

ESSENTIAL CENTRAL SPECTRUM AND RANGE FOR ELEMENTS OF A VON NEUMANN ALGEBRA

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A closed two-sided ideal \mathcal{I} in a von Neumann algebra \mathcal{A} is defined to be a central ideal if $\sum A_i P_i$ is in \mathcal{I} for every set $\{P_i\}$ of orthogonal projections in the center \mathcal{Z} of \mathcal{A} and every bounded subset $\{A_i\}$ of \mathcal{I} . Central ideals are characterized in terms of the existence of continuous fields and their form is completely determined.

If \mathcal{I} is a central ideal of \mathcal{A} and $A \in \mathcal{A}$, then $A_0 \in \mathcal{Z}$ is said to be in the essential central spectrum of A if $A_0 - A$ is not invertible in \mathcal{A} modulo the smallest closed ideal containing \mathcal{I} and ζ for every maximal ideal ζ of \mathcal{Z} . It is shown that the essential central spectrum is a nonvoid, strongly closed subset of \mathcal{Z} and that it satisfies many of the relations of the essential spectrum of operators on Hilbert space. Let \mathcal{A}^{\sim} be the space of all bounded \mathcal{Z} -module homomorphisms of \mathcal{A} into \mathcal{Z} . The essential central numerical range of $A \in \mathcal{A}$ with respect to \mathcal{I} is defined to be $\mathcal{H}_{\mathcal{I}}(A) = \{\phi(A) \mid \phi \in \mathcal{A}^{\sim}, \|\phi\| \leq 1, \phi(1) = P_{\mathcal{I}}, \phi(\mathcal{I}) = (0)\}$. Here $P_{\mathcal{I}}$ is the orthogonal complement of the largest central projection in \mathcal{I} . The essential central numerical range is shown to be a weakly closed, bounded, \mathcal{Z} -convex subset of \mathcal{Z} . It possesses many of the properties of the essential numerical range but in a form more suited to the fact that A is in \mathcal{A} rather than a bounded operator. It is shown that if \mathcal{A} is properly infinite and \mathcal{I} is the ideal of finite elements (resp. the strong radical) of \mathcal{A} , then $\mathcal{H}_{\mathcal{I}}(A)$ is the intersection of \mathcal{Z} with the weak (resp. uniform) closure of the convex hull of $\{UAU^{-1} \mid U \text{ unitary in } \mathcal{A}\}$.

In a final section, we give some applications of these facts. We extend a result of J. G. Stampfli [19] to show that the range of a derivation on a von Neumann algebra is never uniformly dense. We also prove a theorem on self-adjoint commutators using a calculation of M. David [5].

2. Central ideals. Let \mathcal{A} be a von Neumann algebra with center \mathcal{Z} . For any subset \mathcal{B} of \mathcal{A} let (\mathcal{B}) denote the set of all projections of \mathcal{B} . Throughout this paper *all* ideals will be assumed to be closed two-sided ideals. An ideal \mathcal{I} in \mathcal{A} is said to be a central ideal or a \mathcal{Z} -ideal if given a norm bounded set $\{A_i \mid i \in I\}$ of elements of \mathcal{I} and a corresponding set $\{P_i \mid i \in I\}$ of mutually orthogonal projections in \mathcal{Z} , then the sum $\sum A_i P_i$, which exists in the strong topology, is also in \mathcal{I} . (Similar definitions were used by I. Kaplansky

[22, §1] and M. Goldman [13; §4] in the theory of AW^* -modules; however, here there is no canonical inner product.) Any ideal \mathcal{I} in \mathcal{A} is contained in a smallest central ideal $\langle \mathcal{I} \rangle$ given by $\langle \mathcal{I} \rangle = \{\sum\{A_i P_i | i \in I\} | \{A_i | i \in I\} \text{ is a bounded subset of } \mathcal{I} \text{ and } \{P_i | i \in I\} \text{ is a mutually orthogonal subset of } (\mathcal{Z}) \text{ of sum } 1\}$ ([19], remarks preceding corollary to (a5) implies (a1)). If \mathcal{I} a central ideal in \mathcal{A} and if A is an element of \mathcal{A} , then it is clear that there is an element P in (\mathcal{Z}) such that $AP \in \mathcal{I}$ and $AQ \notin \mathcal{I}$ for every Q in (\mathcal{Z}) with $0 < Q \leq 1 - P$. The following definition is now possible.

DEFINITION 2.1. Let \mathcal{A} be a von Neumann algebra and let \mathcal{I} be a central ideal of \mathcal{A} . Then $P_{\mathcal{I}}$ will denote the orthogonal complement of the largest central projection in \mathcal{I} . We notice that $QP_{\mathcal{I}} \in \mathcal{I}$ for a central projection Q implies $QP_{\mathcal{I}} = 0$.

We now describe central ideals with regard to finite element

PROPOSITION 2.2. Let \mathcal{A} be a semi-finite von Neumann algebra with center \mathcal{Z} , let \mathcal{I} be a central ideal of \mathcal{A} , and let $\mathcal{A}P$ be the weak closure of \mathcal{I} , where $P \in (\mathcal{Z})$. Then \mathcal{I} contains every finite projection of \mathcal{A} majorized by P .

Proof. Let F be a finite projection of \mathcal{A} majorized by P . Let Q be an element of (\mathcal{Z}) such that $FQ \in \mathcal{I}$ and $FR \notin \mathcal{I}$ for every R in (\mathcal{Z}) with $0 < R \leq 1 - Q$ (preliminary remarks). We note that $Q' = 1 - Q \leq P$. We obtain a contradiction by assuming that $Q' \neq 0$. Since the weak closure of \mathcal{I} is $\mathcal{A}P$ and since linear combinations of projections are dense in \mathcal{I} , there is a projection E in \mathcal{I} with $EQ' \neq 0$. There is an R in $(\mathcal{Z}Q')$ such that $ER < FR$ and $F(Q' - R) < E(Q' - R)$. Either $ER \neq 0$ or $E(Q' - R) \neq 0$. Now if $ER \neq 0$, there is nonzero S in $(\mathcal{Z}R)$ and projections E_1, \dots, E_n in \mathcal{A} such that $ES = E_1 \sim E_2 \sim \dots \sim E_n$ and $FS - \sum E_i < E_1$. This means that FS is in \mathcal{I} . This is contrary to the choice of Q , so we must assume that $E(Q' - R) \neq 0$. But this also implies that $F(Q' - R)$ is in \mathcal{I} . So we must conclude that $Q' = 0$. Hence, we have shown that every finite projection majorized by P is in \mathcal{I} .

COROLLARY 2.3. An ideal in a finite von Neumann algebra is a central ideal if and only if it is weakly closed.

Proof. If the ideal \mathcal{I} in the finite von Neumann algebra \mathcal{A} is weakly closed, then there is a central projection P in \mathcal{A} such that $\mathcal{I} = \mathcal{A}P$ [9, I, 3, Theorem 2, Corollary 2]. Obviously the ideal

$\mathcal{A}P$ is a central ideal of \mathcal{A} .

Conversely, let \mathcal{I} be a central ideal of \mathcal{A} . Let P be the central projection of \mathcal{A} such that the weak closure of \mathcal{I} is $\mathcal{A}P$. Then \mathcal{I} contains every finite projection majorized by P ; in particular, it contains P itself. So $\mathcal{I} = \mathcal{A}P$ and \mathcal{I} is weakly closed.

We now describe central ideals for an arbitrary von Neumann algebra \mathcal{A} with center \mathcal{Z} . Let P be a projection in \mathcal{Z} and let E be a properly infinite projection in \mathcal{A} majorized by P . (By convention we assume that 0 is a properly infinite projection in a finite algebra \mathcal{Z} .) Let $(\mathcal{I}_P(E))$ be the set of all projections in \mathcal{A} given by $(\mathcal{I}_P(E)) = \{F \in (\mathcal{A}) \mid F \leq P \text{ and } QE < QF \text{ for some } Q \in (\mathcal{Z}) \text{ implies } QE = 0\}$. Let $\mathcal{I}_P(E)$ be the ideal generated by $(\mathcal{I}_P(E))$.

We shall use the following lemma of F. B. Wright [32; §2].

LEMMA. Suppose \mathcal{P} is a set of projections on a von Neumann algebra \mathcal{A} that satisfies the following properties:

- (1) if $E \in (\mathcal{A}), F \in \mathcal{P}$ and $E < F$, then $E \in \mathcal{P}$; and
- (2) if E and F are in \mathcal{P} , then the least upper bound $\text{lub}\{E, F\}$ of E and F is in \mathcal{P} .

Then the set of projections of the ideal generated by \mathcal{P} is exactly \mathcal{P} .

THEOREM 2.4. Let \mathcal{A} be a von Neumann algebra with center \mathcal{Z} . In order for the ideal \mathcal{I} in \mathcal{A} to be a central ideal, it is a necessary and sufficient condition that there exist a projection P in \mathcal{Z} and a properly infinite projection E majorized by P with $\mathcal{I} = \mathcal{I}_P(E)$.

REMARK. The sufficiency is an adaptation of the proof we gave for a special case in an earlier paper [18, Proposition 2.1].

Proof. Let E be a properly infinite projection majorized by the central projection P . We show that $\mathcal{I}_P(E) = \mathcal{I}$ is a central ideal. Let P_1 and P_2 be orthogonal central projections of sum 1 such that $\mathcal{A}P_1$ is a finite algebra and $\mathcal{A}P_2$ is a properly infinite. It is sufficient to show that $\mathcal{I}P_i$ is a central ideal in $\mathcal{A}P_i$ ($i = 1, 2$). However, we have that $\mathcal{I}P_i$ is generated by $(\mathcal{I}P_i) = \{F \in (\mathcal{A}P_i) \mid F \leq PP_i, EQP_i < FQ \text{ for some } Q \text{ in } (\mathcal{Z}P_i) \text{ implies } EP_iQ = 0\}$. Now setting $E_i = EP_i$, we obtain a properly infinite projection in $\mathcal{A}P_i$ so that $\mathcal{I}P_i = \mathcal{I}_{PP_i}(E_i)$. Hence, there is no loss of generality in assuming that \mathcal{A} is either finite or properly infinite.

Let \mathcal{A} be finite. Then $E = 0$ and $(\mathcal{I}_P(0)) = \{F \in (\mathcal{A}) \mid F \leq P\}$. Hence $\mathcal{I} = \mathcal{A}P$ and so \mathcal{I} is a central ideal.

Now assume that \mathcal{A} is properly infinite. There is no loss of

generality in the assumption that $P = 1$. We show that \mathcal{I} satisfies properties (1) and (2) of the lemma of F. B. Wright. By the definition of (\mathcal{I}) is clear that it satisfies property (1). Now let E_1 and E_2 be in (\mathcal{I}) . Since $\text{lub}\{E_1, E_2\} - E_1 < E_2$ [21, Theorem 5.4], we have that $\text{lub}\{E_1, E_2\} - E_1$ is in (\mathcal{I}) by (1). So there is no loss of generality in the assumption that E_1 and E_2 are orthogonal. There is $Q \in (\mathcal{E})$ such that $QE_1 < QE_2$ and $(1 - Q)E_2 < (1 - Q)E_1$. Since $Q(E_1 + E_2) \in (\mathcal{I})$ and $(1 - Q)(E_1 + E_2) \in (\mathcal{I})$ implies that $E_1 + E_2 \in (\mathcal{I})$, there is no loss of generality in the assumption that $E_1 < E_2$. There is a $Q \in (\mathcal{E})$ such that QE_2 is finite and $(1 - Q)E_2$ is properly infinite. Hence, we may assume that either E_2 is finite or properly infinite. If E_2 is finite, then E_1 is finite since $E_1 < E_2$ and so $E_1 + E_2$ is finite. [9, III, 2]. If Q is a central projection with $QE < Q(E_1 + E_2)$, then QE is finite and so $QE = 0$. So we are left with the situation that $E_1 < E_2$, $E_1E_2 = 0$, and E_2 is properly infinite. Because E_2 is properly infinite, there are projections F_1, F_2 satisfying the relations: $F_1 \sim F_2 \sim E_2$, $F_1F_2 = 0$, and $F_1 + F_2 = E_2$. [9; III, 8, Corollary 2]. We have that $E_1 + E_2 \sim E_1 + F_2 < F_2 + F_2 = E_2$. By property (1) of the lemma, we conclude that $E_1 + E_2 \in (\mathcal{I})$. Hence (\mathcal{I}) satisfies properties (1) and (2) of the lemma and this means that the set of projections of the ideal \mathcal{I} generated by (\mathcal{I}) is precisely (\mathcal{I}) . Now we show \mathcal{I} is a central ideal. Let $\{A_i | i \in I\}$ be a bounded set in \mathcal{I} and let $\{Q_i | i \in I\}$ be an orthogonal subset of (\mathcal{E}) of sum 1. For every $\varepsilon > 0$ and every $i \in I$ there is a projection F_i in (\mathcal{I}) such that $\|A_i - A_iF_i\| \leq \varepsilon$. Then $\sum F_iQ_i = F$ is in (\mathcal{I}) . Indeed, if $EQ < FQ$ for some Q in (\mathcal{E}) , then $E(Q_iQ) < F(Q_iQ) = F_i(Q_iQ)$ for every $i \in I$. Thus $(EQ)Q_i = 0$ for every $i \in I$ and $EQ = \sum (EQ)Q_i = 0$. This means that $F \in (\mathcal{I})$. However, we have that

$$\|\sum A_iQ_i - (\sum A_iQ_i)F\| \leq \text{lub} \|A_i - A_iF_i\| \leq \varepsilon.$$

Since $(\sum A_iQ_i)F$ is in \mathcal{I} and since \mathcal{I} is uniformly closed, we have that $\sum A_iQ_i \in \mathcal{I}$. This proves that \mathcal{I} is a central ideal.

We now show that every central ideal \mathcal{I} is of the form $\mathcal{I}_P(E)$. Given a nonzero $P \in (\mathcal{E})$ it is sufficient to prove that there is a properly infinite projection E in \mathcal{A} , a nonzero Q in (\mathcal{E}) , and an $R \in (\mathcal{E})$ with $R \leq Q \leq P$ such that $\mathcal{I}_R(ER)Q = \mathcal{I}Q$. Indeed, suppose we have verified this statement. Let $\{P_i | i \in I\}$ be a maximal set of mutually orthogonal nonzero central projections such that for each P_i there is a properly infinite projection E_i , and a $Q_i \in (\mathcal{E})$ majorized by P_i such that $\mathcal{I}_{Q_i}(EQ_i)P_i = \mathcal{I}P_i$. By the maximality of $\{P_i\}$, we conclude that $\sum P_i = 1$. Setting $E = \sum E_iQ_i$ (resp. $Q = \sum Q_i$) we obtain a properly infinite (resp. central) projection E majorized by Q such that $\mathcal{I}_Q(E) = \mathcal{I}$. In fact, since $\mathcal{I}_Q(E)$ and \mathcal{I} are generated by

their respective projections, it is sufficient to show that $(\mathcal{S}_Q(E)) = (\mathcal{S})$. But we may verify immediately that $(\mathcal{S}_Q(E))P_i = (\mathcal{S}_{Q_i}(E_iQ_i))$, and so we have that $F \in (\mathcal{S}_Q(E))$ if and only if $FP_i \in (\mathcal{S}_{Q_i}(E_iQ_i)) = (\mathcal{S}P_i)$ for every P_i since $\mathcal{S}_Q(E)$ is a central ideal by the first part of this theorem. However, the ideal \mathcal{S} is also a central ideal and thus $F \in (\mathcal{S}_Q(E))$ if and only if $F \in (\mathcal{S})$. So it is sufficient to verify the required statement. We do this in the next paragraph.

