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A THEOREM ON BOUNDED ANALYTIC FUNCTIONS

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The purpose of this paper is to prove the following

THEOREM: **Let $\phi_1, \phi_2, \cdots$ be an infinite sequence of functions in $L^1([0, 2\pi])$ such that $L(f) = \lim_{n\to\infty} \int_0^{2\pi} f(e^{i\theta})\phi_n(\theta)d\theta$ exists for every $f \in H^\infty$. Then there is a $\phi \in L^1([0, 2\pi])$ such that $L(f) = \int_0^{2\pi} f(e^{i\theta})\phi(\theta)d\theta$ for all $f \in H^\infty$.**

Throughout this paper we will use the following notation and conventions: D will denote the unit disc and T its boundary. In order to save time we will avoid making distinctions between T and $[0, 2\pi]$ if no confusion results. Similarly, it will be convenient to treat elements of $H^\infty[= H^\infty(D)$, the bounded analytic functions on $D]$ as though they were the same as those functions on T with which they are naturally identified.

If $w \in D$, the symbol g_w will stand for the function $z \to g(wz)$. $C(T)$ will stand for the usual space of continuous functions on T. A will denote the subspace of $C(T)$ of functions analytically extendable to D. λ will denote ordinary Lebesgue measure divided by 2π and "WLOG" means "without loss of generality".

In their paper [4] Piranian, Shields, and Wells observed that the theorem stated above would imply their result, namely that if $a_0, a_1, \cdots$ was a sequence of complex constants such that $\lim_{r\to 1} \sum_{n=0}^{\infty} a_n b_n r^n$ exists for all $f \in H^\infty$ [with Taylor coefficients $b_0, b_1, \cdots$], then the a_n's are the the nonnegative Fourier coefficients of an $L^1([0, 2\pi])$ function. They also mentioned that our result here was a question raised in [1].

Kahane [3], using a somewhat different method than that in [4] showed that under the hypothesis of our main theorem, there was a $\phi \in L^1([0, 2\pi])$ such that the conclusion held for all $f \in A$. He went further to show that the subset of H^∞ for which the conclusion held was large in some sense. Our proof here makes use of Kahane's result.

2. Remarks and lemmas.

First, given the hypothesis of the main theorem we may assume WLOG that the ϕ_n's are uniformly bounded in L^1 norm. To see why this is so we observe that for each n, $g \to L_n(g) = \int_T g\phi_n$ is a bounded linear functional on A. By the uniform boundedness principle, the norms of the L_n's as elements of A^* are uniformly bounded, say by M. By the Hahn-Banach Theorem, each L_n may be extended to an element of $C(T)^*$with norm less than

M. This extended functional corresponds in the usual way to a Borel measure μ_n on T having variation norm less than M. For each n, $\mu_n - \int \phi_n$ is also a finite Borel measure on T. Since this measure is orthogonal to A, it must be absolutely continuous [by the classical F. and M. Riesz Theorem] and, in turn, so must μ_n. Hence we may replace ϕ_n's with $d\mu_n$'s if necessary. From here on we assume $\|\phi_n\|_1 \leqq 1$, for all n.

Suppose now for purposes of contradiction that there is an $f \in H^\infty$ such that $L(f) \neq \int_T f\phi$ where ϕ is the function referred to in Kahane's result. We may assume WLOG that $\phi \equiv 0$ [simply subtract ϕ from ϕ_n's beforehand and that $|f|_\infty = 1$. We also assert WLOG:

LEMMA 1. *There exists a bounded, increasing function β on T such that*

$$\lim_{n\to\infty} \int_E |\phi_n| = \int_E d\beta$$

whenever E is a finite union of closed subintervals of T.

Proof. Since all our previous assertions remain valid if the ϕ_n's are replaced by an infinite subsequence, we will do this if necessary so that the functions $\int |\phi_n|$'s converge pointwise on T to a function which we call β. This construction and the conclusion of the lemma follow from the Helly's Theorem. [See Zygmund [5] IV-4.6-(p. 137).]

We consider the fact that:

$$\lim_{r\to 1-} \lim_{n\to\infty} \int_T f_r\phi_n = 0 \neq \lim_{n\to\infty} \lim_{r\to 1-} \int_T f_r\phi_n = L(f)$$

despite the fact that f_r's are uniformly bounded and converge to f in measure. It is reasonable to subspect that in some useful sense of the word that the support of $\int f\phi_n$ tends to become concentrated on smaller and smaller sets as $n \to \infty$.

To be more specific, our plan at this point is to produce a sequence of pairwise disjoint "nice" closed sets $E_1, E_2, \cdots$ such that $\int_{E_n} f\phi_n$ tends approximately to $L(f)$ while $\int_{T-E_n} |f\phi_n|$ remains uniformly $< \varepsilon \ll |L(f)|$. [We will find that it is expedient to replace f with $f - f_r$ for some r in order to do this.]

