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ON THE ABSOLUTE MATRIX SUMMABILITY OF FOURIER SERIES

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The paper investigates sufficient conditions under which a summability method of a certain general type absolutely sums the Fourier series of any function of bounded variation. The main theorem includes a recent theorem of M. Izumi and S. Izumi, who considered the problem for the special case of Nörlund summability.

The summability methods considered are those given by a series-to-series transformation $A = (\alpha_{n,k})$. That is to say, given any series

$$\sum_{k=0}^{\infty} a_k , \tag{1}$$

we describe (1) as summable A to s if

$$b_n = \sum_{k=0}^{\infty} \alpha_{n,k} a_k$$

is defined for all n, and if

$$\sum_{n=0}^{\infty} b_n \tag{2}$$

converges to s. We describe (1) as absolutely summable $|A|$ if (2) converges absolutely. Under certain quite weak restrictions on A, necessary and sufficient conditions under which the Fourier series of any function of bounded variation should be absolutely summable $|A|$ have been given by Tripathy [10, Lemma 2]; his result will be stated later as Lemma 1. But the conditions obtained by Tripathy are of such a nature that it is not usually easy in any given example to determine whether they are satisfied or not. The object of the present paper is to obtain sufficient conditions which, while less general, are simpler than those of Tripathy. However, it does not seem possible to obtain reasonably general sufficient condition in any very simple form.

We will be concerned with the case in which A is absolutely conservative, that is to say, it is such that, whenever (1) converges absolutely, so does (2). It is known [4, 6] that in order that this should hold it is necessary and sufficient that, for $k \geqq 0$,

$$\sum_{n=0}^{\infty} |\alpha_{n,k}| = O(1) . \tag{3}$$

We remark that, in order that A should be absolutely regular, that is to say, that in order that, whenever (1) converges absolutely then (2) converges absolutely to the same sum, it is necessary and sufficient that (3) should hold and that, further, for all $k \geqq 0$,

$$\sum_{n=0}^{\infty} \alpha_{n,k} = 1 . \tag{4}$$

2. We now state our main result.

THEOREM. *Let $A = (\alpha_{n,k})$ be an absolutely conservative series-to-series transformation, with $\alpha_{n,k} \geqq 0$ for all n, k. Suppose that either*

(a) *For each fixed n, there is a positive integer $r(n)$ such that $\alpha_{n,k}$ is nondecreasing for $1 \leqq k \leqq r(n)$, and nonincreasing for $k \geqq r(n)$, or*

(b) *For each fixed n, there is a positive integer $s(n)$, such that $\alpha_{n,k}/k$ is nondecreasing for $1 \leqq k \leqq s(n)$, and nonincreasing for $k \geqq s(n)$. Suppose also in case (a) that, for $K \geqq 1$,*

$$\sum_{r(n) \geqq 2K} \frac{1}{r(n)} \sum_{k=r(n)-K}^{r(n)+K} \alpha_{n,k} = O(1) , \tag{5}$$

and in case (b) that, for $K \geqq 1$,

$$\sum_{s(n) \geqq 2K} \frac{1}{s(n)} \sum_{k=s(n)-K}^{s(n)+K} \alpha_{n,k} = O(1) . \tag{6}$$

Then the Fourier series of any function of bounded variation is absolutely summable $|A|$.

REMARK. It is clear that (5) is equivalent to

$$\sum_{r(n) \geqq 2K} \sum_{k=r(n)-K}^{r(n)+K} \frac{\alpha_{n,k}}{k} = O(1) \tag{5'}$$

and it is sometimes more convenient to express (5) in this form. Since there are $2K + 1$ terms in the inner sum in (5), and since the middle term is the greatest, a sufficient condition for (5) is that

$$\sum_{r(n) \geqq 2K} \frac{\alpha_{n,r(n)}}{r(n)} = O\left(\frac{1}{K}\right) . \tag{7}$$