Let P be a nonzero element in (\mathcal{X}) . Since we are looking for a nonzero central projection Q majorized by P , we may assume at the outset that $P = 1$ and that either \mathcal{A} is finite or \mathcal{A} is properly infinite. If \mathcal{A} is finite there is a Q in (\mathcal{X}) with $\mathcal{S} = \mathcal{A}Q$ (corollary 2.3). Then we verify immediately that $\mathcal{S} = \mathcal{S}_Q(0)$. Hence, we may assume that \mathcal{A} is properly infinite. Suppose that there is a projection $P \neq 1$ in (\mathcal{X}) such that $AP = A$ for every A in \mathcal{S} . Then we have that $\mathcal{S}_Q(0)(1 - P) = 0 = \mathcal{S}(1 - P)$. So we may assume that \mathcal{S} is weakly dense in \mathcal{A} . Now suppose that $P_\mathcal{S} \neq 1$. Then the nonzero central projection $Q = 1 - P_\mathcal{S}$ is in \mathcal{S} . This means $\mathcal{S}Q = \mathcal{A}Q = \mathcal{S}_Q(0)$. Hence, we may pass to the case that $P_\mathcal{S} = 1$. By making a further reduction if necessary, we may assume that 1 is the sum of an infinite set $\{E_i | i \in I\}$ of orthogonal, equivalent, σ -finite projections [9, III, 1, Lemma 1]. Let $\mathcal{S}(I)$ be the family of all subsets s of I such that there is a nonzero projection P_s in \mathcal{X} with

$$\sum \{E_i | i \in s\}Q \notin \mathcal{S}$$

for every nonzero $Q \in (\mathcal{X}P_s)$. The family $\mathcal{S}(I)$ is nonvoid since $I \in \mathcal{S}(I)$ with $P_I = 1$. There is an $s_0 \in \mathcal{S}(I)$ such that $\text{Card } s_0 \leq \text{Card } s$ for every $s \in \mathcal{S}(I)$. We may assume that $P_{s_0} = 1$. Let $\sum \{E_i | i \in s_0\} = E$; we notice that E is a properly infinite projection of central support 1. We show that $\mathcal{S}_1(E) = \mathcal{S}(E)$ is equal to \mathcal{S} . First we prove that $(\mathcal{S}) \subset (\mathcal{S}(E))$. Let $F \in (\mathcal{S})$. If $EP < FP$ for some $P \in (\mathcal{X})$, then by choice of s_0 we have that $EP = 0$. So $F \in (\mathcal{S}(E))$ by definition and hence $(\mathcal{S}) \subset (\mathcal{S}(E))$. To show the converse relation $(\mathcal{S}(E)) \subset (\mathcal{S})$ we consider two cases: (i) $\text{Card } s_0$ is finite, and (ii) $\text{Card } s_0$ is infinite. For case (i) we have that E is a σ -finite projection of central support 1. Then we have that $(\mathcal{S}(E))$ is exactly the set of finite projections of \mathcal{A} [9; III, 8, Corollary 5]. But by our preliminary reduction \mathcal{S} is weakly dense in \mathcal{A} and therefore contains all finite projections of \mathcal{A} (Proposition 2.2). So $(\mathcal{S}(E)) \subset (\mathcal{S})$. Now we consider case (ii). Let $F \in (\mathcal{S}(E))$. Since \mathcal{S} is a central ideal, there is a $P \in (\mathcal{X})$ such that $PF \in \mathcal{S}$ and $QF \notin \mathcal{S}$ for every nonzero Q in $(\mathcal{X}(1 - P))$. We obtain a contradiction by assuming $1 - P \neq 0$. Because \mathcal{S} contains all finite projections (Proposition 2.2), we have that $F(1 - P)$ is properly infinite

with central support $1 - P$. We may find a nonzero projection Q in $(\mathcal{K}(1 - P))$ such that FQ is the sum of a set $\{F_i | i \in s\}$ of orthogonal, equivalent, properly infinite σ -finite projections [9; III, 1, Lemma 7]. We have that $F_i \sim E_j Q$ for every $i \in s$ and $j \in s_0$. [9; III, 8, Corollary 5]. Since $\sum \{F_i | i \in s\} = FQ < EQ = \sum \{E_i Q | i \in s_0\}$, and since $\text{Card } s_0$ is infinite, we have that $\text{Card } s \leq \text{Card } s_0$ [9; III, 1, Lemma 6]. If $\text{Card } s_0 \leq \text{Card } s$, we would have a contradiction in that $EQ < FQ$ and $EQ \neq 0$. Thus $\text{Card } s \neq \text{Card } s_0$. But if s' is a subset of $\mathcal{S}(I)$ with $\text{Card } s' = \text{Card } s$, then $\sum \{E_i | i \in s'\} Q' \sim FQ'$ for every Q' in $(\mathcal{K}Q)$ and so $\sum \{E_i | i \in s'\} Q' \notin \mathcal{S}$ for every nonzero Q' in $(\mathcal{K}Q)$. This contradicts the choice of s_0 . Hence, $1 - P = 0$ and $F \in (\mathcal{S})$. So in case (ii) we have $(\mathcal{S}(E)) \subset (\mathcal{S})$. Therefore, we have completed the crucial step, and so there is an $E \in (\mathcal{A})$ and a $Q \in (\mathcal{K})$ with $\mathcal{S} = \mathcal{S}_Q(E)$.

Now let E be a properly infinite projection majorized by the central projection P in the von Neumann algebra \mathcal{A} . Let Q be the central projection of \mathcal{A} such that $\mathcal{A}Q$ is equal to the weak closure of $\mathcal{S}_P(E) = \mathcal{S}$. Then it is clear that $\mathcal{S}_Q(EQ) = \mathcal{S}$. We say a representation $\mathcal{S}_P(E)$ for a central ideal \mathcal{S} is in *canonical form* if $\mathcal{A}P$ is the weak closure of $\mathcal{S}_P(E)$.

PROPOSITION 2.5. *Let $\mathcal{S}_P(E)$ and $\mathcal{S}_Q(F)$ be two central ideals of a von Neumann algebra \mathcal{A} that are represented in canonical form. Then $\mathcal{S}_P(E) = \mathcal{S}_Q(F)$ if and only if $P = Q$ and $E \sim F$.*

Proof. If $P = Q$ and $E \sim F$, then it is clear that $\mathcal{S}_P(E) = \mathcal{S}_Q(F)$.

Now let $\mathcal{S}_P(E) = \mathcal{S}_Q(F) = \mathcal{S}$. Since $\mathcal{A}P = \text{weak closure } \mathcal{S} = \mathcal{A}Q$, we have that $P = Q$. Now let R be the largest central projection majorized by P such that $RE \sim RF$. Suppose $R' = P - R \neq 0$. There is a central projection R'' majorized by R' such that $R''E < R''F$ and $(R' - R'')F < (R' - R'')E$. If $R'' \neq 0$, then $SR''F < SR''E$ for some central projection S implies that $SR''F = 0$. Otherwise, we would have that $SR''E \sim SR''F$ and so R would not be the largest central projection with $RE \sim RF$. This means that $R''E \in \mathcal{S}$. Hence $R''E = 0$ and so $\mathcal{S}R'' = \mathcal{A}R''$. This means that $FR'' = 0$ and consequently that $ER'' \sim FR''$. This is a contradiction. A similar contradiction arises if $R' - R'' \neq 0$. So we must have that $R = P$, i.e., $E \sim F$.

REMARK 2.6. In the sequel we assume *all* representations of central ideals are in *canonical form*.

COROLLARY 2.7. *Let \mathcal{A} be a von Neumann algebra and let \mathcal{S} be a central ideal of \mathcal{A} given by $\mathcal{S} = \mathcal{S}_P(E)$ in canonical form.*

Then in order that $P_{\mathcal{J}} = 1$, a necessary and sufficient condition is that P be the central support of E .

Proof. If the central support of E is Q , then from the definition of $\mathcal{S}_P(E) = \mathcal{S}$, it is clear that $P - Q \in \mathcal{S}$. This means that $P_{\mathcal{J}} \neq 1$ if $P - Q \neq 0$. Conversely, if $1 - P_{\mathcal{J}} \neq 0$, then $(1 - P_{\mathcal{J}})E = 0$. But $(1 - P_{\mathcal{J}}) \leq P$ and thus E cannot have central support P .

Let \mathcal{A} be a von Neumann algebra with center \mathcal{Z} . Let Z be the spectrum of \mathcal{Z} . Let $C_c(Z)$ be the set of order-continuous functions of Z into the set of cardinal numbers. J. Tomiyama [30] showed that there is a dimension function D of \mathcal{A} into $C_c(Z)$ such that $D(E) \leq D(F)$ if and only if $E < F$. W. Wils [31] described the range of D in $C_c(Z)$ as being a certain subset Δ of functions in $C_c(Z)$. Although it is not important in the sequel, one may see that the set of projections of a central ideal $\mathcal{S}_P(E)$ satisfies a certain dimension relation relative to P and E . We therefore feel justified in introducing a name for the following relation.

DEFINITION 2.8. Let \mathcal{S} be a central ideal in a von Neumann algebra \mathcal{A} . Let P be a central projection and let E be a properly infinite projection majorized by P with $\mathcal{S} = \mathcal{S}_P(E)$. A projection F in \mathcal{A} is said to have *dimension* greater than that of \mathcal{S} if F has central support $P_{\mathcal{J}}$ and if $F > EP_{\mathcal{J}}$ (in symbols, $\dim F > \dim \mathcal{S}$).

The following proposition characterizes the projections whose dimension is greater than the dimension of \mathcal{S} .

PROPOSITION 2.9. *Let \mathcal{A} be a von Neumann algebra and let \mathcal{S} be a central ideal of \mathcal{A} . Then a projection F of \mathcal{A} has dimension greater than that of \mathcal{S} if and only if F has central support $P_{\mathcal{J}}$ and $FQ \in \mathcal{S}$ for some central projection Q implies $FQ = 0$.*

Proof. Let \mathcal{Z} be the center of \mathcal{A} . Let $E \in (\mathcal{A})$ and let $P \in (\mathcal{Z})$ so that $\mathcal{S}_P(E)$ represents \mathcal{S} in canonical form. First let $F \in (\mathcal{A})$ with central support $P_{\mathcal{J}}$ such that $QF \in \mathcal{S}$ for some $Q \in (\mathcal{Z})$ implies $QF = 0$. There is an $R \in (\mathcal{Z})$ such that $RE < RF$ and such that $R'E < R'F$ for $R' \in (Z(1 - R))$ implies $R' = 0$. Then $FP(1 - R) \in \mathcal{S}_P(E)$ by definition and so $FP(1 - R) = 0$. Thus we obtain that $FPR = FP$. So $EP_{\mathcal{J}} = EPP_{\mathcal{J}} < FP \leq F$, i.e. $\dim F > \dim \mathcal{S}$.

Conversely, let $\dim F > \dim \mathcal{S}$. Then by definition we have that F has central support $P_{\mathcal{J}}$. Let $Q \in (\mathcal{Z})$ and let $QF \in \mathcal{S}$. We have that $EP_{\mathcal{J}} < F$ implies that $EQP_{\mathcal{J}} \in \mathcal{S}$ (lemma of F. B. Wright). Since $EQP_{\mathcal{J}} < EQP_{\mathcal{J}}$, we have that $EQP_{\mathcal{J}} = 0$ and thus $QPP_{\mathcal{J}} \in \mathcal{S}$.

By definition of $P_{\mathcal{F}}$ we find that $QPP_{\mathcal{F}} = 0$. Also $\mathcal{F}(1 - P) = (0)$ and so $QF = QPF + Q(1 - P)F = 0$.

Now we can give some examples.

EXAMPLE 2.10. In a factor algebra, every ideal is a central ideal.

EXAMPLE 2.11. In a semi-finite algebra \mathcal{A} , the ideal \mathcal{F} generated by all finite projections of \mathcal{A} is a central ideal. If \mathcal{A} is finite, then $\mathcal{F} = \mathcal{A}$; if \mathcal{A} is properly infinite, then $\mathcal{F} = \mathcal{F}_1(E)$, where E is a properly infinite projection of central support 1 for which there is a set $\{P_i\}$ of mutually orthogonal central projections of sum 1 such that EP_i is σ -finite for every P_i [8; III, 1, Lemma 7].

EXAMPLE 2.12. If \mathcal{A} is a properly infinite von Neumann algebra, then the strong radical \mathcal{F} (i.e. the intersection of all maximal ideals) is a central ideal with $\mathcal{F} = \mathcal{F}_1(1)$.

3. The essential central spectrum. Let \mathcal{A} be a von Neumann algebra with center \mathcal{Z} . If \mathcal{F} is an ideal in \mathcal{A} , let $\mathcal{A}(\mathcal{F})$ denote the algebra \mathcal{A} reduced modulo \mathcal{F} , and let $A(\mathcal{F})$ denote the image of an element A under the canonical homomorphism of \mathcal{A} into $\mathcal{A}(\mathcal{F})$. The algebra $\mathcal{A}(\mathcal{F})$ is a C^* -algebra under the norm $\|A(\mathcal{F})\| = \text{glb}\{\|A + B\| \mid B \in \mathcal{F}\}$. If ζ is an element in the spectrum Z of \mathcal{Z} , let $[\zeta]$ denote the smallest ideal in \mathcal{A} containing ζ . For simplicity we let $\mathcal{A}([\zeta])$ and $A([\zeta])$ be denoted by the symbols $\mathcal{A}(\zeta)$ and $A(\zeta)$, respectively. Then J. Glimm [12; Lemma 10] has shown that for fixed $A \in \mathcal{A}$ the map $\zeta \rightarrow \|A(\zeta)\|$ is continuous on the spectrum Z . For every A in \mathcal{A} and ζ in Z , the norm $\|A(\zeta)\|$ is equal to $\|A(\zeta)\| = \text{glb}\{\|AP\| \mid P \in (\mathcal{Z}) \text{ and } P^\wedge(\zeta) = 1\}$. Here P^\wedge denotes the Gelfand transform of P . If \mathcal{F}_1 and \mathcal{F}_2 are ideals in \mathcal{A} , then the algebraic sum $\mathcal{F}_1 + \mathcal{F}_2$ is also an ideal of \mathcal{A} . In the sequel we denote the sum $\mathcal{F} + [\zeta]$ of an ideal \mathcal{F} and the special ideal $[\zeta]$ formed from $\zeta \in Z$ by $\mathcal{F}(\zeta)$. For an element A in \mathcal{A} , we denote the spectrum of $A(\mathcal{F}(\zeta))$ in $\mathcal{A}(\mathcal{F}(\zeta))$ by $\text{Sp}A(\mathcal{F}(\zeta))$.

The next lemma is used repeatedly.

LEMMA 3.1. Let \mathcal{A} be a von Neumann algebra, let \mathcal{Z} be the center of \mathcal{A} , let $P \in (\mathcal{Z})$, let Z be the spectrum of \mathcal{Z} , and let \mathcal{F} be a central ideal of \mathcal{A} . If A is an element of \mathcal{A} such that $f_A(\zeta) = \|A(\mathcal{F}(\zeta))\|$ vanishes for every ζ in the support of P given by $\text{supp } P = \{\zeta \in Z \mid P^\wedge(\zeta) = 1\}$, then the element AP is in \mathcal{F} .

Proof. For every ζ in $\text{supp } P$ and $\varepsilon > 0$ there is a B_ζ in \mathcal{F} such that $\|(A - B_\zeta)(\zeta)\| < \varepsilon$. Hence there is a P_ζ in (\mathcal{Z}) with $P_\zeta^\wedge(\zeta) = 1$

such that $\|(A - B_\zeta)P_\zeta\| < \varepsilon$. Using the fact that $\text{supp } P$ is compact, we may find a set P_1, \dots, P_n of orthogonal projections in \mathcal{K} of sum P and a corresponding set B_1, \dots, B_n in \mathcal{J} such that

$$\|AP - \sum B_i P_i\| = \text{lub} \|(A - B_i)P_i\| < \varepsilon.$$

Since \mathcal{J} is closed, the element AP is in \mathcal{J} .

We characterize those ideals \mathcal{J} for which $\zeta \rightarrow \|A(\mathcal{J}(\zeta))\|$ is continuous on Z for every A in \mathcal{A} .

