

WEIGHTED CONVERGENCE IN LENGTH

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This paper studies the lower semicontinuity of weighted length

$$(*) \quad \liminf_{n \rightarrow \infty} \int_{\gamma_n} f ds \geq \int_{\gamma} f ds ,$$

where the sequence of curves $\{\gamma_n\}$ converges uniformly to the curve γ , and f is a nonnegative lower semicontinuous function. Necessary and sufficient conditions for equality in $(*)$ are obtained, as well as conditions which prevent γ from being rectifiable. Requirements are given for the attainment of the weighted distance, from a point to a set, and the families of functions, for which weighted distance is attained or $(*)$ is satisfied, are shown to be monotone closed from below. Finally, the solutions to the integral inequality

$$(**) \quad |\gamma(t) - \gamma(0)| \geq \int_{\gamma[0,t]} f ds ,$$

are shown to be compact if the initial values $\gamma(0)$ lie in a compact set.

Let γ be a curve in Euclidean m -space E^m and f be a real-valued function on E^m . The (f) -weighted length of γ , $\int_{\gamma} f ds$, has proved of fundamental importance in establishing the path-cut inequality for condensers [2], [3] and the relationship between capacity and extremal length [5], [8]. Theorem (2.4) provides necessary and sufficient conditions for weighted convergence in length, and (2.10) gives conditions under which the weighted distance, from a point to a set, is attained. Corollary (2.6) is a useful special case of [8, Lemma 3.3]. In (3.1) the family of functions, for which weighted distance is attained, is shown to be monotone closed from below, and Theorem (3.2) establishes the compactness of the set of solutions to the contingent equation $(**)$, similar to a result of Filippov [4].

2. Convergence theorems.

NOTATION 2.1. Let E^m denote *Euclidean m -space* consisting of all m -tuples $x = (x_1, \dots, x_m)$ of real numbers with inner product $\langle x, y \rangle = \sum_{i=1}^m x_i y_i$, for all x, y in E^m and norm $|x| = \langle x, x \rangle^{1/2}$. Throughout this paper, points in E^m will often be denoted by the letters x and y , whereas the letters s, t will be reserved for real numbers. The

complex plane is designated by the symbol \mathcal{E} .

Let $\text{Int } A$, $\text{cl } A$, ∂A denote the *interior*, *closure*, and *boundary* of the set A , respectively. The open ball of radius t centered at x will be indicated by the expression $B(x, t)$.

A function $f: E^m \rightarrow E^n$ is *Lipschitz* on the set A in E^m if there is a constant M such that

$$|f(x) - f(y)| \leq M|x - y|,$$

for all x, y in A . If $n = 1$, the *gradient* of f , $\text{grad } f$, will exist L_m -a.e. in A , where L_m is the m -dimensional *Lebesgue measure*. The *Hausdorff 1-dimensional measure* in E^m will be denoted by H^1 (for its definition and properties see [1]). Then $H^1(A)$ represents the *length* of the set A in E^m .

DEFINITIONS 2.2. Two functions $\gamma: [a, b] \rightarrow E^m$, $\gamma^* = [c, d] \rightarrow E^m$ are *Fréchet-equivalent* if

$$\inf_h \sup_t |\gamma(t) - \gamma^*(h(t))| = 0,$$

where $h: [a, b] \rightarrow [c, d]$ is a homeomorphism. A Fréchet equivalence class γ of continuous functions into E^m is called a *curve* in E^m , and each member of the class is called a *parametrization* of γ .

The *length* of a curve γ is given by

$$H_*^1(\gamma) = \sup_{\pi} \sum_i |\gamma(t_{i-1}) - \gamma(t_i)|,$$

where $\gamma: [a, b] \rightarrow E^m$ is any parametrization of γ and π is a partition of $[a, b]$. Note that $H^1(\gamma) < H_*^1(\gamma)$, unless the set of multiple points of γ has H^1 -measure zero (see [7, p. 125]). A curve γ is *rectifiable* if $H_*^1(\gamma) < \infty$. In this case we can write

$$H_*^1(\gamma) = \int_{\gamma} dH^1 = \int_a^b |\text{grad } \gamma(t)| dt.$$

A rectifiable curve can be parametrized with respect to arc-length (see [6, p. 259]); we denote this parametrization by $\gamma(s)$. Note that $|\text{grad } \gamma(s)| = 1$, H^1 -a.e. in $[0, H_*^1(\gamma)]$, since $|\gamma(s) - \gamma(s^*)| \leq |s - s^*|$ implies that $|\text{grad } \gamma(s)| \leq 1$, H^1 -a.e., and

$$H_*^1(\gamma) = \int_0^{H_*^1(\gamma)} |\text{grad } \gamma(s)| ds.$$

If $f: E^m \rightarrow E^1$ is a Borel-measurable function and γ is a rectifiable curve, define (as above)

$$\int_{\gamma} f dH^1 = \int_a^b f(\gamma(t)) |\text{grad } \gamma(t)| dt,$$

then in the event γ is parametrized by arc length,

$$\int_{\gamma} f dH^1 = \int_0^{H^1(\gamma)} f(\gamma(s)) ds .$$

In particular, if $0 \leq S \leq H^1(\gamma)$, we define

$$\int_{\gamma[S]} f dH^1 = \int_0^S f(\gamma(s)) ds .$$

A curve γ is *locally rectifiable* if $H^1(\gamma \cap \text{cl } B(0, k)) < \infty$, for all $k = 1, 2, 3, \dots$, where $\gamma \cap \text{cl } B(0, k)$ are the subcurves of γ with images in $\text{cl } B(0, k)$.