However (7), while much simpler than (5) is less general, and, as will be shown later, fails to be satisfied in some important cases. In a similar way, (6) is equivalent to

$$\sum_{s(n) \geqq 2K} \sum_{k=s(n)-K}^{s(n)+K} \frac{\alpha_{n,k}}{k} = O(1) ; \tag{6'}$$

also, a sufficient condition for (6) is that

$$\sum_{s(n)\geqq 2K} \frac{\alpha_{n,s(n)}}{s(n)} = O\left(\frac{1}{K}\right). \tag{8}$$

It is clear that either one of (a), (b) could be satisfied without the other holding. If, however, they both hold, then (5) is a weaker assumption than (6). Thus, in this case, the first form of the theorem is preferable. To prove this assertion, we write θ_n, ϕ_n for the inner sums in (5′), (6′) respectively, and shall first show that

$$\theta_n \leqq 2\phi_n . \tag{9}$$

To this end, we first note that $s(n) \leqq r(n)$. Consider first the case in which $r(n) - s(n) \geqq K$. Since $\alpha_{n,k}/k$ is nonincreasing for $k \geqq s(n)$, we have[1], for $\mu = 0, 1, \cdots, K-1$,

$$\begin{aligned} &\frac{\alpha(n, r(n)+K-2\mu)}{r(n)+K-2\mu} + \frac{\alpha(n, r(n)+K-2\mu-1)}{r(n)+K-2\mu-1} \\ &\leqq \frac{2\alpha(n, s(n)+K-\mu)}{s(n)+K-\mu} . \end{aligned} \tag{10}$$

Also,

$$\frac{\alpha(n, r(n)-K)}{r(n)-K} \leqq \frac{\alpha(n, s(n))}{s(n)} ,$$

whence

$$\theta_n \leqq 2 \sum_{k=s(n)}^{s(n)+K}{}' \frac{\alpha_{n,k}}{k} \leqq 2\phi_n ,$$

where the dash indicates the term $k = s(n)$, is multiplied by 1/2. If $r(n) - s(n) = t(n) < K$, then (10) still holds for $\mu \leqq t(n) - 1$. Hence

$$\theta_n \leqq 2 \sum_{\mu=0}^{t(n)-1} \frac{\alpha(n, s(n)+K-\mu)}{s(n)+K-\mu} + \sum_{\nu=2t(n)}^{2K} \frac{\alpha(n, r(n)+K-\nu)}{r(n)+K-\nu} \tag{11}$$

where the first sum on the right is taken as 0 if $t(n) = 0$. Since the second sum on the right of (11) can be written

$$\sum_{\mu=t(n)}^{2K-t(n)} \frac{\alpha(n, s(n)+K-\mu)}{s(n)+K-\mu} ,$$

we again deduce (9).

It now follows from (9) that

$$\sum_{s(n)\geqq 2K} \theta_n \leqq 2 \sum_{s(n)\geqq 2K} \phi_n .$$

However, since $s(n) \leqq r(n)$, there may be values of n for which $r(n) \geqq 2K$, but $s(n) < 2K$; these values will occur in the sum (5′), but not

[1] To avoid complicated suffixes, we write $\alpha(n, k)$ for $\alpha_{n,k}$ whenever n, k are replaced by more complicated expressions

in (6′). If we show that, in any case, the contribution of these terms to the sum (5′) is bounded, the conclusion will now follow. If $r(n) \geqq 2K$ but $s(n) \leqq K$, then, since $\alpha_{n,k}/k$ is nonincreasing for $k \geqq K$ we deduce that

$$\theta_n \leqq \frac{(2K+1)\alpha_{n,K}}{K} .$$

If $r(n) \geqq 2K$ and $K < s(n) < 2K$, then $\alpha_{n,k}$ is nondecreasing for $k \leqq 2K$. Hence, for all $k \geqq 1$

$$\frac{\alpha_{n,k}}{k} \leqq \frac{\alpha_{n,s(n)}}{s(n)} \leqq \frac{\alpha_{n,2K}}{K}$$

so that

$$\theta_n \leqq \frac{(2K+1)\alpha_{n,2K}}{K} .$$

Thus the sum of the terms in question does not exceed

$$\frac{(2K+1)}{K} \sum_{n=0}^{\infty} (\alpha_{n,K} + \alpha_{n,2K}) = O(1) ,$$

by (3).

