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SUPERADDITIVITY INTERVALS AND BOAS' TEST

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A test is given for determining maximal intervals of superadditivity for convexo-concave functions. The test is then applied to several families of ogive-shaped functions.

1. Superadditive functions have been widely studied [8, 11] for their own sake but have also found important applications in reliability theory, e.g. [6]. However, tests for superadditivity were non existent in the literature until Bruckner's work [3] in 1962. A more constructive (hence more readily applicable) test due to Boas was given in 1964 in a paper by Beckenbach [2] on analytic inequalities, an area where superadditivity is of use (see [2] for a derivation of Whittaker's inequality [12]). Boas' test is here viewed in the light of Bruckner's result, strengthened, and applied to some families of convexo-concave functions as suggested in [2].

2. Consider a continuous, real-valued function, f, of a real variable, $x \in \boldsymbol{R}$. Then f is called "superadditive" on $[\beta, b] \subset \boldsymbol{R}$ if

$$f(x) + f(y) \leqq f(x + y)$$

for every $x, y, x + y$ in $[\beta, b]$. We normalize to the cases $\beta = 0$, $b > 0$. In this event, superadditivity implies $f(0) \leqq 0$. The following sufficient condition for superadditivity is due to Boas [2]:

THEOREM (Boas' Test). *Assume f is nonnegative on $[0, b]$ with $f(0) = 0$ and f has a continuous derivative on $[0, b]$. If there are numbers $a \leqq b/2$ and $c \leqq a$ such that*

(0) *f is star-shaped*[1] *on $[0, 2a]$,*

(i) *f is concave*[2] *and satisfies $f(x/2) \leqq f(x)/2$ on $[c, b]$,*

(ii) *$f'(0) < f'(b)$,*

(iii) *$f'(x) - f'(b - x)$ has at most one zero in $(0, a)$.*

Then f is superadditive on $[0, b]$.

A proof of the theorem can be made by considering separately the cases:

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[1] f is "star-shaped" on $[0, A]$ means for every $x \in [0, A]$, and every $\alpha \in [0, 1]$ it is true that $f(\alpha x) \leqq \alpha f(x)$. For $f \in C^1[0, A]$ it is necessary and sufficient [4] that $f'(x) \geqq f(x)/x$ for all $x \in (0, A]$.

[2] The function f is called "convex" on $[a, b]$ if for every $x, y \in [a, b]$ it is true that $f((x+y)/2) \leqq (f(x)+f(y))/2$; f is called "concave" if $-f$ is convex.

(I) $0 \leqq x \leqq a, 0 \leqq y \leqq a$;

(II) $x \geqq a, y \geqq a, x + y \leqq b$;

(III) $x < a < y < b, x + y \leqq b$.

It was conjectured that this test could be applied to finding superadditivity intervals of such ogive-shaped functions as $\exp(-1/\alpha x)$ $(0 < \alpha \leqq 1)$; $\ln(1 + x^{\lambda})$ and $\arctan x^{\lambda} (\lambda > 1)$. But it is easy to show that for some of these functions, Boas' test does not apply: consider $\ln(1 + x^2)$. A simple calculation shows that $1 \leqq c \leqq 2\sqrt{2}$ whereas $2a < 2$ and hence $a < c$. It is our primary goal to modify Boas' test so that it can be used to determine intervals of superadditivity for a larger class of functions. Along the way we shall be able to determine conditions giving maximal intervals of superadditivity, and finally a tabulation of intervals of superadditivity is given for some of the functions previously mentioned.

3. We are interested in determining intervals, $[0, b]$, of superadditivity for a special class of functions, the "convexo-concave" functions [1]: f is called convexo-concave on $[0, B]$ if it is convex on $[0, c]$ and concave on $[c, B]$, $0 \leqq c \leqq B$. Already, f is superadditive on $[0, c]$ [4]; that is, $b \geqq c$. Bruckner has characterized superadditivity of such functions in the following way:

THEOREM [3]. *The convexo-concave function,* f, *with* $f(0) \leqq 0$, *is superadditive on* $[0, b]$ *if and only if* $\max_{0 \leqq x \leqq b} [f(x) + f(b - x)] \leqq f(b)$.