THEOREM 3.2. *Let \mathcal{A} be a von Neumann algebra, let \mathcal{K} be the center of \mathcal{A} , and let Z be the spectrum of \mathcal{K} . Let \mathcal{J} be an ideal of \mathcal{A} . In order that $f_A(\zeta) = \|A(\mathcal{J}(\zeta))\|$ be a continuous function on Z for every A in \mathcal{A} , a necessary and sufficient condition is that \mathcal{J} be a central ideal of \mathcal{A} .*

Proof. The sufficiency follows by a proof that is virtually the same as the one we gave in the corollary of (a5) implies (a1) of [19].

Conversely, let f_A be continuous on Z for every A in \mathcal{A} . We show that \mathcal{J} is a central ideal. If $\{A_i | i \in I\}$ is a bounded subset of \mathcal{J} and if $\{P_i | i \in I\}$ is an orthogonal set in (\mathcal{K}) of sum 1, then we prove that $A = \sum A_i P_i$ is in \mathcal{J} . Indeed, the set $\cup_i \{\zeta \in Z | P_i^\wedge(\zeta) = 1\}$ is a dense set of Z on which $f_A(\zeta)$ vanishes since $f_A(\zeta) = \|A_i(\mathcal{J}(\zeta))\| = 0$ whenever $P_i^\wedge(\zeta) = 1$. By the continuity of f_A , we see that f_A vanishes on Z . Hence, the element A is in \mathcal{J} by Lemma 3.1.

REMARK 3.3. If \mathcal{J} is the strong radical of a properly infinite von Neumann algebra, then $\mathcal{J}(\zeta) = \mathcal{J} + [\zeta]$ is the unique maximal ideal which contains ζ [24 and 15, Proposition 2.3].

Now we prove the main result of this section. It is convenient to separate the following lemma.

LEMMA 3.4. *Let \mathcal{K} be a commutative von Neumann algebra and let X_1, \dots, X_n be closed sets which cover the spectrum Z of \mathcal{K} . Then there are orthogonal projections R_1, \dots, R_n in (\mathcal{K}) of sum 1 such that $\{\zeta \in Z | R_i^\wedge(\zeta) = 1\} \subset X_i$ for $1 \leq i \leq n$.*

Proof. Let $\{P_i | i \in I\}$ be a maximal set of nonzero mutually orthogonal projections such that for each $i \in I$ there is an $i(j)$ with $1 \leq i(j) \leq n$ so that $Y_i = \{\zeta \in Z | P_i^\wedge(\zeta) = 1\} \subset X_{i(j)}$. We obtain a contradiction if $P = 1 - \sum P_i \neq 0$. Indeed, the set $Y = \{\zeta \in Z | P^\wedge(\zeta) = 1\}$ is nonvoid and is covered by the closed sets $Y \cap X_1, \dots, Y \cap X_n$. By the Baire category theorem one of the set $Y \cap X_m$ has a nonvoid

interior in Y . This means that there is nonzero projection Q in \mathcal{X} such that $\{\zeta \in Z \mid Q^\wedge(\zeta) = 1\} \subset Y \cap X_m$. This contradicts the maximality of $\{P_i\}$. We must have that $\sum P_i = 1$. The remainder of the proof consists in adding the projections P_i . Let $I_j = \{i \in I \mid Y_i \subset X_j\}$ for $1 \leq j \leq n$ and let $R_j = \sum \{P_i \mid i \in I_j - \cup \{I_k \mid 0 \leq k \leq j - 1\}\}$ for $1 \leq j \leq n$. Here $I_0 = \emptyset$. Then it is clear that R_1, R_2, \dots, R_n satisfy the requirements of the lemma.

THEOREM 3.5. *Let \mathcal{A} be a von Neumann algebra with center \mathcal{Z} , let \mathcal{I} be a central ideal of \mathcal{A} , and let A be an element of \mathcal{A} . Let X_0 be a closed subset of the complex plane C such that the intersection $S(\zeta)$ of X_0 with the spectrum (resp. left-spectrum, right-spectrum, the intersection of the left-spectrum and the right-spectrum) of $A(\mathcal{I}(\zeta))$ is nonvoid for every ζ in the spectrum Z of \mathcal{Z} . Then there is an element A_0 in the center of \mathcal{A} such that $A_0^\wedge(\zeta) \in S(\zeta)$ for every ζ in Z .*

Proof. We first prove that there exists A_0 in \mathcal{Z} such that $A_0^\wedge(\zeta)$ is in the intersection $S(\zeta)$ of X_0 with the spectrum $\text{Sp } A(\mathcal{I}(\zeta))$ of $A(\mathcal{I}(\zeta))$ for every ζ in Z . Since $\text{Sp } A(\mathcal{I}(\zeta))$ is contained in $\text{Sp } A$, there is no loss of generality in assuming $X_0 \subset \text{Sp } (A)$. We prove the theorem by an approximation argument that involves decomposing the space Z .

For every compact set X in the complex plane, let $X(Z) = \{\zeta \in Z \mid X \cap S(\zeta) \neq \emptyset\}$. We show that $X(Z)$ is closed in C . Let $\{\zeta_i\}$ be a net in $X(Z)$ converging to ζ . Let $\alpha_i \in S(\zeta_i) \cap X$; by passing to a subnet, we may assume that $\{\alpha_i\}$ converges to $\alpha \in X \cap X_0$. Arguing by contradiction we show that $\alpha \in \text{Sp } A(\mathcal{I}(\zeta))$. If $\alpha \notin \text{Sp } A(\mathcal{I}(\zeta))$, then there is a $B \in \mathcal{A}$ with

$$\begin{aligned} \|(B(\alpha - A) - 1)(\mathcal{I}(\zeta))\| &= \\ \|((\alpha - A)B - 1)(\mathcal{I}(\alpha))\| &= 0. \end{aligned}$$

By Theorem 3.2, we see that there is a ζ_i and α_i such that

$$\|(B(\alpha_i - A) - 1)(\mathcal{I}(\zeta_i))\| < 1$$

and $\|((\alpha_i - A)B - 1)(\mathcal{I}(\zeta_i))\| < 1$. This means that $\alpha_i \notin \text{Sp } A(\mathcal{I}(\zeta_i))$ and this is contrary to assumption. So $\alpha \in X(Z)$ and $X(Z)$ is closed.

We now begin the approximation argument by decomposing Z into subsets on which we shall approximate A_0 . Suppose we have, for every m less than or equal to the natural number n , constructed sets of integers $I_m = \{1, 2, \dots, p_m\}$ such that for every s in $I_1 \times \dots \times I_m = I(m)$ there is a compact subset $X(s)$ of C of diameter $\leq 2^{-m}$ and a $P(s)$ in (\mathcal{Z}) which satisfies the following properties:

(1) For $s \in I(m)$, $\cup \{X(s; j) \mid j \in I_{m+1}\} = X(s)$ whenever $1 \leq m < n$ and

$$\cup \{X(j) | j \in I(1)\} = X_0 ;$$

(2) $\text{Supp } P(s) = \{\zeta \in Z | P(s)^\wedge(\zeta) = 1\} \subset X(s)(Z)$ for every $s \in I(m)$ ($1 \leq m \leq n$); and

(3) for $s \in I(m)$, $\{P(s; j) | j \in I_{m+1}\}$ is a set of orthogonal projections of sum $P(s)$ whenever $1 < m < n$ and $\{P(j) | j \in I(1)\}$ is a set of orthogonal projections of sum 1.

We shall construct a set $I_{n+1} = \{1, \dots, p_{n+1}\}$, compact sets $X(s) (s \in I(n+1) = I_1 \times \dots \times I_{n+1})$ of diameter $\leq 2^{-(n+1)}$ in the complex plane, and projections $P(s) (s \in I(n+1))$ in \mathcal{X} which satisfy (1), (2), (3). Indeed, let $\{Y_j | j \in I_{n+1}\}$ be compact sets of diameter $\leq 2^{-(n+1)}$ which cover X_0 . Let $X(s, j) = X(s) \cap Y_j$ for $s \in I(n)$ and $j \in I_{n+1}$. Then $\{X(s) | s \in I(m), m = 1, 2, \dots, n+1\}$ satisfies property (1). Now let s be fixed in $I(n)$; we have that $\cup \{X(s; j)(Z) | j \in I_{n+1}\} = X(s)(Z)$. Since $\text{supp } P(s)$ is contained in $X(s)(Z)$, the sets $X(s; j)(Z) \cap \text{supp } P(s) (j \in I_{n+1})$ form a closed cover of $\text{supp } P(s)$. By the Lemma 3.4, there are orthogonal central projections $P(s; j) (j \in I_{n+1})$ of sum $P(s)$ such that

$$\text{supp } P(s; j) \subset X(s; j)(Z)$$

for every $j \in I_{n+1}$. Thus $P(s) (s \in I(n+1))$ satisfies (2) and (3).

We continue by induction to construct $I(n)$, compact sets $X(s) (s \in I(n))$ of diameter $\leq 2^{-n}$, and central projections $P(s) (s \in I(n))$ satisfying (1), (2), and (3) for every $n = 1, 2, \dots$. We notice that if $X(s)$ is void then $P(s) = 0$.

We now construct the approximating elements. Let $n = 1, 2, \dots$ be fixed. If $s \in I(n)$, let $\alpha(s) \in X(s)$ if $X(s)$ is non-void, and $\alpha(s) = 0$ if $X(s)$ is void. Let $A_n = \sum \{\alpha(s)P(s) | s \in I(n)\}$. Then A_n is an element in the center of \mathcal{A} .

We show that $\{A_n\}$ is a Cauchy sequence. Indeed, we have that

$$\|A_n - A_{n+1}\| = \text{lub} \{ \|(A_n - A_{n+1})P(s)\| | s \in I(n+1) \}$$

since $\sum \{P(s) | s \in I(n+1)\} = \sum \{P(s) | s \in I(n)\} = \dots = 1$. However, if $s \in I(n+1)$ is of the form $s = (s'; j)$ with $s' \in I(n)$ and $j \in I_{n+1}$, then

$$\|(A_n - A_{n+1})P(s)\| = \|(\alpha(s') - \alpha(s))P(s)\| \leq 2^{-n}$$

since $\alpha(s) \in X(s')$ whenever $P(s) \neq 0$. Hence, we obtain that

$$\|A_n - A_{n+1}\| \leq 2^{-n}$$

for every $n = 1, 2, \dots$ and so $\{A_n\}$ is a Cauchy sequence in \mathcal{X} .

We show that the limit A_0 of $\{A_n\}$ satisfies the requirements of the Theorem 3.5. Let ζ be an arbitrary point in Z . Given $\epsilon > 0$ we show that there is $\alpha \in S(\zeta)$ such that $|A_0^\wedge(\zeta) - \alpha| \leq \epsilon$. Since $S(\zeta)$ is

closed and since $\varepsilon > 0$ is arbitrary, this will mean that $A_0^\wedge(\zeta) \in S(\zeta)$. Let m be a natural number with $2^{-m+2} < \varepsilon$. Then $|A_0^\wedge(\zeta) - A_m^\wedge(\zeta)| \leq \|A_0 - A_m\| < 2^{-1}\varepsilon$. There is an $s \in I(m)$ such that $P(s)^\wedge(\zeta) = 1$ since $\sum \{P(s) \mid s \in I(m)\} = 1$. By property (2), we have that ζ is in $X(s)(Z)$. So there is an element α in $X(s)$ such that $\alpha \in S(\zeta)$. However we have that $A_m^\wedge(\zeta) = \alpha(s) \in X(s)$, and so $|\alpha(s) - \alpha| < 2^{-m}$ since the diameter of $X(s)$ is less than 2^{-m} . Now we obtain that $|A_0^\wedge(\zeta) - \alpha| \leq \varepsilon$, and by the preceding remarks that $A_0^\wedge(\zeta) \in S(\zeta)$. This completes the proof for the case of $X_0 \cap \text{Sp } A(\mathcal{I}(\zeta)) \neq \emptyset$.

We may prove the existence of an element A_0 in \mathcal{X} such that $(A_0 - A)(\mathcal{I}(\zeta))$ is not left (resp. right, left nor right) invertible in $\mathcal{A}(\mathcal{I}(\zeta))$ and $A_0^\wedge(\zeta) \in X_0$ by the same proof we just gave for an invertible element by using the additional fact that, for any element B in a Banach algebra \mathcal{B} with identity, the set of all complex α such that $\alpha - B$ is not left (resp. right, left nor right) invertible is a non-void compact set ([26; 1.5.4 and 1.4.6]; also cf. [11; Theorem 3.1]).

The following definition is now meaningful.

DEFINITION 3.6. Let \mathcal{A} be a von Neumann algebra, let \mathcal{X} be the center of \mathcal{A} and let Z be the spectrum of \mathcal{X} . Then the *essential central spectrum* $\mathcal{X} - \text{Sp}_{\mathcal{I}} A$ of an element A in \mathcal{A} with respect to the central ideal \mathcal{I} is the set of all A_0 in \mathcal{X} such that $A_0^\wedge(\zeta) \in \text{Sp } A(\mathcal{I}(\zeta))$ for every $\zeta \in Z$. The left-essential (resp. right-essential) central spectral $\mathcal{X} - \text{Sp}_{\mathcal{I}}^e A$ (resp. $\mathcal{X} - \text{Sp}_{\mathcal{I}}^r A$) of A with respect to \mathcal{I} is defined in a similar manner. The intersection $\mathcal{X} - \text{Sp}_{\mathcal{I}}^b A = (\mathcal{X} - \text{Sp}_{\mathcal{I}}^e A) \cap (\mathcal{X} - \text{Sp}_{\mathcal{I}}^r A)$ is called the two-sided essential central spectrum of A with respect to \mathcal{I} .

REMARK 3.7. All sets defined in Definition 3.6 are non-void (Theorem 3.5).

REMARK 3.8. For every $A_0 \in \mathcal{X} - \text{Sp}_{\mathcal{I}} A$, we have that $A_0(1 - P_{\mathcal{I}}) = 0$. Since $(\mathcal{X} - \text{Sp}_{\mathcal{I}}^e A) \cup (\mathcal{X} - \text{Sp}_{\mathcal{I}}^r A) \subset \mathcal{X} - \text{Sp}_{\mathcal{I}} A$, the projection $1 - P_{\mathcal{I}}$ annihilates the other essential central spectrums.

We note that these definitions correspond to the usual ones if \mathcal{A} is the algebra of all bounded operators on a Hilbert space and \mathcal{I} is the ideal of compact operators.

PROPOSITION 3.9. *Let \mathcal{A} be a von Neumann algebra. Then the essential (resp. left-, right-essential) central spectrum of an element A in \mathcal{A} with respect to a central ideal \mathcal{I} is closed in the strong operator topology.*

Proof. Let $\{A_i\}$ be a net in the essential central spectrum of A with respect to \mathcal{I} which converges strongly to A_0 in the center \mathcal{Z} of \mathcal{A} . There is a net $\{P_n\}$ of mutually orthogonal central projections of sum 1 such that for each P_n there is a sequence $\{A_{i(n)}\}$ in $\cup_i \{A_i\}$ with $\lim A_{i(n)}P_n = A_0P_n$ (uniformly) [28; Corollary 13.1]. Since $A_{i(n)}^\wedge(\zeta) \in \text{Sp } A(\mathcal{I}(\zeta))$ for every ζ in the spectrum Z of \mathcal{Z} and since $\text{Sp } A(\mathcal{I}(\zeta))$ is closed, we have that $A_0^\wedge(\zeta) \in \text{Sp } A(\mathcal{I}(\zeta))$ for every ζ in the dense subset $X = \cup_n \{\zeta \in Z \mid P_n^\wedge(\zeta) = 1\}$ of Z [7]. Let $\{\zeta_i\}$ be a net in X which converges to ζ in Z . If $A_0^\wedge(\zeta) \notin \text{Sp } A(\mathcal{I}(\zeta))$, then there is a B in \mathcal{A} with

$$\|(B(A_0 - A) - 1)(\mathcal{I}(\zeta))\| = \|((A_0 - A)B - 1)(\mathcal{I}(\zeta))\| = 0.$$

This means that there is a ζ_i with

$$\|(B(A_0 - A) - 1)(\mathcal{I}(\zeta_i))\| < 1 \quad \text{and} \quad \|(A_0 - A)B - 1)(\mathcal{I}(\zeta_i))\| < 1$$

and thus that $A_0^\wedge(\zeta_i)$ is not in $\text{Sp } A(\mathcal{I}(\zeta_i))$. Hence, we must have that $A_0^\wedge(\zeta)$ is in $\text{Sp } A(\mathcal{I}(\zeta))$ for every ζ in the closure Z of X . This proves that $\mathcal{Z} - \text{Sp}_{\mathcal{I}} A$ is strongly closed.