3. We now state the lemma of Tripathy already mentioned.

LEMMA 1. *Let $A = (\alpha_{n,k})$ be a series-to-series transformation such that*

$$\sum_{n=0}^{\infty} |\alpha_{n,0}| < \infty \tag{12}$$

and such that, for every fixed n,

$$L_n(t) = \sum_{k=1}^{\infty} \alpha_{n,k} \frac{\sin kt}{k} \tag{13}$$

converges boundedly in t. Then in order that the Fourier series of any function of bounded variation should be absolutely summable $|A|$, it is sufficient that

$$\sum_{n=0}^{\infty} |L_n(t)| = O(1) , \tag{14}$$

and necessary that the sum (14) *should be essentially bounded.*

It may be remarked that the result is not quite correctly stated in [10], where it is asserted that the essential boundedness of (14) is necessary and sufficient. But on examining the proof of sufficiency in [10], we find that it requires the boundedness, and not just the essential boundedness, of (14). The point is not of great importance, since if we assume that, for every fixed n,

$$\sum_{k=1}^{\infty} |\alpha_{n,k} - \alpha_{n,k+1}| < \infty ;$$

in other words, that b_n is defined whenever (1) converges, it is easy to prove that the essential boundedness of (14) is equivalent to its boundedness. This result is not, however, required for our present purposes.

In what follows, we will suppose throughout that $0 < t \leqq \pi$. We will apply the hypothesis (5) or (6) with $K = [\pi/t]$; thus, in any equations involving both K and t, it will be assumed that this relation holds.

We require two further lemmas.

LEMMA 2. *Let $A = (\alpha_{n,k})$ be an absolutely conservative series-to-series transformation. If, for every fixed n, $\alpha_{n,k}/k$ is ultimately nonnegative nonincreasing (and thus, in particular, if the hypotheses of the theorem are satisfied) then the hypotheses of Lemma 1 are satisfied.*

Equation (12) follows at once as a special case of (3). Thus, taking n as fixed, we have only to verify that (13) converges boundedly. Suppose that $\alpha_{n,k}/k$ is nonnegative nonincreasing for $k \geqq M$. Then we have, uniformly in k_1, k_2 for $K, M \leqq k_1 \leqq k_2$,

$$\left| \sum_{k=k_1}^{k_2} \alpha_{n,k} \frac{\sin kt}{k} \right| \leqq \frac{\alpha(n, k_1)}{\left(2 \sin \frac{1}{2} t\right) k_1} . \tag{15}$$

But (3) implies that $\alpha(n, k)$ is bounded; hence the expression on the right of (15) is $O(1)$ uniformly in the range considered, and, for fixed t, tends to 0 (uniformly in k_2) as $k_1 \to \infty$. Since M is a constant,

$$\sum_{k=1}^{M-1} \left| \alpha_{n,k} \frac{\sin kt}{k} \right|$$

is bounded; also, if $K \geqq M$,

$$\sum_{k=M}^{K} \left| \alpha_{n,k} \frac{\sin kt}{k} \right| \leqq t \sum_{k=M}^{K} |\alpha_{n,k}| = O(1)$$

(by the boundedness of $\alpha_{n,k}$ and the definition of K). Hence the result.

LEMMA 3. *Suppose that $\theta_k \geqq 0$. Suppose that θ_k is nondecreasing* *for* $1 \leqq k \leqq s$, *and nonincreasing for* $k \geqq s$. *Then, for any positive integers* a, b, *and any* t *with* $0 < t \leqq \pi$,

$$\left|\sum_{k=a}^{b} \theta_k e^{ikt}\right| \leqq A \sum_{k=\max(1,s-K)}^{s+K} \theta_k \, , \tag{16}$$

where A is an absolute constant.