The main difficulties in applying Bruckner's test are first in obtaining the quantity "b", and second in taking the maximum on the lefthand side. By requiring $f \in C^1[0, b]$ we can ameliorate the second objection and turning to Boas' test we obtain a candidate for b: namely, let b be the smallest positive root of $f(x) = 2f(x/2)$.

THEOREM. *Let* $f \in C^1[0, b]$ *be convexo-concave on* $[0, b]$ $(0 < b < \infty)$ *with* $f(0) \leqq 0$ *and*[3]

(i) $f(b) \geqq 2f(b/2)$,

(ii) $f'(0) < f'(b)$,

(iii-a) $f'(x) = f'(b - x)$ *no more than once on* $(0, b/2)$. *Then* f *is superadditive on* $[0, b]$.

Proof. Consider the function $g(x) \equiv f(x) + f(b - x) - f(b)$. Then $f(0) \leqq 0$ implies $g(0) \leqq 0$. By (i) and (ii), $g(b/2) \leqq 0$ and $g'(0) < 0$, respectively. Suppose g is positive on $(0, b/2)$. Then it has a positive

[3] It is important for generalizing to higher dimensions that condition (0) in Boas' test has been deleted. See [6].

maximum on $(0, b/2)$. Therefore $g'(x) = f'(x) - f'(b - x)$ has at least *two* zeros on $(0, b/2)$, contrary to (iii-a). Finally, then, $g(x) \leqq 0$ on $[0, b/2]$ and—by symmetry of g about $x = b/2$,

$$\max_{0 \leqq x \leqq b} [f(x) + f(b - x)] \leqq f(b)$$

which, by Bruckner's theorem, shows f superadditive on $[0, b]$.

For the function $f(x) \equiv \ln(1 + x^2)$ it is easy to check that (i), (ii) are satisfied for $b = 2\sqrt{2}$. Condition (iii-a) is also straight forward: it is true by Descartes' rule of signs.

Notice that for $f(0) < 0$, f is superadditive at least as long as it is merely nondecreasing and nonpositive. This relatively arbitrary state of affairs will be avoided by assuming $f(0) = 0$ in what follows. For a further appreciation of (iii) we give a corollary to Bruckner's theorem.

COROLLARY. *Suppose convexo-concave f, with $f(0) = 0$, is continuously differentiable. Then f is superadditive on $[0, b]$ if and only if for every x_0 in $[0, b]$ such that $f'(x_0) = f'(b - x_0)$, it is true that $f(x_0) + f(b - x_0) \leqq f(b)$.*

Thus we see how the maximizing duties in Bruckner's theorem have been replaced by a zero-*counting* operation in the other two theorems. The fourth condition in Boas' test is less restrictive than (iii-a) above since b is not less than $2a$. But it is not hard to see that (iii-a) can be replaced by

(iii-b) $f'(x) = f'(b - x)$ no more than once on the *smaller* of the two intervals $(0, c)$, (c, b),

which is a less restrictive condition than even Boas' fourth condition. (Here "c" is the inflection point of f.)

Perhaps a computational note is in order here. If we refer generically to conditions (iii), (iii-a), (iii-b) as "root conditions", then in applications the root condition can often be tested by Sturm's theorem [7]. For example, the functions $\ln(1 + x^n)$ $(n = 2, 3, 4, \cdots)$ have as derivatives rational functions with denominators not vanishing for positive arguments. Verifying a root condition is then a matter of counting the number of zeros of *polynomials* in a finite interval. Sturm sequences can also be readily computed for rational functions [10], and Sturm's idea can be extended to counting real zeros of even more general functions [5]. Finally, upon observing that f' is unimodal[4], an optimum strategy for localizing the inflection point c (as used in (iii-b)) is well-known [9].