The statements concerning the left- and right-essential central spectra are proved in an analogous fashion.

For future reference we note some simple facts in the following proposition.

PROPOSITION 3.10. *Let \mathcal{A} be a von Neumann algebra with center \mathcal{Z} on the Hilbert space H , let \mathcal{I} be a central ideal in \mathcal{A} , let P_1 and P_2 be orthogonal projections of sum 1 in \mathcal{Z} , and let A be an element of \mathcal{A} . Let \mathcal{A}_i be the von Neumann algebra $\mathcal{A}P_i$ with center $\mathcal{Z}_i = \mathcal{Z}P_i$ on the Hilbert space P_iH , let \mathcal{I}_i be the central ideal $\mathcal{I}P_i$ in \mathcal{A}_i , and let A_i be the element AP_i in \mathcal{A}_i for $i = 1, 2$. Then $\mathcal{Z} - \text{Sp}_{\mathcal{I}} A = \{B_1 + B_2 \mid B_i \in \mathcal{Z}_i - \text{Sp}_{\mathcal{I}_i} A_i, i = 1, 2\}$.*

REMARK. A similar statement holds for the left- and right-essential central spectrums.

Proof. This follows from the fact that the spectrum of \mathcal{Z}_i is $\{\zeta P_i \mid \zeta \in Z, P_i^\wedge(\zeta) = 1\}$, where Z is the spectrum of \mathcal{Z} , and thus that $[\zeta P_i]$ in $\mathcal{A}P_i$ is equal $[\zeta]P_i$.

We now restrict our attention to self-adjoint elements. We note that the essential central spectrum of a self-adjoint element consists of self-adjoint elements.

PROPOSITION 3.11. *Let \mathcal{A} be a von Neumann algebra, let \mathcal{Z} be*

the center of \mathcal{A} , and let A be a self-adjoint element of \mathcal{A} . Then there are elements C_u and C_l in the essential central spectrum $\mathcal{K} - \text{Sp}_{\mathcal{F}} A$ of A with respect to the central ideal \mathcal{F} such that $C_l \leq C \leq C_u$ for every C in $\mathcal{K} - \text{Sp}_{\mathcal{F}} A$.

Proof. The set $\mathcal{K} - \text{Sp}_{\mathcal{F}} A$ is a monotonely increasing net in \mathcal{K} . Indeed, if C and C' are in $\mathcal{K} - \text{Sp}_{\mathcal{F}} A$, then there is a $P \in (\mathcal{K})$ such that $\text{lub}\{C, C'\} = PC + (1 - P)C'$. Since $PC + (1 - P)C'$ is in $\mathcal{K} - \text{Sp}_{\mathcal{F}} A$ (by 3.10), the set $\mathcal{K} - \text{Sp}_{\mathcal{F}} A$ is monotonely increasing. Then the least upper bound C_u of $\mathcal{K} - \text{Sp}_{\mathcal{F}} A$ is the strong limit of elements in $\mathcal{K} - \text{Sp}_{\mathcal{F}} A$ and so C_u is in the essential central spectrum of A with respect to \mathcal{F} (Proposition 3.8).

In an analogous manner, we may show that $\mathcal{K} - \text{Sp}_{\mathcal{F}} A$ is monotonely decreasing and thus we may find a greatest lower bound C_l for $\mathcal{K} - \text{Sp}_{\mathcal{F}} A$ in $\mathcal{K} - \text{Sp}_{\mathcal{F}} A$.

PROPOSITION 3.12. *Let \mathcal{A} be a von Neumann algebra with center \mathcal{K} , let \mathcal{F} be a central ideal of \mathcal{A} , and let A be a self-adjoint element of \mathcal{A} . Let C_u and C_l be the least upper bound and the greatest lower bound of the essential central spectrum of A with respect to \mathcal{F} , respectively. Then $C_u^\wedge(\zeta) = \text{lub Sp } A(\mathcal{F}(\zeta))$ and $C_l^\wedge(\zeta) = \text{glb Sp } A(\mathcal{F}(\zeta))$ for every ζ in the spectrum Z of \mathcal{K} .*

Proof. Since $C_u^\wedge(\zeta) \in \text{Sp } A(\mathcal{F}(\zeta))$ for every $\zeta \in Z$, we have that $C_u^\wedge(\zeta) \leq \alpha_\zeta = \text{lub Sp } A(\mathcal{F}(\zeta))$, for every $\zeta \in Z$. Conversely, we obtain a contradiction if we assume that $\alpha_\zeta - C_u^\wedge(\zeta) = 2\varepsilon > 0$ for some $\zeta \in Z$. Indeed, let E be the spectral projection of $A - C_u$ corresponding to the interval $[\varepsilon, +\infty)$. Because $(A - C_u)(1 - E) \leq \varepsilon(1 - E)$, we have that $E(\mathcal{F}(\zeta)) \neq 0$. Hence, there is a $P \in (\mathcal{K})$ such that $P^\wedge(\zeta) = 1$ and $E(\mathcal{F}(\zeta')) \neq 0$ for all ζ' in $\text{supp } P = \{\zeta' \in Z \mid P^\wedge(\zeta') = 1\}$ (Theorem 3.2). Since $\varepsilon E \leq (A - C_u)E$, we have that $\text{Sp } (A - C_u)(\mathcal{F}(\zeta')) \cap [\varepsilon, +\infty) \neq \emptyset$ for all $\zeta' \in \text{supp } P$. Reducing to the algebra $\mathcal{A}P$ with center $\mathcal{K}P$, we see that $S(\zeta') = \text{Sp } (A - C_u)P((\mathcal{F}P)(\zeta')) \cap [\varepsilon, +\infty)$ is non-void for every ζ' in the spectrum X of $\mathcal{K}P$. Because $\mathcal{F}P$ is a central ideal in $\mathcal{A}P$, we may find a B in \mathcal{K} such that $(BP)^\wedge(\zeta') \in S(\zeta')$ for every $\zeta' \in X$ (Theorem 3.5). If D is an arbitrary element in $\mathcal{K} - \text{Sp}_{\mathcal{F}}(A - C_u)$, then $PB + (1 - P)D = B'$ is in $\mathcal{K} - \text{Sp}_{\mathcal{F}}(A - C_u)$ (Proposition 3.10), and consequently, the element $B'' = B' + C_u$ is in $\mathcal{K} - \text{Sp}_{\mathcal{F}} A$. But we have that $B''P + C_u(1 - P)$ is in $\mathcal{K} - \text{Sp}_{\mathcal{F}} A$ (Proposition 3.10) and that $B''P + C_u(1 - P) \geq C_u + \varepsilon P$. This contradicts the definition of C_u . Thus we must have that $C_u^\wedge(\zeta) = \text{lub Sp } A(\mathcal{F}(\zeta))$ for every $\zeta \in Z$.

A similar proof holds for C_l .

The following proposition shows that if A_0 is in the essential central spectrum of A with respect to \mathcal{I} , then $A_0 - A$ is small on a large subspace with respect to \mathcal{I} .

PROPOSITION 3.13. *Let \mathcal{A} be a von Neumann algebra, let \mathcal{I} be a central ideal of \mathcal{A} , let A be a self-adjoint element of \mathcal{A} , let A_0 be an element of the essential central spectrum of A with respect to \mathcal{I} , and let $\varepsilon > 0$. If F is the spectral projection of $A_0 - A$ corresponding to the interval $[-\varepsilon, \varepsilon]$, then $FP \in \mathcal{I}$ for some central projection P implies $P \in \mathcal{I}$ (i.e. $P \leq 1 - P_{\mathcal{I}}$).*

Proof. Let P be a central projection with $PF \in \mathcal{I}$. We show $P \in \mathcal{I}$. We may assume that $P \neq 0$. Let ζ be a point in the spectrum of the center of \mathcal{A} such that $P^\wedge(\zeta) = 1$. We have that

$$(A_0 - A)(\mathcal{I}(\zeta)) = (A_0 - A)(1 - F)(\mathcal{I}(\zeta)).$$

If $(1 - F)(\mathcal{I}(\zeta)) \neq 0$, then $(A_0 - A)(\mathcal{I}(\zeta))$ is invertible in $\mathcal{A}(\mathcal{I}(\zeta))$. Since this is not possible, we have that $1(\mathcal{I}(\zeta)) = 0$. This means that $P \in \mathcal{I}(\zeta)$. Since ζ with the property $P^\wedge(\zeta) = 1$ is arbitrary in the last relation, we have that $P \in \mathcal{I}$ by Lemma 3.1.

We now characterize the essential central spectrum of a self-adjoint element in terms of the canonical form of a central ideal (cf. Remark 2.6ff. and Definition 2.8).

PROPOSITION 3.14. *Let \mathcal{A} be a von Neumann algebra with no finite type I direct summand, let \mathcal{I} be a central ideal of \mathcal{A} , and let A be a self-adjoint element in \mathcal{A} . An element A_0 is in the essential central spectrum of A with respect to \mathcal{I} if and only if there is an orthogonal sequence $\{E_n\}$ of projections in \mathcal{A} of dimension greater than $\dim \mathcal{I}$ such that $AE_n(\mathcal{I}) = E_nA(\mathcal{I})$ and $\|(A_0 - A)E_n(\mathcal{I})\| \leq n^{-1}$ for every $n = 1, 2, \dots$ and $A_0 = A_0P_{\mathcal{I}}$.*

Proof. Let A_0 be in the essential central spectrum of A with respect to \mathcal{I} . There is no loss of generality in the assumption that $P_{\mathcal{I}} = 1$ and that $A_0 = 0$. [9; III, 5, Problem 7]. Let F_n be the spectral projection of A corresponding to the interval $[-n^{-1}, n^{-1}]$ for $n = 1, 2, \dots$; then we have that $\{F_n\}$ is a monotonely decreasing sequence of projections such that $\dim F_n > \dim \mathcal{I}$ (Propositions 3.13 and 2.9).

Let \mathcal{I} be represented in the form $\mathcal{I} = \mathcal{I}_S(E)$ (2.4-2.6). Now let $\{P_i\}$ be a maximal set of mutually orthogonal central projections such that for each i there is a natural number $j(i)$ with $(F_k - F_{k+1})P_i \in \mathcal{I}$ whenever $k \geq j(i)$. This means that $AF_{j(i)}P_i \in \mathcal{I}$ since

$$\|AF_{j(i)}P_i(\mathcal{S})\| = \|AF_kP_i(\mathcal{S})\| \leq k^{-1}$$

for arbitrary $k \geq j(i)$. Hence, setting $F' = \sum F_{j(i)}P_i$ and $P = \sum P_i$, we obtain a projection F' of central support P such that $AF' \in \mathcal{S}$ and $EP < F'$ (Proposition 2.9). Since \mathcal{A} has no finite type I direct summands, we may find a sequence $\{G'_n\}$ of orthogonal projections of sum FP such that the central support of G'_n is P and such that $EP < G'_n$. Indeed, there is a central projection R majorized by P such that FR is properly infinite and $F(P - R)$ is finite. In the first instance FR is the sum of a sequence of mutually orthogonal projections each equivalent to FR [9; III, 8, Corollary 2]. In the second instance, we have that $E(P - R) = 0$. Indeed, E is a properly infinite projection and $E(P - R)$ is finite since $E(P - R) < F(P - R)$. Now $F(P - R)$ may be written as the sum of a sequence of orthogonal projections of central support $P - R$ [9; III, 1, Theorem 1, Corollary 3].

Now, for every nonzero central projection Q majorized by $P' = 1 - P$ and for every $n = 1, 2, \dots$, there is a nonzero central projection Q' with $Q' \leq Q$ and a natural number $m \geq n$ such that $(F_m - F_{m+1})Q'$ has central support Q' and $EQ' < (F_m - F_{m+1})Q'$ (Proposition 2.9). By induction we may find sets $\{G_{ni} | i \in I_n\}$ ($1 \leq n < \infty$) of projections with the following properties:

- (1) if Q_{ni} denotes the central support of G_{ni} , then $EQ_{ni} < G_{ni}Q_{ni}$ ($i \in I_n; n = 1, 2, \dots$);
- (2) $\{Q_{ni} | i \in I_n\}$ is a mutually orthogonal set of sum P' ;
- (3) for each $i \in I_n$ there is a natural number $s = s(i) \geq n$ with $G_{ni} = (F_s - F_{s+1})Q_{ni}$; and
- (4) if $i \in I_m, j \in I_n$, and $Q_{mi}Q_{nj} \neq 0$ then $s(i) < s(j)$ whenever $m < n$. Here I_n is a countable indexing set with $I_m \cap I_n \neq \emptyset$ for $m \neq n$.

Indeed, at the $(n + 1)$ -st stage of the induction we work in algebras of the form $\mathcal{A}Q_{i_1} \cdots Q_{i_n}$ ($i_j \in I_j$) and then sum the appropriate pieces together by summing over those pieces corresponding to the same $s(i)$. Setting $G''_n = \sum \{G_{ni} | i \in I_n\}$, we obtain sequence of mutually orthogonal projections of central support P' such that $EP' < G''_nP'$, $AG''_n = G''_nA$, and $\|AG''_n\| \leq n^{-1}$ for every $n = 1, 2, \dots$. Setting $E_n = G'_n + G''_n$ for $n = 1, 2, \dots$, we obtain a sequence $\{E_n\}$ of mutually orthogonal projections of central support 1 such that

$$E < E_n, AE_n(\mathcal{S}) = E_nA(\mathcal{S}), \text{ and } \|AE_n(\mathcal{S})\| \leq n^{-1}$$

for every n .

Conversely, let $\{E_n\}$ be a sequence of (not necessarily orthogonal) projections which satisfy the conditions of the proposition for the central element A_0 . Suppose there is a B in \mathcal{A} with $B(A_0 - A)(\mathcal{S}(\zeta)) = 1 \neq 0$ for some ζ in the spectrum of the center. Then we have

that

$$\|E_n(\mathcal{S}(\zeta))\| = \|B(A_0 - A)E_n(\mathcal{S}(\zeta))\| \leq n^{-1}\|B\|,$$

for every $n = 1, 2, \dots$ implies $\|E_n(\mathcal{S}(\zeta))\| = 0$ for all sufficiently large n . However, this means that $\|E_n(\mathcal{S}(\zeta'))\| = 0$ for all ζ' in a neighborhood of ζ since $\zeta' \rightarrow \|E_n(\mathcal{S}(\zeta'))\|$ is a continuous function of the spectrum of the center into $\{0, 1\}$ (Theorem 3.2). So there is a projection P in the center with $P^\wedge(\zeta) = 1$ such that $E_n P \in \mathcal{S}$ (Lemma 3.1). But this contradicts the hypothesis that $\dim E_n > \dim \mathcal{S}$ (Proposition 2.9). Consequently, the element A_0 is in the essential spectrum of A with respect to \mathcal{S} .