That portion (if any) of the sum on the left for which $s - K \leqq k \leqq s + K$ clearly satisfies the required inequality. Also, by partial summation, that portion (if any) for which $k > s + K$ does not, in modulus, exceed

$$\frac{2}{|1 - e^{it}|}\theta_{s+K} \leqq \frac{2}{|1 - e^{it}|(K + 1)}\sum_{k=s}^{s+K}\theta_k$$
$$\leqq \frac{2t}{\pi|1 - e^{it}|}\sum_{k=s}^{s+K}\theta_k \, .$$

That portion of the sum (if any) for which $k < s - K$ may be dealt with in a similar way, and the conclusion follows.

This lemma is a slight generalisation of a lemma due to McFadden [5] which has been much used in investigations on the Nörlund summability of Fourier series.

4. We now come to the proof of the theorem. It follows from Lemmas 1 and 2 that it is enough to show that the hypotheses of the theorem imply (14). Consider first those values of n (if any) for which $r(n) < 2K$ in case (a), and for which $s(n) < 2K$ in case (b). In case (b), we are given that $\alpha_{n,k}/k$ is nonincreasing for $k \geqq 2K$; in case (a), we are given that $\alpha_{n,k}$ is nonincreasing for $k \geqq 2K$; hence, a fortiori, so is $\alpha_{n,k}/k$. Thus, in either case, since the partial sums of $\sum \sin kt$ are $O(1/t)$, we have

$$\sum_{k=2K}^{\infty} \alpha_{n,k}\frac{\sin kt}{k} = O\left(\frac{\alpha_{n,2K}}{2Kt}\right) = O(\alpha_{n,2K})$$

by definition of K. For those terms in the sum (13) for which $k \leqq 2K$, we use $|\sin kt| \leqq kt$; and it follows that

$$|L_n(t)| = O\left\{t\sum_{k=1}^{2K-1}\alpha_{n,k}\right\} + O(\alpha_{n,2K}) \, .$$

Hence the contribution to the sum (14) of those values of n now under consideration is

$$O\left\{t\sum_{k=1}^{2K-1}\sum_{n=0}^{\infty}\alpha_{n,k}\right\} + O\left\{\sum_{n=0}^{\infty}\alpha_{n,2K}\right\} = O(1) \tag{17}$$

by (3) and the definition of K.

We now investigate the remaining values of n. Consider first case (b). For any fixed n, we apply Lemma 3 with $\theta_k = \alpha_{n,k}/k$, and take the imaginary part of (16). It follows at once that

$$L_n(t) = O\left\{\sum_{k=s(n)-K}^{s(n)+K} \frac{\alpha_{n,k}}{k}\right\};$$

and (14) therefore follows from (6′) and (17).

Now consider case (a). Since $\alpha_{n,k}$ is nonincreasing for $k \geqq r(n)$ so is $\alpha_{n,k}/k$; thus the part of the sum (13) for which $k > r(n) - K$ may be dealt with as in case (b). The part for which $k < K$ may be dealt with by using $|\sin kt| \leqq kt$, as in the proof of (17). Thus, writing

$$R_n(t) = \sum_{k=K}^{r(n)-K} \alpha_{n,k}\frac{\sin kt}{k},$$

it remains only to show that

$$\sum_{r(n)\geqq 2K} |R_n(t)| = O(1). \tag{18}$$

Now,

$$\begin{aligned} R_n(t) &= \frac{1}{2\sin\frac{1}{2}t}\sum_{k=K}^{r(n)-K}\frac{\alpha_{n,k}}{k}\left[\cos\left(k-\frac{1}{2}\right)t - \cos\left(k+\frac{1}{2}\right)t\right] \\ &= \frac{1}{2\sin\frac{1}{2}t}\left\{-\sum_{k=K}^{r(n)-K}\cos\left(k+\frac{1}{2}\right)t\Delta_k\left(\frac{\alpha_{n,k}}{k}\right)\right. \\ &\quad + \frac{\alpha_{n,K}}{K}\cos\left(K-\frac{1}{2}\right)t \\ &\quad \left. - \frac{\alpha(n, r(n)-K+1)}{r(n)-K+1}\cos\left(r(n)-K+\frac{1}{2}\right)t\right\}. \end{aligned}$$

