi(x, T^*y)$$

for all x in E and all y in $\mathrm{dom}(T^*)$. First let y be in $\mathrm{dom}(T^*)$. Then x is in $\ker(f_y)$ if and only if $f_y(x) = 0$ if and only if $\Phi(Tx, y) = 0$ if and only if $\Phi(x, T^*y) = 0$ if and only if x is in $(kT^*y)^\perp$. Thus

$$\ker(f_y) = (kT^*y)^\perp$$

is closed.

Conversely, let y be a vector such that f_y is an orthocontinuous linear functional. Then by Frechet-Riesz, there is a unique vector y^* such that $f_y(x) = \Phi(x, y^*)$ for all x in E. That is, $\Phi(Tx, y) = \Phi(x, y^*)$ for all x in E. Thus

$$\Phi \oplus \Phi((y, y^*), (-Tx, x)) = -\Phi(y, Tx) + \Phi(y^*, x) = 0$$

for all x in E so that (y, y^*) is in $U(G(T))^\perp = G(T^*)$. This means y is in $\mathrm{dom}(T^*)$.

We are now in a position to relate orthocontinuity to the adjoint. We first state a lemma whose proof will be omitted.

LEMMA 5.4. *Let T be an everywhere defined linear operator on E. Define the linear functional f_y by $f_y(x) = \Phi(Tx, y)$ for all x in E. Then* $\ker(f_y) = T^{-1}((ky)^\perp)$.

THEOREM 5.5. *Let T be an everywhere defined linear operator on E. Then T is orthocontinuous if and only if T^* is everywhere* *defined.*

Proof. Let T be orthocontinuous. Then $T^{-1}((ky)^{\perp})$ is closed for all y in E. Thus by (5.3) and (5.4), y is in $\mathrm{dom}(T^*)$.

Conversely if $\mathrm{dom}(T^*) = E$, then for

$$M = M^{\perp\perp},\ T^{-1}(M) = (T^*(M^{\perp}))^{\perp}$$

by (5.2) and this is closed so T is orthocontinuous.

COROLLARY 5.6. *T is orthocontinuous if and only if $T^{-1}((ky)^{\perp})$ is closed for all y in E.*

We close by remarking that the algebra of bounded operators on Hilbert space is a well studied object. The algebraic analogue for a quadratic space is the *adjoint algebra*, *$Ad(E, \Phi)$*, of all linear operators on E that have everywhere defined adjoints.

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Received June 24, 1971.

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