COROLLARY 3.15. *Let \mathcal{A} be a von Neumann algebra with no finite type I direct summands and let \mathcal{S} be a central ideal of \mathcal{A} . Then the essential central spectrum with respect to \mathcal{S} of a self-adjoint element A contains A_0 if and only if there is a sequence $\{E_n\}$ of mutually orthogonal projections of dimension greater than $\dim \mathcal{S}$ such that $\|(A_0 - A)E_n\| \leq n^{-1}$ for every $n = 1, 2, \dots$ and $A_0 = A_0 P_{\mathcal{S}}$.*

Proof. There is no loss of generality in the assumption that $A_0 = 0$ since every element in the essential central spectrum of A is self-adjoint. Then there are orthogonal projections $\{F_n\}$ such that $\dim F_n > \dim \mathcal{S}$, $AF_n(\mathcal{S}) = F_n A(\mathcal{S})$ and $\|AF_n(\mathcal{S})\| < (2n)^{-1}$ for every $n = 1, 2, \dots$ (Proposition 3.14). For every n there is a $B_n \in \mathcal{S}$ with $\|AF_n - B_n F_n\| < (2n)^{-1}$. There is a projection $G_n \in \mathcal{S}$ such that $G_n \leq F_n$ and $\|B_n F_n(1 - G_n)\| \leq (2n)^{-1}$. Let $E_n = F_n - G_n$. If Q is a central projection with $QE_n \in \mathcal{S}$, then $QF_n \in \mathcal{S}$ and $QP_{\mathcal{S}} = 0$ and so $\dim E_n > \dim \mathcal{S}$ (Proposition 2.9.). But we have that

$$\|AE_n\| \leq \|(A - B_n)E_n\| + \|B_n E_n\| \leq n^{-1}.$$

Thus $\{E_n\}$ is the required sequence.

The converse is derived from Proposition 3.14 since $\|B(\mathcal{S})\| \leq \|B\|$ for every $B \in \mathcal{A}$.

COROLLARY 3.16. *Let \mathcal{A} be a von Neumann algebra with no finite type I direct summand and let \mathcal{S} be a central ideal in \mathcal{A} . If the left-essential (resp. right-essential) central spectrum of an element A in \mathcal{A} contains A_0 , then there is a sequence $\{E_n\}$ of orthogonal projections in \mathcal{A} such that $\dim E_n > \dim \mathcal{S}$ and $\|(A_0 - A)E_n\| \leq n^{-1}$ (resp. $\|(A_0 - A)^* E_n\| \leq n^{-1}$) for every $n = 1, 2, \dots$.*

Proof. Since the essential central spectrum of $(A_0 - A)^*(A_0 - A)$ contains 0, the Corollary 3.15 can be applied.

REMARK 3.17. If \mathcal{A} is a finite type I algebra, \mathcal{I} is a central ideal, $A \in \mathcal{A}$, and A_0 in the essential central spectrum of A with respect to \mathcal{I} , then D. Deekard and C. Pearcy [6] showed that there is an abelian projection E of central support $P_{\mathcal{I}}$ in \mathcal{A} with $(A_0 - A)E = 0$.

4. The essential central range. Let \mathcal{A} be a von Neumann algebra with center \mathcal{Z} . Then \mathcal{A} may be considered as a module over \mathcal{Z} . Let \mathcal{A}^{\sim} be the \mathcal{Z} -module of all bounded module homomorphisms of \mathcal{A} into \mathcal{Z} and let $\mathcal{A}^{\sim+}$ be the set of all elements of \mathcal{A}^{\sim} which map \mathcal{A}^+ into \mathcal{Z}^+ . For a central ideal \mathcal{I} of \mathcal{A} , let $E_a(\mathcal{I}) = \{\phi \in \mathcal{A}^{\sim+} \mid \phi(\mathcal{I}) = (0) \text{ and } \phi(P_{\mathcal{I}}) = P_{\mathcal{I}}\}$. Here $P_{\mathcal{I}}$ is the orthogonal complement of the largest central projection in \mathcal{I} . We notice that $E_a(\mathcal{I})$ is the set of all states (i.e. elements ϕ of $\mathcal{A}^{\sim+}$ with $\phi(1) = 1$) of \mathcal{A}^{\sim} which vanish on \mathcal{I} whenever $P_{\mathcal{I}} = 1$, or equivalently, if $\mathcal{I} = \mathcal{I}_P(E)$ (Remark 2.6), whenever the central support of E is equal to P (Corollary 2.7). In particular, the set $E_a(\mathcal{I})$ is equal to the set of all states which vanish on \mathcal{I} whenever \mathcal{I} is the ideal generated by the set of all finite projections or \mathcal{I} is the strong radical of a properly infinite von Neumann algebra (Examples 2.11 and 2.12). It is clear that $E_a(\mathcal{I})$ is compact in the topology of pointwise convergence on \mathcal{A} where \mathcal{Z} is taken with the weak topology, i.e., in the $\sigma_w(\mathcal{A}^{\sim}, \mathcal{A})$ -topology of \mathcal{A}^{\sim} . If $\{\phi_i \mid i \in I\}$ is any subset of $E_a(\mathcal{I})$ and $\{P_i \mid i \in I\}$ is a set of orthogonal central projections of sum 1, then $\phi(A) = \sum P_i \phi_i(A)$ defines an element ϕ in $E_a(\mathcal{I})$. Furthermore, we see that $E_a(\mathcal{I})$ is *central-convex* in the sense that $C\phi_1 + (1 - C)\phi_2$ is in $E_a(\mathcal{I})$ for every ϕ_1 and ϕ_2 in $E_a(\mathcal{I})$ and C in \mathcal{Z} with $0 \leq C \leq 1$.

DEFINITION 4.1. Let \mathcal{A} be a von Neumann algebra, let \mathcal{I} be a central ideal of \mathcal{A} , and let A be an element of \mathcal{A} . The set $\mathcal{K}_{\mathcal{I}}(A) = \{\phi(A) \mid \phi \in E_a(\mathcal{I})\}$ will be called the essential central range of A with respect to \mathcal{I} . We notice that $\mathcal{K}_{\mathcal{I}}(A)$ is a central-convex, weakly compact (and consequently uniformly closed) subset of the sphere in the center of \mathcal{A} of radius $\|A\|$ about the origin.

PROPOSITION 4.2. Let \mathcal{A} be a von Neumann algebra, let \mathcal{I} be a central ideal of \mathcal{A} , and let A be an element of \mathcal{A} . Then for every ζ in the spectrum of the center of \mathcal{A} , the set $\mathcal{K}_{\mathcal{I}}(A)(\zeta) = \{B^{\wedge}(\zeta) \mid B \in \mathcal{K}_{\mathcal{I}}(A)\}$ is a compact set of complex numbers.

Proof. Since $\mathcal{K}_{\mathcal{I}}(A)(\zeta)$ is bounded, it is sufficient to show that $\mathcal{K}_{\mathcal{I}}(A)(\zeta)$ is closed. If α is the limit of a sequence $\{\phi_n(A)^{\wedge}(\zeta)\}$ where $\phi_n \in E_a(\mathcal{I})$ for every $n = 1, 2, \dots$, we show that $\alpha \in \mathcal{K}_{\mathcal{I}}(A)(\zeta)$. There is no loss of generality in assuming that $\alpha = 0$. We may assume

that $|\phi_n(A)^\wedge(\zeta)| < n^{-1}$ for every $n = 1, 2, \dots$. There is a sequence $\{P_n\}$ of central projections with $\|\phi_n(A)P_n\| \leq n^{-1}$, and $P_n^\wedge(\zeta) = 1$ for every $n = 1, 2, \dots$. Let $Q_0 = \text{glb}\{P_1 \cdots P_n \mid n = 1, 2, \dots\}$ and let $Q_i = P_1(1 - P_2)$, $Q_2 = P_1P_2(1 - P_3)$, \dots ; then $\{Q_i \mid i = 0, 1, 2, \dots\}$ is a sequence of orthogonal central projections of sum P_1 . The homomorphism

$$\psi_n = (1 - P_1)\phi_1 + Q_0\phi_n + \sum \{Q_i\phi_i \mid i = 1, 2, \dots\}$$

is an element of $E_a(\mathcal{F})$ and so

$$A_0 = \lim \psi_n(A) = (1 - P_1)\phi_1(A) + \sum \{Q_i\phi_i(A) \mid i = 1, 2, \dots\}$$

is in $\mathcal{K}(A)$. Since $(1 - P_1)^\wedge(\zeta) = 0$ and $Q_i^\wedge(\zeta) = 0$ for all $i \geq 1$, either $Q_0^\wedge(\zeta) = 1$ or $\sum \{Q_i \mid i \geq n\}^\wedge(\zeta) = 1$ for all $n = 1, 2, \dots$. In either case $A_0^\wedge(\zeta) = 0$ since $\|\sum \{Q_i \mid i = 0, n, n + 1, \dots\}A_0\| \leq n^{-1}$. This means that $0 \in \mathcal{K}_\zeta(A)(\zeta)$.

We need the following lemma. Its proof is a simple reworking of [19; proof of corollary to (a5) implies (a1)].

Lemma 4.3. *Let \mathcal{A} be a von Neumann algebra, let \mathcal{F} be a central ideal of \mathcal{A} , and let E be a projection in \mathcal{A} . There is positive module homomorphism of the module \mathcal{A} into its center which vanishes on \mathcal{F} and satisfies the relation $\phi(1) = \phi(E) = 1 - Q$ where Q is the largest central projection of \mathcal{A} such that $EQ \in \mathcal{F}$.*

THEOREM 4.4. *Let \mathcal{A} be a von Neumann algebra. The essential central range of a self-adjoint element A of \mathcal{A} with respect to a central ideal \mathcal{F} is the smallest central-convex subset of \mathcal{A} which contains the essential central spectrum of A with respect to \mathcal{F} .*

Proof. Let \mathcal{Z} be the center of \mathcal{A} , let ζ be in the spectrum of \mathcal{Z} , and let ϕ be an element of $E_a(\mathcal{F})$. Let ϕ_ζ be the bounded linear functional on \mathcal{A} defined by $\phi_\zeta(B) = \phi(B)^\wedge(\zeta)$ for all $B \in \mathcal{A}$. If B_1, \dots, B_n are in \mathcal{A} and C_1, \dots, C_n are in ζ , then

$$\phi_\zeta(\sum B_i C_i) = \sum C_i^\wedge(\zeta)\phi_\zeta(B_i) = 0.$$

This proves that ϕ_ζ vanishes on a dense subset of $[\zeta]$ and so vanishes on $[\zeta]$. Hence ϕ_ζ vanishes on $\mathcal{F}(\zeta)$. Now let $C_l = \text{glb } \mathcal{Z} - \text{Sp}_\mathcal{F} A$ and $C_u = \text{lub } \mathcal{Z} - \text{Sp}_\mathcal{F} A$. We have that $C_l^\wedge(\zeta) \leq A(\mathcal{F}(\zeta)) \leq C_u^\wedge(\zeta)$ in $\mathcal{A}(\mathcal{F}(\zeta))$. (Proposition 3.12). This means that

$$\phi(C_l)^\wedge(\zeta) = C_l^\wedge(\zeta) \leq \phi(A)^\wedge(\zeta) \leq C_u^\wedge(\zeta) = \phi(C_u)^\wedge(\zeta)$$

for all ζ with $P_\zeta^\wedge(\zeta) = 1$. Consequently, we have that $C_l = C_l P_\mathcal{F} \leq \phi(A) \leq C_u P_\mathcal{F} = C_u$. So we may find a C in \mathcal{Z} with $0 \leq C \leq 1$ such

that $CC_i + (1 - C)C_u = \phi(A)$. Hence, the smallest central convex set containing $\mathcal{K} - \text{Sp}_{\mathcal{S}}A$ contains $\mathcal{K}_{\mathcal{S}}(A)$.

Conversely, to show the opposite relation we simply must show that C_i and C_u are in $\mathcal{K}_{\mathcal{S}}(A)$. We work with C_u . Given $\varepsilon > 0$, there is a projection E in \mathcal{A} such that E commutes with A , $\|(C_u - A)E\| < \varepsilon$, and if EP is in \mathcal{S} for a central projection P then P is in \mathcal{S} (Proposition 3.13). There is a ϕ in $E_a(\mathcal{S})$ such that $\phi(E) = P_{\mathcal{S}}$ (Lemma 4.3). From the Cauchy-Schwarz inequality for elements of $A^{\sim+}$, we obtain

$$\begin{aligned} \|\phi(A) - C_u\| &= \|\phi(A - C_u)\| = \\ &\|\phi((A - C_u)E)\| + \|\phi((A - C_u)(1 - E))\| \\ &\leq \varepsilon + \|A - C_u\| \|\phi(1 - E)\| = \varepsilon. \end{aligned}$$

Because $\mathcal{K}_{\mathcal{S}}(A)$ is uniformly closed and because $\varepsilon > 0$ is arbitrary, we have that $C_u \in \mathcal{K}_{\mathcal{S}}(A)$. By a similar argument $C_i \in \mathcal{K}_{\mathcal{S}}(A)$.

COROLLARY 4.5. *Let \mathcal{A} be a von Neumann algebra. The essential central range of an element A in \mathcal{A} with respect to a central ideal \mathcal{S} is equal to a set $\{A_0\}$ if and only if $A_0P_{\mathcal{S}} = A_0$ and $A - A_0 \in \mathcal{S}$.*

Proof. First let the essential central range $\mathcal{K}_{\mathcal{S}}(A)$ of A be equal to A_0 . Then $\phi(A) = A_0$ for every $\phi \in E_a(\mathcal{S})$. Hence $\phi(A + A^*) = A_0 + A_0^*$ for every $\phi \in E_a(\mathcal{S})$. This means that the essential central spectrum of $A + A^* - (A_0 + A_0^*)$ with respect to the ideal \mathcal{S} is equal to $\{0\}$ (Theorem 4.4). Hence $A + A^* - (A_0 + A_0^*) \in \mathcal{S}$ (Proposition 3.12 and Lemma 3.1). Similarly we find that $(A - A^*) - (A_0 - A_0^*) \in \mathcal{S}$. Consequently, we have that $A - A_0 \in \mathcal{S}$.

The converse is obvious.

The following remarks lead to a characterization of the essential central range. This reduces to the known characterization of the essential numerical range of the algebra of all bounded operators on a separable Hilbert space [11; 5.1]. Let \mathcal{A} be a von Neumann algebra on the Hilbert space H and let \mathcal{K} be the center of \mathcal{A} . Let E be an abelian projection with central support P in the commutant \mathcal{K}' of \mathcal{K} [cf. 9; I, §7]. For every $A \in \mathcal{A}$, there is a unique $\tau_E(A)$ in $\mathcal{K}P$ with $EAE = \tau_E(A)E$. Then $A \rightarrow \tau_E(A)$ defines an element in $\mathcal{A}^{\sim+}$ with $\tau_E(1) = P$. For every projection P in \mathcal{K} let $V_P(\mathcal{A}) = \{\tau_E \in \mathcal{A}^{\sim+} | E \text{ is an abelian projection in } \mathcal{K}' \text{ of central support } P\}$; for every $A \in \mathcal{A}$ let $W_P(A) = \text{uniform closure } \{\phi(A) | \phi \in V_P(\mathcal{A})\}$.

We now need a version of the Toeplitz-Hausdorff Theorem.

LEMMA 4.6. *Let \mathcal{A} be a von Neumann algebra. Then, for*

every $A \in \mathcal{A}$ and central projection P , the set $\{\phi(A) \mid \phi \in V_P(\mathcal{A})\}$ is central-convex.

Proof. There is no loss of generality in the assumption that $P = 1$. Let E_1 and E_2 be maximal abelian projections (i.e. abelian projections with central support 1) in the commutant \mathcal{K}' of the center \mathcal{K} of \mathcal{A} and let $C \in \mathcal{K}$ with $0 \leq C \leq 1$. Setting $E = \text{lub}\{E_1, E_2\}$, we obtain a projection E such that the reduced algebra \mathcal{K}'_E is the product of homogeneous algebras of type I_n where $n \leq 2$. Indeed, we have that $\text{lub}\{E_1, E_2\} - E_1 \prec E_1$ and so $\text{lub}\{E_1, E_2\} - E_1$ is abelian. So there is no loss of generality in the assumption that $\mathcal{A} = \mathcal{K}'$ is homogeneous of degree 2 since the degree 1 case requires no further proof. Now we may write $\phi(B) = C\tau_{E_1}(B) + (1 - C)\tau_{E_2}(B)$ as $\phi(B) = A_1\tau_{F_1}(B) + A_2\tau_{F_2}(B)$ where F_1, F_2 are orthogonal maximal abelian projections of sum 1 and A_1, A_2 are elements in \mathcal{K}^+ with $A_1 + A_2 = 1$ [14; §4]. So we may assume that E_1 and E_2 are orthogonal of sum 1. Let $\tau_i = \tau_{E_i}$ ($i = 1, 2$). Since it is sufficient to find a maximal abelian projection E with $\tau_E(A - \tau_2(A)) = C\tau_1(A - \tau_2(A))$, we may assume that $\tau_2(A) = 0$. Now there is a sequence $\{P_n\}$ of orthogonal projections in \mathcal{K} such that $\tau_1(A)P_n$ is invertible in $\mathcal{K}P_n$ and $\tau_1(A)(1 - \sum P_n) = 0$. Because the sum of abelian projections with orthogonal central supports is again abelian, there is no loss of generality in the assumption that $\tau_1(A) = 1$.