Ultimately we will construct $g \in H^\infty$ so that g is approximately $(-1)^n$ on E_n. The function gf [actually we will look at $g \times (f - f_r)$] will give us a counterexample to the condition that $L(h)$ exists for all $h \in H^\infty$, and hence we will have a contradiction to the assumption

$L(f) \neq 0$.

Let $\varepsilon_0 = (1/10)|L(f)|$. In order to prove Lemma 2, it will be desirable to keep the singular part of β small, say less than $\varepsilon_0/2$. To be sure of this we can choose a closed subset E of the support of the singular part of β such that outside of E, the singular part of β has variation norm less than $\varepsilon_0/2$.

Let g denote a Rudin-Carleson type function such that $g \in A$, g is zero on E, and g is close to 1 outside some neighborhood of E. Such functions were used in both [3] and [4], and a proof of their existence is available in Hoffman [2] p. 80, 81. [See also [2], Notes on p. 95.] If the original ϕ_n's are replaced by $g\phi_n$'s, we may proceed as before with our new set of ϕ_n's, ϕ, β, etc. The new $d\beta = |g|$ times the old $d\beta$, and hence the singular part of the new β will have variation norm less than $\varepsilon_0/2$. This process gives us a new value for $L(f)$, however, and we must be sure that the new value is close enough to the old that our assertion is still valid when the new value of $L(f)$ is used in the expression for ε_0. To do this we observe that the functions $f\phi_n$ also satisfy the hypothesis of our Theorem [in place of the ϕ_n's] and that by Kahane's Theorem, there is a $\psi \in L^1([0, 2\pi])$ such that

$$\lim_{n\to\infty} \int_T hf\phi_n = \int_T h\psi \quad \text{for all} \quad h \in A .$$

In particular this is true when $h = g$. Since ψ is absolutely continuous and since we can make g uniformly as close to 1 as we like outside neighborhoods of E taken as small as we like, the new $L(f) = \int_T g\psi$ can be taken as close to the old $L(f) = \int_T \psi$ as we like. Hence WLOG we may assume that the singular part of β has variation norm less than $\varepsilon_0/2$. Let us now choose $\delta > 0$ such that

$$\lambda(E) < \delta \Rightarrow \int_E d\beta_a < \varepsilon_0/2 - \int_T d\beta_S$$

where β_a and β_s are the absolutely continuous and singular parts of β respectively. We note that if J is a finite union of closed intervals, and $\lambda(J) < \delta$, then for n sufficiently large $\int_J |\phi_n| < \varepsilon_0/2$.

Choose $r \in (0, 1)$ such that $\lambda(F) < \delta$ where

$$F = \{\theta \,|\, \theta \in [0, 2\pi],\ |f(e^{i\theta})| - f_r(e^{i\theta})| \geqq \varepsilon_0\} .$$

Let G be an open subset of T such that $F \subset G$ and $\lambda(G) < \delta$.

Since

$$L(f_r) = 0,\ L(f) = L(f - f_r) = \lim_{n\to\infty} \int_G (f - f_r)\phi_n + \lim_{n\to\infty} \int_{T-G} (f - f_r)\phi_n .$$

[We may choose subsequences of the original ϕ_n's if necessary in order to guarantee the limits exist.] Now for each n, $\int_{T-G} |(f - f_r)\phi_n| \leqq \varepsilon_0$. Hence $\left| \int_G (f - f_r)\phi_n - L(f) \right| < \varepsilon_0$ for all sufficiently large n.

LEMMA 2. *There exists a sequence of sets $E_1, E_2, \cdots$; a sequence of positive numbers $\delta_1, \delta_2, \cdots$; and an increasing sequence of positive integers $j_1, j_2, \cdots$ such that:*

(a) *Each E_n is a finite union of closed intervals.*

(b) *Let E_j' denote the closure of the δ_j neighborhood of E_j. Then $E_j' \subset G$.*

(c) $j \neq k \Rightarrow E_j' \cap E_k' = \varnothing$. [*Note that this* $\Rightarrow \lambda(E_j) \to 0$, *and* $\lambda(E_j') \to 0$.]

(d) $\int_{G-E_k} |\phi_{j_k}| < \varepsilon_0/2$ *for* $k = 1, 2, \cdots$.

(e) $\int_{E_n} (f - f_r)\phi_{j_n} \to x_0$ *where* $|x_0 - L(f)| < 2\varepsilon_0$.

Proof. Construction using mathematical induction and the following scheme: After the first k, E_j's, δ_j's and j_n's are constructed, we pick j_{k+1}, E_{k+1}, and δ_{k+1} in the order.

