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THE DISAPPEARING CLOSED SET PROPERTY

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A topological space X is said to have the disappearing closed set (DCS) property or to be a DCS space, if for every proper closed subset C there is a family of open sets $\{U_i\}_{i=1}^{\infty}$ such that $U_{i+1} \subseteq U_i$ and $\bigcap_{i=1}^{\infty} U_i = \emptyset$, and there is also a sequence $\{h_i\}$ of homeomorphisms on X onto X such that $h_i(C) \subseteq U_i$, for all i. Properties of DCS spaces are studied as are connections between this and other related definitions.

I. Simple examples of sets with the DCS property are the n-sphere, $n > 0$, and the open n-cell, $n > 0$. This definition was formulated in an attempt to generalize the definition of invertible set which has been extensively studied by Doyle, Hocking and others [1, 2, 3, 4, 6]. A space X is said to be invertible if for every proper closed subset C of X there is a homeomorphism h on X onto X such that $h(C) \subseteqq X - C$. Neither of these definitions implies the other. For example, an open n-cell is not invertible, and on the other hand, the 0-sphere is invertible but does not satisfy the DCS property. However, both definitions require that closed sets can be made "small" or "thin."

It is proved in [5] that compact n-manifolds have the DCS property. It is the purpose of this paper to investigate some other topological properties of DCS spaces.

II. THEOREM 1. *Any disconnected DCS space X must have an infinite number of components.*

Proof. Suppose X has a finite number of components, A_j, $j = 1, \cdots, n$. Each A_j is both open and closed. Consider the DCS property applied to $\bigcup_{j=2}^{n} A_j = B$, a closed set. There are open sets $\{U_i\}_{i=1}^{\infty}$ and homeomorphisms $\{h_i\}_{i=1}^{\infty}$ such that $h_i(B) \subseteqq U_i$, $U_{i+1} \subseteqq U_i$, and $\bigcap_{i=1}^{\infty} U_i = \emptyset$. Since there are at most a finite number of components A_i and since the U_i form a decreasing sequence of open sets whose intersection is empty, there must be an m such that for each $j = 1, \cdots, n$, there are $x_j \in A_j$ such that $x_j \notin U_m$. But $X - U_m \subseteqq h_m(A_1)$, since $h_m(B) \subseteqq U_m$ and $X = A_1 \cup B$, $A_1 \cap B = \emptyset$. Thus $x_j \in h_m(A_1)$, $j = 1, \cdots, n$. But this is a contradiction unless $n = 1$, since $h_m(A_1)$ is connected, but intersects all components of X.

An example of a DCS space which is not connected is the product space obtained by crossing the real numbers with the rationals.

One method of constructing DCS spaces is given by the following:

THEOREM 2. *If X and Y are DCS spaces, so is $X \times Y$.*

Proof. Let C be a proper closed subset of $X \times Y$, and let $P \subseteqq X$, $Q \subseteqq Y$ be open sets in X and Y, respectively, such that $P \times Q \subseteqq X \times Y - C$. Let $\{U_i\}_{i=1}^{\infty}$, $\{h_i\}_{i=1}^{\infty}$ and $\{V_i\}_{i=1}^{\infty}$, $\{k_i\}_{i=1}^{\infty}$ be the open sets and homeomorphisms for $X - P$ and $Y - Q$ in X and Y, respectively. If $(x, y) \in X \times Y$, define $\phi_i(x, y) = \{h_i(x), k_i(y)\}$. Now $\{W_i\}_{i=1}^{\infty} = \{(U_i \times Y) \cup (X \times V_i)\}_{i=1}^{\infty}$ is a decreasing sequence of open sets in $X \times Y$, with empty intersection. Also, $\phi_i(C) \subseteqq W_i$. Thus, $X \times Y$ has the *DCS* property.

The relation between invertible spaces and spaces with the *DCS* property can be seen more clearly in the following analysis.

If an invertible T_1 space X has the property that the intersection of all neighborhoods of any point is that point, and if any closed set C in an open set U may be "moved" so as to miss any given $x \in U$, without moving outside U, then X has the *DCS* property. (If U is open, $U - \{x\}$ is also.)