The rest of this lemma is the classical Toeplitz-Hausdorff theorem. Let U be a partial isometry of \mathcal{A} with $U^*U = E_1$ and $UU^* = E_2$ and let $A = E_1 + A_1U + A_2U^*$, where $A_1, A_2 \in \mathcal{K}$. There is a unitary operator V in \mathcal{K} with $V|A_1 - A_2^*| = A_1 - A_2^*$. Let $T = V^*A_1 + VA_2$. There is a $D \in \mathcal{K}$ with $-1 \leq D \leq 1$ such that

$$D^2 + D(1 - D^2)^{1/2}T = C$$

[6]. Now, by direct calculation, we find that

$$E = D^2E_1 + VD(1 - D^2)^{1/2}U + V^*D(1 - D^2)^{1/2}U^* + (1 - D^2)E_2$$

is a projection in \mathcal{A} of central support 1 that vanishes on the range of $(1 - C^2)^{1/2}E_1 - VCE_2$. So E must be a maximal abelian projection. Finally, by another calculation, we obtain that $EAE = CE$.

Let \mathcal{A} be a von Neumann algebra with center \mathcal{K} . Let \mathcal{A} be considered as a \mathcal{K} -module and let \mathcal{A}_\sim be the \mathcal{K} -module of all σ -weakly continuous module homomorphisms of \mathcal{A} into \mathcal{K} . Let $\mathcal{A}_\sim^+ = \mathcal{A}^+ \cap \mathcal{A}_\sim$ be the set of all *normal* (i.e. positive σ -weakly continuous) module homomorphisms of \mathcal{A} into \mathcal{K} .

Now we can extend Lemma 4.6.

LEMMA 4.7. *Let \mathcal{A} be a von Neumann algebra, let P be a central projection of \mathcal{A} , and let $A \in \mathcal{A}$; then*

$$W_P(A) = \{\phi(A) \mid \phi \in \mathcal{A}^+, \phi(1) = P\}.$$

Proof. First let $\phi \in \mathcal{A}^+$ with $\phi(1) = P$. We show that $\phi(A) \in W_P(A)$. There is a monotonely decreasing sequence $\{A_n\}$ of positive elements in the center \mathcal{Z} of \mathcal{A} , and a sequence $\{E_n\}$ of orthogonal abelian projections in the commutant \mathcal{Z}' of \mathcal{Z} with central supports $\{P_n\}$ respectively such that $\lim A_n = 0$ (uniformly), $\sum A_n = P$ (strongly), $E_{n+1} \leq E_n$, $\text{supp } P_n^\wedge = \text{supp } A_n^\wedge$ ($n = 1, 2, \dots$), and $\phi(B) = \sum A_n \tau_{E_n}(B)$ (strongly) for all $B \in \mathcal{A}$ ([16; Theorem 2] and [14; §4]). There is a mutually orthogonal set $\{Q_i\}$ in (\mathcal{Z}) of sum P such that

$$\lim_m \sum \{A_n Q_i \mid 1 \leq n \leq m\} = P Q_i$$

uniformly (cf. [14, Theorem 4.1]). For each Q_i we may therefore find an m_i with $\|B_i Q_i\| \leq \varepsilon$, where $B_i = \sum \{A_n \mid n \geq m_i\}$ and where $\varepsilon > 0$ is a preassigned constant. Now there are abelian projections F_k ($1 \leq k \leq m_i = m$) of central support $P Q_i$ such that $E_k Q_i \leq F_k$. Since $\text{supp } P_k^\wedge = \text{supp } A_k^\wedge$, we have that $\phi_i = \sum \{A_k \tau_{F_k} \mid 1 \leq k \leq m - 1\} + B_i \tau_{F_m}$ is equal to $\sum \{A_k Q_i \tau_{E_k} \mid 1 \leq k \leq m - 1\} + B_i \tau_{F_m}$. Since $\sum \{A_k Q_i \mid 1 \leq k \leq m - 1\} + B_i Q_i = P Q_i$, there is an abelian projection G_i in \mathcal{Z}' of central support $P Q_i$ such that $\tau_{G_i}(A) = \phi_i(A)$ (Lemma 4.6). Notice that

$$\|(\phi_i(A) - \phi(A)) Q_i\| \leq \|B_i \tau_{F_m}(A)\| + \|\sum \{A_n \tau_{E_n}(A) \mid n \geq m\} Q_i\| \leq 2\varepsilon \|A\|.$$

Now $\sum G_i = G$ is an abelian projection of central support P and

$$\|\phi(A) - \tau_G(A)\| \leq \text{lub} \|(\phi(A) - \tau_{G_i}(A)) Q_i\| \leq 2\varepsilon \|A\|.$$

So $\phi(A) \in W_P(A)$ since $\varepsilon > 0$ is arbitrary and $W_P(A)$ is closed.

The converse relation is obvious since τ_E is a normal module homomorphism.

PROPOSITION 4.8. *Let \mathcal{A} be a von Neumann algebra. Then the essential central range of an element A in \mathcal{A} with respect to the central ideal \mathcal{I} is equal to $\cap \{W_P(A + B) \mid B \in \mathcal{I}\}$. Here $P = P_{\mathcal{I}}$.*

Proof. Let $\phi \in E_n(\mathcal{I})$. Let Q be the central projection in \mathcal{A} such that $\mathcal{A}Q$ is a discrete algebra and $\mathcal{A}(1 - Q)$ is a continuous algebra. There is a net $\{\phi'_n\}$ (resp. $\{\phi''_m\}$) of elements of $(\mathcal{A}Q)^\pm$ (resp. $(\mathcal{A}(1 - Q))^\pm$) with $\phi'_n(Q) = P_{\mathcal{I}}Q$ (resp. $\phi''_m(1 - Q) = P_{\mathcal{I}}(1 - Q)$) such that $\lim \phi'_n(BQ) = \phi(BQ)$ (resp. $\lim \phi''_m(B(1 - Q)) = \phi(B(1 - Q))$) uniformly for every $B \in \mathcal{A}$. This follows from Theorem 5.4 (resp. Theorem 5.1) of [17]. Then setting $\phi_{nm}(B) = \phi'_n(BQ) + \phi''_m(B(1 - Q))$, we obtain a

net $\{\phi_{nm}\}$ in \mathcal{A}^+ with $\phi_{nm}(1) = P_{\mathcal{F}}$ for all m, n and $\lim \phi_{nm}(B) = \phi(B)$ (uniformly) for all $B \in \mathcal{A}$. Let $B \in \mathcal{F}$ and let $\varepsilon > 0$; then there is a ϕ_{mn} with $\|\phi_{mn}(B)\| \leq \varepsilon$ and $\|\phi_{mn}(A) - \phi(A)\| \leq \varepsilon$ since $\phi(B) = 0$. Since $\varepsilon > 0$ is arbitrary and since $W_P(A + B)$ is closed, we have that $\phi(A) \in W_P(A + B)$ by Lemma 4.7. Since $B \in \mathcal{F}$ is arbitrary $\phi(A) \in \bigcap \{W_P(A + B) | B \in \mathcal{F}\}$. So $\mathcal{N}_{\mathcal{F}}(A) \subset \bigcap \{W_P(A + B) | B \in \mathcal{F}\}$.

We now prove that the opposite inclusion relation is true. First let A be self-adjoint. We show that $0 \in \bigcap \{W_P(A + B) | B = B^* \in \mathcal{F}\}$ implies that $0 \in \mathcal{N}_{\mathcal{F}}(A)$. Let \mathcal{Z} be the center of \mathcal{A} and let $C_u = \text{lub } \mathcal{Z} - \text{Sp}_{\mathcal{F}} A$. Suppose there is an $\alpha > 0$ and a nonzero projection Q in \mathcal{Z} with $Q \leq P$ and $C_u Q \leq -2\alpha Q$. We have that $(C_u - A)(\mathcal{F}(\zeta)) \geq 0$ for every ζ in the spectrum of \mathcal{Z} (Proposition 3.12). If f_+ (resp. $-f_-$) is the function that is identity on the real interval $[0, \infty)$ (resp. $(-\infty, 0]$) and 0 on the complement, we have that $f_-(C_u - A)$ is a self-adjoint element in \mathcal{F} (Lemma 3.1). However, by hypothesis there is an abelian projection E in \mathcal{Z}' of central support P with $\|\tau_E(f_-(C_u - A) - A)\| \leq \alpha$. On the other hand, we have that

$$Q\tau_E(f_-(C_u - A) - A) = Q\tau_E(f_+(C_u - A) - C_u) \geq 2\alpha Q .$$

This is a contradiction. Hence, we find that $C_u P \geq 0$. Since $1 - P \in \mathcal{F}$, we have that $C_u(1 - P) = 0$ and so $C_u \geq 0$ (cf. Remark 3.8). Similarly, we obtain $C_l = \text{glb } \mathcal{Z} - \text{Sp}_{\mathcal{F}}(A) \leq 0$ and finally that $0 \in \mathcal{N}_{\mathcal{F}}(A)$ (Theorem 4.4).

Now let A be an arbitrary element of \mathcal{A} with $0 \in \bigcap \{W_P(A + B) | B \in \mathcal{F}\}$. Let $\mathcal{S} = \{|B| = (B^*B)^{1/2} | B \in \mathcal{N}_{\mathcal{F}}(A)\}$. We note that \mathcal{S} is a monotonely decreasing net in \mathcal{Z}^+ . Indeed, let B and C be in $\mathcal{N}_{\mathcal{F}}(A)$. There is a central projection Q with $Q|B| + (1 - Q)|C| = \text{glb}\{|B|, |C|\}$. But the set $\mathcal{N}_{\mathcal{F}}(A)$ is central-convex and so $\mathcal{N}_{\mathcal{F}}(A)$ contains $D = QB + (1 - Q)C$. Thus, we have that $|D| = Q|B| + (1 - Q)|C|$ is in \mathcal{S} . Thus \mathcal{S} has a greatest lower bound B_0 in \mathcal{Z}^+ . We show $B_0 = 0$ by arguing by contradiction. Suppose there is a point ζ in the spectrum Z of \mathcal{Z} with $B_0^\wedge(\zeta) > 0$. Then we may assume that $B_0^\wedge(\zeta) = \text{glb}\{C^\wedge(\zeta) | C \in \mathcal{S}\}$ since $B_0^\wedge(\zeta) = \text{glb}\{C^\wedge(\zeta) | C \in \mathcal{S}\}$ holds on a dense open set of Z [7]. There is a $C \in \mathcal{N}_{\mathcal{F}}(A)$ such that $|C^\wedge(\zeta)| = B_0^\wedge(\zeta)$ (Proposition 4.2). Then we may find a unitary U in \mathcal{Z} such that $UC = |C|$. We have that $0 \in \bigcap \{W_P(UA + B) | B \in \mathcal{F}\}$ since $U(W_P(A + U^*B)) = W_P(UA + B)$ and that $B_0 = \text{glb}\{|B| | B \in \mathcal{N}_{\mathcal{F}}(UA)\}$ since $\mathcal{N}_{\mathcal{F}}(UA) = U\mathcal{N}_{\mathcal{F}}(A)$. Furthermore, we have that $|C| \in \mathcal{N}_{\mathcal{F}}(UA)$. Hence, there is no loss in generality in assuming that there is a $C \in \mathcal{N}_{\mathcal{F}}(A)$ with $C^\wedge(\zeta) = B_0^\wedge(\zeta)$. Now let $A_1 = (A + A^*)/2$ and $A_2 = (A - A^*)/2i$. We show that $0 \in \bigcap \{W_P(A_j + B) | B = B^* \in \mathcal{F}\}$ ($1 \leq j \leq 2$). In fact, given $\varepsilon > 0$ and $B = B^* \in \mathcal{F}$, there is an abelian projection E

with central support P in the commutant of \mathcal{K} such that $\|\tau_E(A+B)\| \leq \varepsilon$. Hence, we have that

$$\|\tau_E(A + A^* + 2B)\| = \|\tau_E(A + B) + \tau_E(A + B)^*\| \leq 2\varepsilon .$$

Similarly, we may find an abelian projection F of central support P such that $\|\tau_F(A - A^* + 2iB)\| \leq 2\varepsilon$. Now by the preceding paragraph we conclude that $0 \in \mathcal{K}_{\mathcal{S}}(A_j)$ ($1 \leq j \leq 2$). Let ϕ be an element of \mathcal{A}^+ with $\phi(1) = P$, $\phi(\mathcal{S}) = 0$, and $\phi(A_1) = 0$. However, every element of the form $\alpha\phi(A) + (1 - \alpha)C$ ($0 \leq \alpha \leq 1$) is in $\mathcal{K}_{\mathcal{S}}(A)$ and so there is at least one α with $0 \leq \alpha \leq 1$ such that

$$|\alpha\phi(A)^\wedge(\zeta) + (1 - \alpha)C^\wedge(\zeta)| < C^\wedge(\zeta) = B_0^\wedge(\zeta) .$$

Indeed $\phi(A)^\wedge(\zeta)$ is pure imaginary. This contradicts the choice of B_0 . Hence, we must have that $0 \in \mathcal{K}_{\mathcal{S}}(A)$.

PROPOSITION 4.9. *Let \mathcal{A} be a von Neumann algebra; then A_0 is in the essential central range of $A \in \mathcal{A}$ with respect to the central ideal \mathcal{S} if $A_0P_{\mathcal{S}} = A_0$ and if, given $\varepsilon > 0$, there is a projection E with $\dim E > \dim \mathcal{S}$ such that $\|E(A_0 - A)E\| \leq \varepsilon$. Conversely, if $A \in \mathcal{A}$ is self-adjoint and if A_0 is in the essential central range of A with respect to \mathcal{S} , then there is a projection E in \mathcal{A} with $\dim E > \dim \mathcal{S}$ such that $\|E(A_0 - A)E\| \leq \varepsilon$.*

Proof. The first statement follows from Lemma 4.3 and Proposition 2.9 since the essential central range $\mathcal{K}_{\mathcal{S}}(A)$ of A with respect to \mathcal{S} is uniformly closed.

Now let A be self-adjoint and let $A_0 \in \mathcal{K}_{\mathcal{S}}(A)$. There is no loss of generality in assuming at the outset that $A_0 = 0$ and that $P_{\mathcal{S}} = 1$. Let \mathcal{S} have the canonical form $\mathcal{S} = \mathcal{S}_P(F)$ (Remark 2.6). Let $\varepsilon > 0$ be given. Let $C_l = \text{glb } \mathcal{K} - \text{Sp}_{\mathcal{S}} A$ and let $C_u = \text{lub } \mathcal{K} - \text{Sp}_{\mathcal{S}} A$ where \mathcal{K} is the center of \mathcal{A} . Since $0 \in \mathcal{K}_{\mathcal{S}}(A)$, we have that $C_l \leq 0 \leq C_u$ (Theorem 4.4).

Now let R be the largest central projection such that $\mathcal{A}R$ is of type I and $\mathcal{S}R = 0$. Consequently, if G is a finite type I projection majorized by $1 - R$, then $G \in \mathcal{S}(1 - R)$ (Proposition 2.2). By Proposition 3.10 we may assume that either $R = 1$ or $1 - R = 1$.