Using the fact that $\lim_{n\to\infty} \int_{\cup_{p=1}^k E_p'} |\phi_n| < (1/2)\varepsilon_0$ [since $\lambda(\bigcup_{p=1}^k E_p') < \lambda(G) < \delta$] and the fact that $\int_G (f - f_r)\phi_n$ eventually comes within ε_0 of $L(f)$, we have that for j_{k+1} sufficiently large: $\int_{\cup_{p=1}^k E_p'} |\phi_{j_{k+1}}| < (1/2)\varepsilon_0$ and $\int_{G-\cup_{p=1}^k E_p'} (f - f_r)\phi_{j_{k+1}}$, is within $2\varepsilon_0$ of $L(f)$.

We now choose E_{k+1} inside the open set $G - \bigcup_{p=1}^k E_p'$. Using the absolute continuity of $\phi_{j_{k+1}}$ we can choose E_{k+1} large enough that (d) holds, and that $\int_{E_{k+1}} (f - f_r)\phi_{j_{k+1}}$ is within $2\varepsilon_0$ of $L(f)$.

δ_{k+1} will now be chosen so that (b) and (c) satisfied. Obviously our construction will satisfy (a), (b), (c), (d). We may choosen an appropriate subsequence if necessary in order that (e) be satisfied as well.

3. Construction of the counterexample function.

LEMMA 3. *Let E be a closed subset of T, $\varepsilon > 0$. Then there is a function, s, analytic on D such that:*

(a) *s has positive real part and* $|s|_\infty < 1$

(b) $\theta \in E \Rightarrow |s(e^{i\theta}) - 1| < \varepsilon$

(c) $\theta \notin E \Rightarrow |s(e^{i\theta})| < 2\lambda(E)/\varepsilon \cdot \mathrm{dist}\,(\theta, E)$

(d) $|s(0)| < \lambda(E)/\varepsilon$.

Proof. Let $U = (1/\varepsilon)\pi_E$ on T [π_E denotes characteristic function for E]. Let u be the harmonic function on D corresponding to U on the boundary [u is the integral of U with respect to Poisson's kernel]. Let v be the conjugate harmonic function for u such that $v(0) = 0$. Let $g = u + iv$. [g is analytic on D with positive real part.]

Note that for $\theta \notin E$, $|g(e^{i\theta})| = |v(e^{i\theta})|$ where

$$v(e^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(e^{i\phi})\frac{\sin(\theta-\phi)}{1-\cos(\theta-\phi)}d\phi = \frac{1}{2\pi\varepsilon}\int_E \frac{\sin(\theta-\phi)}{1-\cos(\theta-\phi)}d\phi \,.$$

The maximum modulus of the function inside the integral occurs when $|\theta - \phi| = \operatorname{dist}(\theta, E)$. In order not to be troubled by awkward trigonometric expressions in the material to follow, we observe by some elementary calculations that $|\sin x|/(1-\cos x) < 2/|x|$ for $|x| < \pi$. Hence we may assert that $|v(e^{i\theta})| < 2\lambda(E)/\varepsilon\cdot\operatorname{dist}(\theta, E)$. Now let

$$s = g/(1+g) = 1 - 1/(1+g)\,.$$

(a) Since g is of positive real part, the range of $1/(1+g)$ is contained in the disc $\{z \mid |z - 1/2| < 1/2\}$. So is the range of s.

(b) For $\theta \in E$, $\operatorname{Re}(g(e^{i\theta})) = 1/\varepsilon$ and hence $\operatorname{Re}(1 + g(e^{i\theta})) = 1 + 1/\varepsilon$. This makes $|1 + g(e^{i\theta})| \geqq 1 + 1/\varepsilon$ and in turn $|1/(1+g(e^{i\theta}))| \leqq \varepsilon/(1+\varepsilon) < \varepsilon$ whence $|s(e^{i\theta}) - 1| = |1/(1 + g(e^{i\theta}))| < \varepsilon$.

(c) For $\theta \notin E$, $|s(e^{i\theta})| = |g(e^{i\theta})|/|1 + g(e^{i\theta})|$ where

$$|g(e^{i\theta})| < 2\lambda(E)/\varepsilon\cdot\operatorname{dist}(\theta, E) \quad\text{and}\quad |1 + g(e^{i\theta})| \geqq 1$$

(d) $s(0) = g(0)/(1 + g(0))$, where $g(0) = \lambda(E)/\varepsilon$ and the proof is complete.

Construction. Given $\varepsilon_1 > 0$, $\varepsilon_2 > 0$; a sequence of functions $s_1, s_2, \cdots$ is to be constructed as follows:

Suppose $s_1, s_2, \cdots, s_k$ have been chosen and that $S_k = \sum_{j=1}^k s_j$ is such that $|S_k|_\infty = M_k < \infty$, s_{k+1} will be of the form $c_{k+1}s$ where c_{k+1} is a positive real number and s is related to $E_{n_{k+1}}$ in the same manner that s is related to E in Lemma 3.