Since

$$\Delta_k\left(\frac{\alpha_{n,k}}{k}\right) = \frac{\alpha_{n,k}}{k(k+1)} + \frac{\Delta_k(\alpha_{n,k})}{k+1},$$

it follows that

$$\begin{aligned} R_n(t) &= O\left\{\frac{1}{t}\left[\sum_{k=K}^{r(n)-K}\frac{\alpha_{n,k}}{k(k+1)} + \sum_{k=K}^{r(n)-K}\frac{|\Delta_k(\alpha_{n,k})|}{k+1}\right.\right. \\ &\quad \left.\left. + \frac{\alpha_{n,K}}{K} + \frac{\alpha(n, r(n)-K+1)}{r(n)-K+1}\right]\right\} \\ &= O\{R_n^1(t) + R_n^2(t) + R_n^3(t) + R_n^4(t)\}, \end{aligned}$$

say. Now, since $\alpha_{n,k}$ is nondecreasing in the relevant range

$$\begin{aligned} R_n^{n_2}(t) &= -\frac{1}{t}\sum_{k=K}^{r(n)-K}\frac{\Delta_k(\alpha_{n,k})}{k+1} \\ &= \frac{1}{t}\sum_{k=K}^{r(n)-K}\frac{\alpha_{n,k}}{k(k+1)} - \frac{1}{t}\frac{\alpha_{n,k}}{K} + \frac{1}{t}\frac{\alpha(n, r(n)-K+1)}{r(n)-K+1} \\ &= R_n^1(t) - R_n^3(t) + R_n^4(t)\ , \end{aligned}$$

so that

$$R_n(t) = O\{R_n^1(t) + R_n^4(t)\}\,.$$

Next,

$$\sum_{r(n)\geqq 2K} R_n^1(t) \leqq \frac{1}{t}\sum_{k=K}^{\infty}\frac{1}{k(k+1)}\sum_{n=0}^{\infty}\alpha_{n,k} = O(1)$$

by (3) and the definition of K. Finally, if $r(n) \geqq 2K$,

$$\begin{aligned} R_n^4(t) &= O\left\{\frac{\alpha(n, r(n)-K+1)}{tr(n)}\right\} \\ &= O\left\{\frac{1}{tKr(n)}\sum_{k=r(n)-K+1}^{r(n)}\alpha_{n,k}\right\} \\ &= O\left\{\frac{1}{r(n)}\sum_{k=r(n)-K+1}^{r(n)}\alpha_{n,k}\right\}\ , \end{aligned}$$

so that

$$\sum_{r(n)\geqq 2K} R_n^4(t) = O(1)\ ,$$

by (5). The proof of the theorem is thus completed.

5. We now consider an application of our general theorem to the special case of Nörlund summability. We recall that, given a sequence $p = \{p_n\}$, Nörlund summability (N, p) is defined as given by the sequence-to-sequence transformation

$$t_n = \frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}s_k\ , \tag{19}$$

where we write

$$P_n = p_0 + p_1 + \cdots p_n\ ;$$

it is assumed that p is such that, for all n, $P_n \neq 0$. If we write

$$t_n = b_0 + b_1 + \cdots b_n;\ s_k = a_0 + a_1 + \cdots a_k$$

we see that (19) can be expressed as the series-to-series transformation

$$b_0 = a_0\ ;$$

$$b_n = \sum_{k=1}^{n} \left(\frac{P_{n-k}}{P_n} - \frac{P_{n-k-1}}{P_{n-1}} \right) a_n \qquad (n \geqq 1) ,$$