III. This suggests a relationship with another concept, also studied by Doyle and Hocking. A space X is near-homogeneous if for any $x \in X$ and any open set U such that $x \in U$, for every $y \in X$ there is a homeomorphism on X onto X such that $h(y) \in U$.

Once again, the 0-sphere is a space that does not satisfy the *DCS* property, but is near-homogeneous. However, the following converse is true:

THEOREM 3. *Every DCS space X is near-homogeneous.*

Proof. Let $x \in X$ and U an open set containing x. Let $y \in X$. Consider $C = X - U$, a proper closed subset of X. Since X has the *DCS* property, there is a sequence of homeomorphisms $\{h_i\}_{i=1}^{\infty}$ on X onto X such that $\bigcap_{i=1}^{\infty} h_i(C) = \varnothing$, a somewhat weaker statement than the *DCS* property allows. There is some j such that $y \notin h_j(C)$. But then $y \in h_j(U)$, so $h_j^{-1}(y) \in U$. Thus, X is near-homogeneous.

In the preceding proof, it is seen that near-homogeneity does not require that closed sets get "thin," but that they move around enough. An equivalent form of the definition of near-homogeneity, related to the *DCS* property, is of interest here.

THEOREM 4. *Let $H(X)$ be the family of all homeomorphisms on X onto X. X is near-homogeneous iff, for every proper closed set $C \subseteqq X$, $\bigcap_{h \in H(X)} h(C) = \varnothing$.*

Proof. If X is near-homogeneous, let C be a closed subset of X, and let $U = X - C$. Let $y \in C$. Then there is an $h \in H(X)$ such that $h(y) \in U$, by near-homogeneity and thus $\bigcap_{h \in H(X)} h(C) = \emptyset$.

Conversely, let $x, y \in X$, and let U be an open set such that $x \in U$. Let $C = X - U$. If $y \notin C$, there is nothing to show, so suppose $y \in C$. Then there is an $h \in H(X)$ such that $h(y) \notin C$. Otherwise $\bigcap_{h \in H(X)} h(C)$ would not be empty. But this is the desired homeomorphism.

IV. Another definition relating to invertibility that has been studied is that of local invertibility. A space X is said to be invertible at a point $x \in X$ if for every open set U containing x there is a homeomorphism h on X onto X such that $h(X - U) \subseteqq U$. In [2] it was proved that for such a space certain local properties become global properties. For example, if X is invertible and locally compact at x, then X is compact. The corresponding definition here is the following. A space X has the DCS/x property for all closed sets which miss x. It is evident that a space X has the DCS property, iff it has the DCS/x property for each $x \in X$. Examples of spaces with the DCS/x property include the closed n-cell, the n-leafed rose and, in fact any space that is invertible at x in such a way that the inverting homeomorphism may be taken to fix x. A space that is not invertible at any point but which does have the DCS/x property is the "half-open" annulus $[0, 1) \times S_1$. It will have the DCS/x property for every point of $\{0\} \times S_1$.

Since the DCS/x definition cannot guarantee that any part of the closed set will be carried close to x under any of the homeomorphisms, theorems as sweeping as those of local invertibility cannot be obtained. However, the following is true:

THEOREM 5. *Let X be a space that has the DCS/x property at x and suppose X is locally T_i, $i = 0, 1, 2$, in a neighborhood P of x. Then X is T_i.*

Proof. Let $y, z \in X$, $y \neq z$ (perhaps one is x). Let $\{U_i\}_{i=1}^{\infty}$ and $\{h_i\}_{i=1}^{\infty}$ be the open sets and homeomorphisms given by the DCS/x property for the closed set $X - P$. There is a j such that $y, z \notin U_j$. Then $y, z \notin h_j(X - P)$, so $y, z \in h_j(P)$. But then $h_j(y)$ and $h_j(z)$ have the separation property required and thus y and z do also.

Note that this kind of argument is an improvement on near-homogeneity, since it makes it possible to bring two points (or any finite number of points) into a neighborhood of x at once.

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