We want c_{k+1} sufficiently large and ε [in Lemma 3] sufficiently small that:

(a) $\theta \in E_{n_{k+1}} \Rightarrow \varepsilon_2 \log|S_{k+1}(e^{i\theta})| = (-1)^{k+1}(\pi/2)(\operatorname{mod} 2\pi) - \pi/2$ within an error of magnitude not more than ε_1. Note that we can pick ε dependent only on ε_1 and ε_2 [independent of $k+1$], and $c_{k+1} \gg M_k$ so as to make the ratio between $|s_{k+1} + S_k|$ and $|\operatorname{Re}(s_{k+1})|$ small enough to make $\log|S_{k+1}|$ close enough to $\log(c_{k+1})$ on $E_{n_{k+1}}$ for this purpose. Furthermore, the choice of c_{k+1} depends only on $E_{n_1}, E_{n_2}, \cdots, E_{n_k}$. We wish further to have:

(b) $\theta \in E_{n_k} = \varepsilon_2 \log|S_p(e^{i\theta})| = (-1)^k\pi/2(\operatorname{mod} 2\pi) - \pi/2$ within an error of magnitude not more than ε_1 for all $p > k$. To do this, we use the fact that for $\theta \in E_{n_k}$, $p > k$; then dist $(\theta, E_{n_p}) > \delta_{n_k}$ [independent of p-note]. Hence $|s_p(e^{i\theta})| < c_p\lambda(E_{n_p})/\varepsilon \cdot \text{dist}\,(\theta, E_{n_p}) < c_p\lambda(E_{n_p})/\varepsilon\delta_{n_k}$. Recall that the choice of c_p depends only on S_{p-1} and is independent of E_{n_p}. Hence we may require that $\lambda(E_{n_p}) \to 0$ sufficiently rapidly to guarantee that $\sum_{p=k+1} c_p\lambda(E_{n_p})/\varepsilon\delta_{n_k}$ is always small enough that (b) is satisfied. The above requirement also guarantees that $\sum_{k=1}^{\infty} c_k\lambda(E_{n_k})/\varepsilon$ converges.

Each s_p has positive real part and hence by Harnack's principal the S_p's must either converge to an analytic function, S, of positive real part on D, or diverge to ∞ on D. The latter is impossible since each $|S_p(0)| < \sum_{k=1}^{p} |s_k(0)| \leqq \sum_{k=1}^{p} c_k\lambda(E_{n_k})/\varepsilon < \sum_{k=1}^{\infty} c_k\lambda(E_{n_k})/\varepsilon < \infty$. We also note that our requirement in (b) above also guarantees that the S_p's converge absolutely on each E_{n_k} and hence we also have: $\theta \in E_{n_k} \Rightarrow$ $\varepsilon_2 \log |S| = (-1)^k\pi/2(\text{mod}\, 2\pi) - \pi/2$ within an error of magnitude not more than ε_1.

Let $g = e^{i\varepsilon_2 \log S}$. Then g is bounded on D [in fact: $e^{-\varepsilon_2\pi/2} < |g(z)| < e^{\varepsilon_2\pi/2}$ for all $z \in D$]. $\theta \in E_{n_p} \Rightarrow$ argument $(g(e^{i\theta})) = ((-1)^p\pi/2)(\text{mod}\, 2\pi) - \pi/2 +$ error not larger than ε_1. This is, given $\varepsilon_3 > 0$ we may choose $\varepsilon_1, \varepsilon_2$ so that $1 - \varepsilon_3 < |g(z)| < 1 + \varepsilon_3$ for all $z \in D$ and such that $|g(e^{i\theta}) - (-1)^p| < \varepsilon_3$ for all $\theta \in E_{n_p}$. Now:

$$\int_T g(f - f_r)\phi_{j_{n_k}} = \int_{E_{n_k}} g(f - f_r)\phi_{j_{n_k}} + \int_{G-E_{n_k}} g(f - f_r)\phi_{j_{n_k}} + \int_{T-G} g(f - f_r)\phi_{j_{n_k}}\,.$$

Recalling Lemma 2, we see that the first of these three integrals is within $2\varepsilon_0(1 + \varepsilon_3)$ of $(-1)^pL(f)$; the second has magnitude less than $\varepsilon_0(1 + \varepsilon_3)$ [by (d), Lemma 2] and the third also has magnitude less than $\varepsilon_0(1 + \varepsilon_3)$ [from the way in which f_r and G were chosen]. If ε_3 is chosen small enough, $\int_T g(f - f_r)\phi_{j_{nk}}$ fails to have a limit as $k \to \infty$ and we have our contradiction.

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