where we adopt the convention that $P_{-1} = 0$. Thus we have, with the notation of our main theorem, $\alpha_{n,k} = 0$ for $k > n$, while, for $1 \leqq k \leqq n$

$$\begin{aligned} \alpha_{n,k} &= \frac{P_{n-k}}{P_n} - \frac{P_{n-k-1}}{P_{n-1}} \\ &= \frac{P_n p_{n-k} - P_{n-k} p_n}{P_n P_{n-1}} . \end{aligned} \tag{20}$$

Now consider the case in which $\{p_n\}$ is nonnegative nonincreasing. We remark that, since $P_0 \neq 0$, we then have $p_0 > 0$. Further (since $p_n \geqq 0$) $\{P_n\}$ is nondecreasing; thus it follows from (20) that $\alpha_{n,k} \geqq 0$. Thus we may omit the modulus signs in (3); and it is now easy to see that (4), and hence (3), holds. Thus, in the case now considered, (N, p) is absolutely regular. Further, for fixed n, p_{n-k} is nondecreasing and P_{n-k} nonincreasing as k increases from 1 to n. Since $\alpha_{n,k} = 0$ for $k > n$, it follows that condition (a) is satisfied, with $r(n) = n$. Also equation (5) becomes

$$\sum_{n=2K}^{\infty} \frac{1}{nP_n P_{n-1}} \sum_{k=n-K}^{n} (P_n p_{n-k} - P_{n-k} p_n) = O(1) . \tag{21}$$

The inner sum in (21) does not exceed

$$\sum_{k=n-K}^{n} P_n p_{n-K} = P_n P_K ,$$

and thus a sufficient condition for (21) to hold is that

$$\sum_{n=2K}^{\infty} \frac{1}{nP_{n-1}} = O\left(\frac{1}{P_K}\right) . \tag{22}$$

However, since the hypotheses on p imply that $P_{n-1} \sim P_n$, and that ${}_K P \leqq P_{2K} \leqq 2P_K$, it is easily seen that (22) is equivalent to the slightly simpler condition

$$\sum_{n=K}^{\infty} \frac{1}{nP_n} = O\left(\frac{1}{P_K}\right) . \tag{23}$$

Thus our theorem includes the following result;

THEOREM A. *Suppose that $\{p_n\}$ is nonnegative nonincreasing, and that (23) holds. Then the Fourier series of any function of bounded variation is absolutely summable $|N, p|$.*

The assumption that $\{p_n\}$ is nonnegative nonincreasing is not, without some further condition, sufficient for the conclusion, for it has been shown by Pati [8] that, when $p_n = 1/(n+1)$, it is not true that the Fourier series of any function of bounded variation is absolutely summable $|N, p|$. This example also shows that, in our main theorem, the assumptions that A is absolutely conservative and that (a) holds would not alone suffice for the conclusion.

Theorem A is included in a recent, slightly more general, theorem of M. Izumi and S. Izumi [3]. It includes earlier theorems of H. P. Dikshit [2, Theorem 2] and T. Singh [9]; the result of Singh itself generalises a theorem of Pati [7]. The theorems of Dikshit and of Singh are respectively as follows.

THEOREM B. *Suppose that* $p_n > 0$, *and that* p_{n+1}/p_n *is non decreasing, and less than or equal to* 1 *for all* n. *Suppose that* (23) *holds. Then the Fourier series of any function of bounded variation is absolutely summable* $|N, p|$.

THEOREM C. *Suppose that, for all* n, $p_n \geqq p_{n+1} > 0$, *and that* $p_n - p_{n+1}$ *is nonincreasing. Suppose also that*

$$\sum_{n=0}^{K} \frac{P_n}{n+1} = O(P_K) \ . \tag{24}$$

Then the fourier series of any function of bounded variation is absolutely summable $|N, p|$.