First suppose that $R = 1$. We may assume that \mathcal{A} is equal the commutant of its center [9; I, 8, Theorem 1]. Then there are abelian projections E_1 and E_2 of central support 1 in \mathcal{A} such that

$$\|\tau_{E_1}(A) - C_l\| + \|\tau_{E_2}(A) - C_u\| \leq \varepsilon .$$

(Theorem 4.4 and Proposition 4.8). There is a C in \mathcal{K} with $0 \leq C \leq 1$ such that $CC_l + (1 - C)C_u = 0$, and there is an abelian projection E

of central support 1 in \mathcal{A} such that $\tau_E(A) = C\tau_{E_1}(A) + (1 - C)\tau_{E_2}(A)$ (Lemma 4.6). Thus, we obtain

$$\|EAE\| \leq \|C\| \|\tau_{E_1}(A) - C_l\| + \|1 - C\| \|\tau_{E_2}(A) - C_u\| \leq \varepsilon.$$

So we may assume that $1 - R = 1$. Because the closure of every open subset of the spectrum Z of \mathcal{Z} is open, we may find a sequence $\{P_n | n = 0, 1, 2, \dots\}$ of mutually orthogonal central projections of sum 1 such that

$$C_l P_n \leq -n^{-1} P_n < 0 < n^{-1} P_n \leq C_u P_n$$

for $n = 1, 2, \dots$, and $C_l C_u P_0 = 0$. We shall find projections E_n of central support P_n such that $FP_n < E_n$ and $\|E_n A E_n\| \leq 4\varepsilon$. Then we shall have that $E = \sum E_n$ has central support 1, $F < E$, and $\|EAE\| = \text{lub}_n \|E_n A E_n P_n\| \leq 4\varepsilon$ (cf. [9, III, §1]). Now, we have that $\mathcal{S}_n = \mathcal{S} P_n = \mathcal{S}_{P_n}(EP_n)$ is a representation of the central ideal \mathcal{S}_n of $\mathcal{A} P_n$ in canonical form. Since $C_l C_u P_0 = 0$, there is a P'_0 in $(\mathcal{Z} P_0)$ with $P'_0 C_l + (P_0 - P'_0) C_u = 0$ (Lemma 3.4). Thus, we see that $0 \in \mathcal{Z} P_0 - \text{Sp}_{\mathcal{S}_0}(AP_n)$ (Proposition 3.10) and so we may find the projection E_0 (Proposition 3.13). By reducing to an algebra $\mathcal{A} P_n$, we may assume that $C_l \leq -\alpha < 0 < \alpha \leq C_u$ (Proposition 3.10).

It is sufficient to show that every nonzero $Q \in (\mathcal{Z})$ majorizes a nonzero $R \in (\mathcal{Z})$ such that there is a $G \in (\mathcal{A})$ of central support R with $FR < G$ and $\|GAG\| \leq 4\varepsilon$. Then the usual maximality argument for the projections R may be employed to find the projection E_n . By making yet another reduction to a direct summand of \mathcal{A} , we may assume, without loss of generality, that there are natural numbers m, n , and p such that

$$\|mp^{-1} + C_l\| \leq p^{-1} \leq \varepsilon \quad \text{and} \quad \|np^{-1} - C_u\| \leq \varepsilon.$$

We now find n (resp. m) orthogonal projections F_i of dimension greater than $\dim \mathcal{S}$ such that $\|(C_l - A)F_i\| \leq \varepsilon$ (resp. $\|(C_u - A)F_i\| \leq \varepsilon$). We normally would apply Proposition 3.14, however it is necessary for the combined set of $m + n$ projections to be orthogonal and so the following additional argument is required. Let A_1 and A_2 be elements of \mathcal{A}^+ such that $A_1 - A_2 = A$ and $A_1 A_2 = 0$. For every $\zeta \in Z$, we have that $-C_l^\wedge(\zeta) = \|A_2(\mathcal{S}(\zeta))\|$ and $C_u^\wedge(\zeta) = \|A_1(\mathcal{S}(\zeta))\|$ (Proposition 3.12). Let G_1 and G_2 be the domain projections of A_1 and A_2 , respectively. For definiteness, let $G = G_1$. If Q is a central projection with $GQ \in \mathcal{S}$, then $Q = 0$; otherwise, there is a $\zeta \in Z$ with $G(\mathcal{S}(\zeta)) = GQ(\mathcal{S}(\zeta)) = 0$ and consequently with $\|A_1(\mathcal{S}(\zeta))\| = 0$. This implies that $\dim G > \dim \mathcal{S}$. So there is a projection G' with $F \sim G' \leq G$. We now restrict \mathcal{A} to the subspace of the Hilbert space determined by G to obtain the von Neumann algebra $\mathcal{A}_G =$

$G\mathcal{A}G$ (cf. [9; I, § 2]). The set $G\mathcal{S}G = \mathcal{S}_G$ is easily seen to be a central ideal of \mathcal{A}_G since the center of \mathcal{A}_G is $\mathcal{Z}G = \mathcal{Z}_G$ [9; III, 5, Problem 7]. We have that $A_1 \in \mathcal{A}_G^+$. Since the spectrum of \mathcal{Z}_G is the set $X = \{\zeta G \mid \zeta \in Z\}$, we have that the smallest ideal $[\zeta G]$ of \mathcal{A}_G which contains ζG is $G[\zeta]G$. We may now easily show that

$$\|A_1(\mathcal{S}_G + [\zeta G])\| = \inf \{\|A + B + C\| \mid B \in \mathcal{S}_G, C \in [\zeta G]\}$$

is equal to $\|A_1(\mathcal{S}(\zeta))\|$ for every $\zeta \in Z$. This means that $C_u G = \text{lub } \mathcal{Z}_G - \text{Sp}_{\mathcal{S}_G} A_1$ (Proposition 3.12). Therefore, we may find a set F_1, F_2, \dots, F_m of mutually orthogonal projections in \mathcal{A}_G of dimension greater than $\dim \mathcal{S}_G$ such that

$$A_1 F_i(\mathcal{S}_G) = F_i A_1(\mathcal{S}_G) \quad \text{and} \quad \|(C_u G - A_1) F_i(\mathcal{S}_G)\| \leq \varepsilon$$

for every $i = 1, 2, \dots, m$ (Proposition 3.14). Indeed, the algebra \mathcal{A}_G has no finite type I direct summands. Thus, we may find orthogonal projections F_1, F_2, \dots, F_m majorized in \mathcal{A} by $G = G_1$ such that

$$F \sim G' < F_i, AF_i(\mathcal{S}) = F_i A(\mathcal{S}), \quad \text{and} \quad \|(C_u - A) F_i(\mathcal{S})\| \leq \varepsilon$$

for every $i = 1, 2, \dots, m$. Likewise, we may find orthogonal projections F_{m+1}, \dots, F_{m+n} majorized by G_2 such that $F < F_i, AF_i(\mathcal{S}) = F_i A(\mathcal{S})$, and $\|(C_l - A) F_i(\mathcal{S})\| \leq \varepsilon$ for every $i = m + 1, \dots, m + n$. Since G_1 and G_2 are orthogonal, the projections F_1, \dots, F_{m+n} are mutually orthogonal. There are partial isometries $U_{ij} (1 \leq i, j \leq m + n)$ of \mathcal{A} which satisfy the following properties:

- (1) $U_{ij} U_{kl} = \delta_{il} U_{kj}$ (δ = Kronecker delta);
- (2) $U_{ij} = U_{ji}^*$; and
- (3) U_{ii} is a projection with $F \sim U_{ii} < F_i$, for all i, j, k, l .

The element $E' = (m + n)^{-1} \sum U_{ij}$ is a projection in \mathcal{A} with $E' \sim F$, i.e. $\dim E' > \dim \mathcal{S}$. Here, indeed, a calculation using $(m + n) \times (m + n)$ complex matrices suffices. Furthermore, using the fact that $AF_i(\mathcal{S}) = F_i A(\mathcal{S})$ for every i , we have that

$$\begin{aligned} & \|E' A E'(\mathcal{S})\| \\ &= \|E'(A - (C_u \sum \{U_{ii} \mid i \leq m\} + C_l \sum \{U_{ii} \mid i > m\})) E'(\mathcal{S})\| \\ & \quad + \|E'(\sum \{C_u U_{ii} \mid i \leq m\} + \sum \{C_l U_{ii} \mid i > m\}) E'(\mathcal{S})\| \\ & \leq \|(m + n)^{-2} \sum_{ij} (\sum \{U_{ki}(A - C_u) U_{jk} \mid k \leq m\} \\ & \quad + \sum \{U_{ki}(A - C_l) U_{jk} \mid k > m\}) (\mathcal{S})\| \\ & \quad + \|(m + n)^{-1} (m C_u + n C_l) E'(\mathcal{S})\| \leq 2\varepsilon. \end{aligned}$$

Now there is a $B \in \mathcal{S}$ with $\|E' A E' - E' B E'\| \leq 3\varepsilon$. In the ideal \mathcal{S} , we may find a spectral projection E'' for $E' B^* B E'$ majorized by E' so that $\|B E'(1 - E'')\| \leq \varepsilon$. If $Q(E' - E'') \in \mathcal{S}$ for some $Q \in (\mathcal{Z})$, then $Q E' \in \mathcal{S}$ and consequently $Q = 0$. This means that

$$\dim(E' - E'') > \dim \mathcal{I}$$

(Proposition 2.9). Setting $E = E' - E''$, we obtain the relation

$$\|EAE\| \leq \|E(A - B)E\| + \|EBE\| \leq 4\epsilon .$$

REMARK 4.10. If \mathcal{A} is the algebra of all bounded operators on a separable Hilbert space H and \mathcal{I} is the ideal of completely continuous operators, then Fillmore, Stampfli, and Williams [11, Theorem 5.1, Corollary] have obtained Proposition 4.8 without the added restriction that A is self-adjoint. The theorem of Fillmore, et al., depends on properties of Hilbert-Schmidt operators on separable H ; however, it is likely that the restriction can also be removed here.

Let \mathcal{A} be a von Neumann algebra. Let $U(\mathcal{A})$ be the group of unitary operators of \mathcal{A} and let \mathcal{E} be the set of positive real-valued functions f of finite support such that $\sum \{f(U) \mid U \in U(\mathcal{A})\} = 1$. For each $f \in \mathcal{E}$ and A in \mathcal{A} , let $f \cdot A = \sum \{f(U)U^*AU \mid U \in U(\mathcal{A})\}$ and let $\mathcal{K}'(A)$ be the uniform closure of $\{f \cdot A \mid f \in \mathcal{E}\}$. If $B \in \mathcal{K}'(A)$, then $\mathcal{K}'(B) \subset \mathcal{K}'(A)$. Then the intersection $\mathcal{K}(A)$ of $\mathcal{K}'(A)$ with the center is a nonvoid closed convex subset of the center ([8]; cf. also [9; III, §5]). Furthermore the set $\mathcal{K}'(A)$ (resp. $\mathcal{K}(A)$) is central-convex in the sense that $CC_1 + (1 - C)C_2$ is in $\mathcal{K}'(A)$ (resp. $\mathcal{K}(A)$) for every C_1 and C_2 in $\mathcal{K}'(A)$ (resp. $\mathcal{K}(A)$) and C in the center with $0 \leq C \leq 1$ [19; proof, Lemma 6].

The following forms the basis for our analysis of $\mathcal{K}(A)$.

PROPOSITION 4.11. *Let \mathcal{A} be a von Neuman algebra and let A be an element in \mathcal{A} . Let ζ be a point in the spectrum of the center of \mathcal{A} . Then the set $\mathcal{K}(A)(\zeta) = \{B^\wedge(\zeta) \mid B \in \mathcal{K}(A)\}$ is a compact subset of the complex plane.*

Proof. Because $\mathcal{K}(A)(\zeta)$ is bounded, it is sufficient to show that $\mathcal{K}(A)(\zeta)$ contains an arbitrary limit point α . Due to the fact that $\mathcal{K}(A - \alpha)(\zeta) = \mathcal{K}(A)(\zeta) - \alpha$, there is no loss in generality in proving that $0 \in \mathcal{K}(A)(\zeta)$ whenever 0 is a limit point of $\mathcal{K}(A)(\zeta)$. We proceed to do this. For every $n = 1, 2, \dots$, there is a function f_n in the subset \mathcal{E} of real-valued functions on the unitary operators of \mathcal{A} and a central projection P_n of \mathcal{A} with $P_n^\wedge(\zeta) = 1$ and $\|(f_n \cdot A)P_n\| \leq 2^{-n}$. Let $\{Q_i\}$ be the sequence of orthogonal projections defined by $Q_1 = P_1 - P_1P_2, Q_2 = P_1P_2 - P_1P_2P_3, \dots$, and let $B \in \mathcal{K}'(A)$. Then let $C_n = B(1 - P_1) + \sum \{(f_i \cdot A)Q_i \mid 1 \leq i \leq n\} + (f_{n+1} \cdot A)Q'_n$ ($n = 1, 2, \dots$). Here $Q'_n = P_1 \cdots P_{n+1}$ is the orthogonal complement of $(1 - P_1) + \sum \{Q_i \mid 1 \leq i \leq n\}$. We notice that $C_n \in \mathcal{K}'(A)$ for every n since $\mathcal{K}'(A)$ is central-convex. However, the sequence $\{C_n\}$ is Cauchy since $\|C_n - C_{n+1}\| \leq \max\{\|(f_{n+1} \cdot A)Q'_{n+1}\|, \|(f_{n+2} \cdot A)Q'_{n+1}\|\} \leq 2^{-n-1}$. This means that $\{C_n\}$ con-

verges to an element C in $\mathcal{K}'(A)$. We have that

$$\|C(\zeta)\| = \lim \|C_n(\zeta)\| = \lim \|((f_{n+1} \cdot A)Q_n)(\zeta)\| \leq \limsup 2^{-n} = 0$$

and thus C is in the ideal $[\zeta]$. This means that $\mathcal{K}'(C) \subset [\zeta]$. However, we have that $\mathcal{K}'(C) \subset \mathcal{K}'(A)$ because $C \in \mathcal{K}'(A)$. This means that $\mathcal{K}(A) \cap [\zeta] \neq \emptyset$, or equivalently, that $0 \in \mathcal{K}(A)(\zeta)$.

THEOREM 4.12. *Let \mathcal{A} be a properly infinite von Neumann algebra, let \mathcal{I} be the strong radical of \mathcal{A} , and let A be an element of \mathcal{A} . Then the set $\mathcal{K}(A)$ is equal to the set $\mathcal{K}_{\mathcal{I}}(A) = \{\phi(A) \mid \phi \text{ is a state of } \mathcal{A} \sim \text{ with } \phi(\mathcal{I}) = (0)\}$.*

REMARK 4.13. Here notice $E_n(\mathcal{I})$ is the set of all states of $\mathcal{A} \sim$ which vanish on \mathcal{I} .

Proof. First let A be self-adjoint. We show that every element C in the essential central spectrum of A with respect to \mathcal{I} is in $\mathcal{K}(A)$. There is no loss of generality in assuming for this that $C = 0$. Then for every $\varepsilon > 0$, there is a projection E in \mathcal{A} such that

$$\|AE\| \leq \varepsilon \quad \text{and} \quad E \sim 1$$

(Example 2.12 and Corollary 3.15).