4. Now it is quite striking that the choice of b as the smallest positive root, σ, of $2f(x/2) = f(x)$ often turns out to be maximal. Certainly σ is an upper bound on the interval of superadditivity. Consider the quantity $\min\{\sigma, \tau\}$ where σ, τ are the smallest positive, odd zeros of $2f(x/2) - f(x)$, $f'(0) - f'(x)$, respectively. Then we may be assured of a maximal interval of superadditivity.

THEOREM. *Suppose $f \in C^1[0, b]$ is superadditive on $[0, b]$ where $b \equiv \min\{\sigma, \tau\} < \infty$. Then f is not superadditive on any larger interval, $[0, B]$, $B > b$.*

The proof is immediate by failure of superadditivity near $x = 0$ ($b = \tau$ case) and $x = B/2$($b = \sigma$ case) where $B = b + \varepsilon$, $\varepsilon > 0$ arbitrary. In our example, $2\sqrt{2}$ is the largest value of b so that $\ln(1 + x^2)$ is superadditive on $[0, b]$. With this optimality result, then, we turn to computing intervals of superadditivity in the next section.

5. Tables of $\hat{b}$ are now given where $\hat{b}$ is the largest 7D approximation smaller or equal to b and $[0, b]$ is the maximum interval of superadditivity for the function indicated.

λ	$\arctan x^\lambda$	$\ln(1 + x^\lambda)$	$\exp(-\lambda/x)$	λ
1.1	.5852351	.3425001	1.586964	1.1
1.2	.8532410	.7280202	1.731234	1.2
1.3	1.051079	1.104767	1.875503	1.3
1.4	1.205188	1.452478	2.019773	1.4
1.5	1.328208	1.764139	2.164042	1.5
1.6	1.427957	2.039063	2.308312	1.6
1.7	1.509790	2.279467	2.452581	1.7
1.8	1.577572	2.488734	2.596851	1.8
1.9	1.634178	2.670539	2.741120	1.9
2	1.681792	2.828427	2.885390	2
3	1.906368	3.634241	4.328085	3
4	1.966894	3.868672	5.770780	4
5	1.987133	3.948700	7.213475	5
6	1.994715	3.978890	8.656170	6
7	1.997751	3.991011	10.09886	7
8	1.999019	3.996080	11.54156	8
9	1.999565	3.998260	12.98425	9
10	1.999804	3.999218	14.42695	10

[4] A function $f(x)$ is "unimodal" if there is a ξ so that f is either strictly increasing for $x \leqq \xi$ and strictly decreasing for $x > \xi$, or else strictly increasing for $x < \xi$ and strictly decreasing for $x \geqq \xi$.

Entries above or to the left of the stepped line were unattainable by Boas' original test.

For $\exp(-\lambda/x)$ ($\lambda \geqq 1$) it is easy to verify (in this case, Boas' test is sufficient) that the intervals of superadditivity $[0, b(\lambda)]$ are determined by $b(\lambda) = \lambda/\ln 2$.

In [2] it is suggested that maximum intervals of superadditivity be computed not only for $f = f_\lambda$ but also for the "average function of f", $F = F_\lambda$, and for the "inverse average function," $\phi = \phi_\lambda$, where

$$F_\lambda(x) \equiv \begin{cases} 0 & x = 0 , \\ \dfrac{1}{x}\displaystyle\int_0^x f_\lambda(t)dt & x > 0 ; \end{cases}$$

$$\phi_\lambda(x) \equiv f_\lambda(x) + xf'_\lambda(x) \qquad x \geqq 0 .$$

For the case $f_\lambda(x) \equiv \exp(-\lambda/x)$ we can give the following maximum intervals of superadditivity:

Function	$\hat{b}(\lambda)$-end point
ϕ_λ	$\lambda/1.116845$
f_λ	$\lambda/.6931472$
F_λ	$\lambda/.4243251$

where Boas' test was inapplicable to the ϕ_λ-case.

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