It is immediately evident that Theorem A includes Theorem B. The result that Theorem A includes Theorem C follows from the following lemma, which shows that, in Theorem C, we may replace (24) by (23).

LEMMA 4. *Suppose that* $p_0 > 0$, $p_n \geqq 0$. *Then* (23), (24) *are equivalent. In fact, either is equivalent to the assertion*

(c) *There is a constant integer* $r > 1$, *and a constant* $\lambda > 1$ *such that, for all sufficiently large* n,

$$P_{rn} \geqq \lambda P_n \ . \tag{25}$$

We first prove that (23) implies (c). Suppose, then, that (23) holds. Thus there is a constant M such that, for all sufficiently large K,

$$\sum_{n=K}^{\infty} \frac{1}{nP_n} \leqq \frac{M}{P_K} \ .$$

Since P_n is nondecreasing, this gives

$$\frac{M}{P_K} > \sum_{n=K}^{rK} \frac{1}{nP_n} \geqq \frac{1}{P_{rK}} \sum_{n=K}^{rK} \frac{1}{n} \ .$$

But

$$\sum_{n=K}^{rK} \frac{1}{n} \longrightarrow \log r$$

as $K \to \infty$, and (c) therefore follows if r has been chosen so that

(26) $$\log r > M .$$

If (24) holds, we have, for all sufficiently large K,

$$\sum_{n=0}^{K} \frac{P_n}{n+1} \leqq MP_K .$$

Thus, replacing K by rK,

$$MP_{rK} > \sum_{n=K}^{rK} \frac{P_n}{n+1} \geqq P_K \sum_{n=K}^{rK} \frac{1}{n+1} ,$$

and we again deduce (c) if r has been chosen so that (26) holds.

We now consider the converse implications. Suppose, then, that (25) holds for $n \geqq n_0$. Then, for $\nu \geqq n_0$

$$\sum_{n=r\nu}^{r(\nu+1)-1} \frac{1}{nP_n} \leqq \frac{r}{r\nu P_{r\nu}} \leqq \frac{1}{\nu\lambda P_\nu} .$$

Hence, for $K \geqq n_0$ and $s \geqq 1$,

(27) $$\sum_{n=r^sK}^{r^{s+1}K-1} \frac{1}{nP_n} \leqq \frac{1}{\lambda} \sum_{n=r^{s-1}K}^{r^sK-1} \frac{1}{nP_n} .$$

By successive applications of (27), we deduce that for $s \geqq 0$,

$$\sum_{n=r^sK}^{r^{s+1}K-1} \frac{1}{nP_n} \leqq \frac{1}{\lambda^s} \sum_{n=K}^{rK-1} \frac{1}{nP_n} \leqq \frac{1}{\lambda^s P_K} \sum_{n=k}^{rK-1} \frac{1}{n} = O\left(\frac{1}{\lambda^s P_K}\right) .$$

Hence

$$\sum_{n=K}^{\infty} \frac{1}{nP_n} = O\left(\frac{1}{P_K} \sum_{s=0}^{\infty} \frac{1}{\lambda^s}\right) = O\left(\frac{1}{P_K}\right) ,$$

which gives (23). To prove (24), we have, for $\nu \geqq n_0$,

$$\sum_{n=r\nu}^{r(\nu+1)-1} \frac{P_n}{n+1} \geqq \frac{rP_{r\nu}}{r(\nu+1)} \geqq \frac{\lambda P_\nu}{\nu+1} .$$

Hence, for $s \geqq 1$,

$$\sum_{n=r^sn_0}^{r^{s+1}n_0-1} \frac{P_n}{n+1} \geqq \lambda \sum_{n=r^{s-1}n_0}^{r^sn_0-1} \frac{P_n}{n+1} ,$$

so that, for $0 \leqq s \leqq t-1$,

$$\sum_{n=r^s n_0}^{r^{s+1}n_0-1} \frac{P_n}{n+1} \leqq \frac{1}{\lambda^{t-1-s}} \sum_{n=r^{t-1}n_0}^{r^t n_0-1} \frac{P_n}{n+1}. \tag{28}$$