There are orthogonal projections E' and E'' of sum E such that $E' \sim E'' \sim E$ [9; III, 8, Corollary 2]. By replacing E by E' , we may assume that $\|AE\| \leq \varepsilon$ and $E \sim 1 - E \sim 1$. Then the element

$$2^{-1}((E - (1 - E))A(E - (1 - E)) + A) = EAE + (1 - E)A(1 - E)$$

is in $\mathcal{K}'(A)$. Now let E_1, \dots, E_n be orthogonal projections of sum E with $E_1 \sim \dots \sim E_n \sim E$, and let U_1, \dots, U_n be unitary operators in \mathcal{A} so that the domain support of $(1 - E)U_i$ equals E_i . For every unit vector x in the Hilbert space, we have

$$\begin{aligned} & \|\sum \{n^{-1}U_i^{-1}(EAE + (1 - E)A(1 - E))U_i x \mid 1 \leq i \leq n\}\| \\ & \|\sum n^{-1}U_i^{-1}EAEU_i x\| + \|\sum n^{-1}U_i^{-1}(1 - E)A(1 - E)U_i x\| \\ & \leq n(n^{-1}\varepsilon) + \|\sum n^{-1}E_i U_i^{-1}(1 - E)A(1 - E)U_i E_i x\| \\ & \leq \varepsilon + n^{-1}\|A\|. \end{aligned}$$

This proves that $\mathcal{K}'(A)$ contains an element of norm less than or equal to $\varepsilon + n^{-1}\|A\|$. Because $\varepsilon > 0$ and n are arbitrary, the set $\mathcal{K}(A)$ contains 0. This means that the essential central spectrum of A with respect to \mathcal{I} is contained in $\mathcal{K}(A)$. Hence, the least upper bound C_u and the greatest lower bound C_l of the essential central spectrum are in $\mathcal{K}(A)$. Since $\mathcal{K}_{\mathcal{I}}(A)$ is the smallest central-

convex set containing C_l and C_u (Theorem 4.4) and since $\mathcal{K}(A)$ is central convex, we have that $\mathcal{K}_{\mathcal{F}}(A) \subset \mathcal{K}(A)$.

Now let A be an arbitrary element of \mathcal{A} and let $\phi \in E_a(\mathcal{F})$. We may assume that $\phi(A) = 0$. We show that 0 is in $\mathcal{K}(A)(\zeta) = \{B^\wedge(\zeta) \mid B \in \mathcal{K}(A)\}$ for every ζ in the spectrum of the center. Since $\mathcal{K}(A)(\zeta)$ is compact (Proposition 4.11), there is a C in $\mathcal{K}(A)$ with $|C^\wedge(\zeta)| = \text{glb}\{|\alpha| \mid \alpha \in \mathcal{K}(A)(\zeta)\}$. There is no loss of generality in assuming $C^\wedge(\zeta) \geq 0$. We obtain a contradiction by assuming $C^\wedge(\zeta) > 0$. Indeed, we have that $\phi(A + A^*) = \phi(A) + \phi(A)^* = 0$. By the preceding paragraph we conclude that $0 \in \mathcal{K}(2^{-1}(A + A^*))$ and so there is a sequence $\{f_n\}$ in the subset \mathcal{E} of functions on the unitary operators of \mathcal{A} with $\lim f_n \cdot (2^{-1}(A + A^*)) = 0$. We may also assume that $\{f_n \cdot ((2i)^{-1}(A - A^*))\}$ converges to a self-adjoint element B in the center [9; III, §5, Problem 2]. Hence, the element iB is in $\mathcal{K}(A)$. However, we must have that $B^\wedge(\zeta) = 0$. Indeed, if $B^\wedge(\zeta) \neq 0$, then the distance to the origin of the line segment L in the complex plane with end-points $C^\wedge(\zeta)$ and $iB^\wedge(\zeta)$ is less than $C^\wedge(\zeta)$. However, this contradicts the definition of C since $L \subset \mathcal{K}(A)(\zeta)$. So we must have that $C^\wedge(\zeta) = 0$, and hence $0 \in \mathcal{K}(A)(\zeta)$. The proof is now completed by a compactness argument. Let $\varepsilon > 0$ be given. For every ζ in the spectrum of the center, there a C_ζ in $\mathcal{K}(A)$ and a central projection P_ζ with $P_\zeta^\wedge(\zeta) = 1$ such that $\|C_\zeta P_\zeta\| \leq \varepsilon$. Due to the compactness of the spectrum of the center, we may find C_1, \dots, C_n in $\mathcal{K}(A)$ and orthogonal central projections P_1, \dots, P_n of sum 1 such that

$$\|\sum C_i P_i\| \leq \varepsilon .$$

However, $\mathcal{K}(A)$ is central-convex and so $\sum C_i P_i \in \mathcal{K}(A)$. Since $\varepsilon > 0$ is arbitrary and since $\mathcal{K}(\mathcal{A})$ is closed, we have that $0 \in \mathcal{K}(A)$. This completes the first part of the proof.

Conversely, let $C \in \mathcal{K}(A)$. There is no loss of generality in assuming $C = 0$. We find ϕ in $E_a(\mathcal{F})$ with $\phi(A) = 0$: Let ϕ_0 be a state of \mathcal{A}^\sim that vanishes on \mathcal{F} (Lemma 4.3). Let $\{f_n\}$ be a sequence of functions in \mathcal{E} such that $\lim f_n \cdot A = 0$. Let ϕ_n be the state of $E_a(\mathcal{F})$ given by $\phi_n(B) = \phi_0(f_n \cdot B)$ for every B in \mathcal{A} . Due to the compactness of the state space of \mathcal{A}^\sim in the $\sigma_w(\mathcal{A}^\sim, \mathcal{A})$ -topology, there is a subnet $\{\phi_{n_j}\}$ of $\{\phi_n\}$ and a state ϕ of \mathcal{A}^\sim such that $\{\phi_{n_j}(B)\}$ converges weakly to $\phi(B)$ for every B in \mathcal{A} . Clearly, the state ϕ vanishes on \mathcal{F} . However, for every x and y in the Hilbert space, we have that

$$|(\phi(A)x, y)| = \lim_j |(\phi_{n_j}(A)x, y)| \leq \limsup \|\phi_0\| \|f_{n_j} \cdot A\| \|x\| \|y\| = 0 .$$

This proves that $\phi(A) = 0$, and so $0 \in \mathcal{K}_{\mathcal{F}}(A)$.

COROLLARY 4.14. *Let \mathcal{A} be a properly infinite von Neumann*

algebra and let A be an element of \mathcal{A} . Then the convex subset $\mathcal{K}(A)$ of the center is weakly compact.

Proof. For any central ideal \mathcal{I} , the set $\mathcal{K}_{\mathcal{I}}(A)$ is weakly compact (Introduction, §4).

Let A be an element in the von Neumann algebra \mathcal{A} . Define $\mathcal{E}(A)$ to be the intersection of the weak closure of $\mathcal{K}'(A)$ with the center of \mathcal{A} . Using the tools we developed here, we can extend the theorem of J. Conway [4] from the case of properly infinite factors to properly infinite algebras with arbitrary centers. For this extension the following lemma is needed.

LEMMA 4.15. *Let \mathcal{A} be a von Neumann algebra on the Hilbert space H . Let f be a $\sigma_w(\mathcal{A}^{\sim}, \mathcal{A})$ -continuous hermitian functional on \mathcal{A}^{\sim} (i.e. $f(\phi)$ is real for every ϕ in \mathcal{A}^{\sim} which takes hermitian elements of \mathcal{A} into hermitian elements of the center). Then there is an $x \in H$ and a self-adjoint $A \in \mathcal{A}$ such that $f(\phi) = (\phi(A)x, x)$ for every $\phi \in \mathcal{A}^{\sim}$.*

Proof. There are $x_1, \dots, x_n, y_1, \dots, y_n$ in H and A_1, \dots, A_n in \mathcal{A} such that $f(\phi) = \sum (\phi(A_i)x_i, y_i)$ for all ϕ in \mathcal{A}^{\sim} [17; §2, Introduction]. For each i there are z_{ij} ($1 \leq j \leq 4$) such that

$$w_{x_i, y_i} = w_{z_{i1}} + w_{z_{i2}} + i(w_{z_{i3}} - w_{z_{i4}})$$

where $w_{x_i, y_i}(B) = (Bx_i, y_i)$ and $w_z = w_{z, z}$ on the center of \mathcal{A} [9; I, 4, Theorem 6 and III, 1, Theorem 4, Corollary]. Then there is an x in H with $w_x = \sum_{i,j} w_{z_{ij}}$ [9; III, 1, Theorem 4, Corollary]. For each ij , there is a positive element C_{ij} in the center with $(BC_{ij}x, x) = (Bz_{ij}, z_{ij})$ for all B in the center (Radon-Nikodym theorem). Thus there is an element $B = \sum A_i(C_{i1} - C_{i2} + i(C_{i3} - C_{i4}))$ in \mathcal{A} with $f(\phi) = (\phi(B)x, x)$ for every ϕ in \mathcal{A}^{\sim} . If $\phi^*(B) = \phi(B^*)^*$ for $\phi \in \mathcal{A}^{\sim}$, then

$$(\phi(B^*)^*x, x) = f(\phi^*) = f(\phi)^- = (\phi(B)x, x)^-$$

for every ϕ in \mathcal{A}^{\sim} implies that $f(\phi) = (\phi(B^*)x, x) = (\phi(B)x, x)$ for every ϕ in \mathcal{A} . Hence, $f(\phi) = (\phi(A)x, x)$ for every ϕ in \mathcal{A}^{\sim} . Here $A = 2^{-1}(B + B^*)$.

THEOREM 4.16. *Let \mathcal{A} be a properly infinite von Neumann algebra, and let \mathcal{I} be the ideal of finite elements of \mathcal{A} ; then $\mathcal{E}(A) = \mathcal{K}_{\mathcal{I}}(A)$ for every A in \mathcal{A} .*

Proof. One may prove the theorem using the same steps (with

appropriate modifications) that Conway [4] employed in his proof for factor algebras. We content ourselves with pointing out the appropriate steps. Let \mathcal{C} be the set of all states of \mathcal{A}^\sim such that $\phi(A) \in \mathcal{C}(A)$ for all A in \mathcal{A} . For every $A \in \mathcal{A}$ and $A_0 \in \mathcal{C}(A)$, there is a $\phi \in \mathcal{C}$ such that $\phi(A) = A_0$. This uses the $\sigma_w(\mathcal{A}^\sim, \mathcal{A})$ -topology instead of the weak $*$ -topology of the dual of \mathcal{A} [4; Lemma 5]. The set $\mathcal{C}(A)$ is equal to $\{0\}$ for every $A \in \mathcal{I}$ [4; Lemma 6]. Hence, the set \mathcal{C} is a subset of $E_a(\mathcal{I})$. But if A is self-adjoint and $\phi \in E_a(\mathcal{I})$, then $\phi(A) \in \mathcal{C}(A)$ since the least upper bound and the greatest lower bound of the essential central spectrum of A with respect to \mathcal{I} are in $\mathcal{C}(A)$ (argue as in [4; Lemma 4] based on Proposition 3.13) and since $\mathcal{C}(A)$ is central-convex (use the fact that $\mathcal{K}'(A)$ is central-convex). If there is ϕ_0 in $E_a(\mathcal{I})$ but not in the $\sigma_w(\mathcal{A}^\sim, \mathcal{A})$ -compact convex set \mathcal{C} , then there is a $\sigma_w(\mathcal{A}^\sim, \mathcal{A})$ -continuous hermitian functional on \mathcal{A}^\sim which strongly separates ϕ_0 from \mathcal{C} . However, every $\sigma_w(\mathcal{A}^\sim, \mathcal{A})$ -continuous hermitian functional f of \mathcal{A}^\sim is of the form $f(\phi) = (\phi(A)x, x)$ for some fixed self-adjoint A in \mathcal{A} and some vector x in the Hilbert space. This contradicts the fact that $\phi_0(A) \in \mathcal{C}(A)$ and so that $\phi_0(A) = \phi(A)$ for some $\phi \in \mathcal{C}$. Hence, $\mathcal{C} = E_a(\mathcal{I})$ and $\mathcal{K}_{\mathcal{I}}(A) = \mathcal{C}(A)$.

COROLLARY 4.17. *Let \mathcal{A} be a σ -finite properly infinite von Neumann algebra; then $\mathcal{K}(A) = \mathcal{C}(A)$ for every A in \mathcal{A} .*

Proof. The ideal generated by the finite elements of \mathcal{A} is the strong radical of \mathcal{A} . The corollary then follows from Theorems 4.12 and 4.16.

5. Applications. Using the notions of essential central spectrum and essential numerical range, we can extend some theorems on commutators and derivations to arbitrary properly infinite von Neumann algebras. These theorems are known for the algebra of all bounded linear operators on a Hilbert space, which is generally assumed to be separable, but the techniques employed there also suffice here.

A linear map δ of an algebra is said to be a *derivation* if $\delta(AB) = A\delta(B) + \delta(A)B$ for every A and B in the algebra. S. Sakai [27] proved that every derivation δ of a von Neumann algebra \mathcal{A} is inner in the sense that there is an A in \mathcal{A} such that $\delta(B) = AB - BA$ for every B in \mathcal{A} . The next proposition is due to J. G. Stampfli [29] for the algebra of bounded linear operators on a Hilbert space. His technique suffices here.

PROPOSITION [Stampfli] 5.1. *The range of a derivation on a von*

Neumann algebra is not uniformly dense in the algebra.

Proof. Since every von Neumann algebra may be written as a product of a finite and a properly infinite von Neumann algebra, it is sufficient to consider these two cases separately. If the algebra is finite, then the range of the derivation is contained in the set of elements whose canonical operator-valued trace vanishes. So the range of a derivation cannot be dense. If the von Neumann algebra \mathcal{A} is properly infinite and the derivation δ on \mathcal{A} is given by $\delta(B) = AB - BA$, then we construct an operator that is not in the closure of the range of δ . Let A_0 be a central element such that $(A - A_0)(\mathcal{J}(\zeta))$ is neither left nor right invertible for all ζ in the spectrum of the center. Here \mathcal{J} is the strong radical of \mathcal{A} (Theorem 3.5). Because $\delta(B) = (A - A_0)B - B(A - A_0)$ for all $B \in \mathcal{A}$, we may assume $A_0 = 0$. There are sequences $\{E_n\}$ and $\{F_n\}$ of mutually orthogonal projections in \mathcal{A} such that $E_n \sim 1 \sim F_n$, $\|AE_n\| \leq n^{-1}$, and $\|F_nA\| \leq n^{-1}$ for every $n = 1, 2, \dots$ (Example 2.12 and Corollary 3.16). Then there is a partial isometry U in \mathcal{A} with domain support $E = \sum E_i$ and range support $F = \sum F_i$ such that $UE_i = F_iU$. We show that $\alpha = \|U - \delta(B)\| \geq 1$ for every $B \in \mathcal{A}$. Indeed, for every $n = 1, 2, \dots$, we have that

$$1 = \|F_nUE_n\| \leq \|F_n(U - \delta(B))E_n\| + \|F_n\delta(B)E_n\| \leq \alpha + 2n^{-1}\|B\|.$$

Hence the open ball of radius 1 about U does not meet the range of δ .

In [18], we showed that an element A in a properly infinite von Neumann algebra \mathcal{A} is a *commutator* in \mathcal{A} (i.e. there are elements B and C with $A = BC - CB$) provided $0 \in \mathcal{N}(A)$. We can also prove that $0 \in \mathcal{N}(A)$ provided $A = BC - CB$ and $\pm(B^*B - BB^*)$ is a positive operator in \mathcal{A} . Now an element A is said to be a self-adjoint commutator if $A = BC - CB$ with $B = B^*$. H. Radjavi [25] characterized those self-adjoint elements in the algebra $B(H)$ of all bounded linear operators on a separable Hilbert space H which are self-adjoint commutators and J. Anderson [1] recently announced that he has completely characterized self-adjoint commutators in $B(H)$. We prove a proposition in this direction for properly infinite von Neumann algebras using a matrix calculation of M. David [5].

PROPOSITION 5.2. *Let \mathcal{A} be a properly infinite von Neumann algebra and let A be a self-adjoint element in \mathcal{A} . If 0 is in the essential central spectrum of A with respect to the strong radical of \mathcal{A} , then A is a self-adjoint commutator in \mathcal{A} .*

Proof. There is a sequence $\{E_n\}$ of orthogonal projections with $E_n \sim 1$ and $\|AE_n\| \leq 1/n!$ for all $n = 1, 2, \dots$ (Lemma 3.16 and Example 2.12). Thus, $\|E_m AE_n\| \leq \min\{1/m!, 1/n!\}$. Then the matrix calculation of M. David [5; Theorem 3] is applicable.

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Added in Proof, August 24, 1972. We have obtained a better version of Proposition 5.2 by showing that A is a self-adjoint commutator whenever 0 is in the essential central range of A .

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