Now take any $K \geqq n_0$. Choose t so that $r^t n_0 \leqq K < r^{t+1} n_0$. Then, by (28),

$$\sum_{n=0}^{K} \frac{P_n}{n+1} \leqq \sum_{n=0}^{n_0-1} \frac{P_n}{n+1} + \sum_{s=0}^{t-1} \frac{1}{\lambda^{t-1-s}} \sum_{n=r^{t-1}n_0}^{r^t n_0-1} \frac{P_n}{n+1} + \sum_{n=r^t n_0}^{K} \frac{P_n}{n+1}, \tag{29}$$

where the second term on the right is omitted when $t = 0$. The first term on the right of (29) is a constant, and is thus certainly $O(P_K)$, since $P_K \geqq p_0 > 0$. Also

$$\sum_{n=r^{t-1}n_0}^{r^t n_0-1} \frac{P_n}{n+1} \leqq P_K \sum_{n=r^{t-1}n_0}^{r^t n_0-1} \frac{1}{n+1} = O(P_K);$$

$$\sum_{n=r^t n_0}^{K} \frac{P_n}{n+1} \leqq P_K \sum_{n=r^t n_0}^{K} \frac{1}{n+1} = O(P_K)$$

(since $K < r^{t+1} n_0$). Thus (24) follows.

The conditions (7), (8) have been mentioned as giving simple sufficient conditions. But, while simpler than (5) or (6), they appear to be insufficiently general to be of great use. Consider, for example, the case of Cesàro summability (C, δ). This is a Nörlund method with

$$p_n = \binom{n+\delta-1}{n}.$$

If $0 < \delta \leqq 1$, then the conditions of Theorem A are satisfied. Thus that theorem includes the result that the Fourier series of any function of bounded variation is absolutely summable $|C, \delta|$; this result was long ago proved by Bosanquet [1]. Now, in this case, $\alpha_{n,k} = 0$ for $k > n$, while, for $1 \leqq k \leqq n$,

$$\alpha_{n,k} = \frac{k}{n} \frac{\binom{n-k+\delta-1}{n-k}}{\binom{n+\delta}{n}}.$$

Thus (a), (b) are both satisfied, with $r(n) = s(n) = n$. But either (7) or (8) reduces to

$$\sum_{n=2K}^{\infty} \frac{1}{n\binom{n+\delta}{n}} = O\left(\frac{1}{K}\right),$$

and this is satisfied only if $\delta = 1$.

6. As another application of our main theorem, we let $\{k(n)\}$ be an increasing sequence of nonnegative integers, with $k(0) = 0$, and define

$$\alpha_{n,k} = \begin{cases} 1 & (k(n) \leqq k < k(n+1); \\ 0 & \text{otherwise.} \end{cases}$$

Thus absolute summability $|A|$ of a given series reduces to the absolute convergence of the series formed from it by bracketing together, for every n, those terms whose suffixes k satisfy $k(n) \leqq k < k(n+1)$. It is clear that (a), (b) are both satisfied, with $r(n) = s(n) = k(n)$ (except when $n = 0$). In this case, the weaker conditions (7), (8) still give a significant result. Either of these conditions is equivalent to

$$\sum_{k(n) \geqq K} \frac{1}{k(n)} = O\left(\frac{1}{K}\right) . \tag{30}$$

We note that (30) is satisfied, in particular, if

$$\liminf_{n\to\infty} \frac{k(n+1)}{k(n)} > 1 . \tag{31}$$

Thus our theorem includes the following result. *Suppose that* (31) *holds. Let us bracket together, in the way indicated, the terms of the Fourier series of any function of bounded variation. Then the resulting series is absolutely convergent.*

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