THEOREM 2.3. *Let $\{\gamma_n(s)\}$ be a sequence of rectifiable curves in E^m , such that $H^1(\gamma_n) \geq L > 0$ and $\gamma_n(0) \rightarrow \gamma_0$. Let γ_S be an accumulation point of the set $\{\gamma_n(S)\}$, $0 < S \leq L$. Then some subsequence $\{\gamma_{n_j}\}$ converges uniformly on $[0, S]$ to a curve γ containing γ_0 and γ_S such that for every nonnegative lower semicontinuous function $f: E^m \rightarrow E^1$,*

$$(1) \quad \liminf_{j \rightarrow \infty} \int_{\gamma_{n_j}[S]} f dH^1 \geq \int_{\gamma} f dH^1 .$$

Proof. Since all but finitely many points of $\{\gamma_n(S)\}$ lie in $B(\gamma_0, S + 1)$, so does γ_S . By selecting a subsequence and reindexing we can assume $\gamma_n(S) \rightarrow \gamma_S$. Each γ_n is Lipschitzian with constant 1, so $\{\gamma_n\}$ is equicontinuous on $[0, S]$, and uniformly bounded by $|\gamma_0| + S + 1$. By Ascoli's Theorem, some subsequence $\{\gamma_{n_j}\}$ converges uniformly on $[0, S]$ to a function $\gamma: [0, S] \rightarrow E^m$. Clearly γ is a curve from γ_0 to γ_S , and is Lipschitzian with constant 1. Thus, $|\text{grad } \gamma| \leq 1$, H^1 - a.e., and by Fatou's lemma and the lower semicontinuity of f

$$\begin{aligned} \liminf_{j \rightarrow \infty} \int_{\gamma_{n_j}[S]} f dH^1 &\geq \int_0^S \liminf_{j \rightarrow \infty} f(\gamma_{n_j}(t)) dt , \\ &\geq \int_0^S f(\gamma(t)) dt \geq \int_{\gamma} f dH^1 . \end{aligned}$$

COROLLARY 2.4. *Assuming the hypotheses in Theorem (2.3), the condition*

$$(2) \quad \lim_{j \rightarrow \infty} \int_0^S |\text{grad } \gamma_{n_j}(t) - \text{grad } \gamma(t)| dt = 0 ,$$

holds if and only if

$$(3) \quad \lim_{j \rightarrow \infty} \int_{\gamma_{n_j}[S]} f dH^1 = \int_{\gamma} f dH^1 ,$$

for every continuous function $f: E^m \rightarrow E^1$.

Proof. Let M be a bound for f on $B(\gamma_0, S + 1)$. Given $\varepsilon > 0$, the uniform convergence of $\{\gamma_{n_j}\}$ and (2) imply that, for sufficiently large j ,

$$\begin{aligned} \left| \int_{\gamma_{n_j}} f dH^1 - \int_{\gamma} f dH^1 \right| &\leq \int_0^S |f \circ \gamma_{n_j} - f \circ \gamma| |\text{grad } \gamma_{n_j}| dt \\ &+ \int_0^S |f \circ \gamma| |\text{grad } \gamma_{n_j} - \text{grad } \gamma| dt \leq \varepsilon(S + M). \end{aligned}$$

Thus (2) implies (3). Conversely, if $f \equiv 1$, then

$$S = \lim_{j \rightarrow \infty} \int_{\gamma_{n_j}[S]} dH^1 = \int_0^S |\text{grad } \gamma(t)| dt \leq S,$$

and it follows that $|\text{grad } \gamma| = 1$, H^1 - a.e. By the triangle inequality,

$$\begin{aligned} &|\text{grad } \gamma_{n_j} - \text{grad } \gamma|^2 \\ &= 4 - |\text{grad } \gamma_{n_j} + \text{grad } \gamma|^2 \leq 4(2 - |\text{grad } \gamma_{n_j} + \text{grad } \gamma|), \end{aligned}$$

so Schwarz's inequality yields

$$(4) \quad \left[\int_0^S |\text{grad } \gamma_{n_j} - \text{grad } \gamma| dt \right]^2 \leq 4S \left(2S - \int_0^S |\text{grad } \gamma_{n_j} + \text{grad } \gamma| dt \right).$$

But $\{\gamma_{n_j} + \gamma\}$ converges uniformly to 2γ on $[0, S]$, so Theorem (2.3) implies

$$(5) \quad \liminf_{j \rightarrow \infty} \int_0^S |\text{grad } \gamma_{n_j} + \text{grad } \gamma| dt \geq 2 \int_0^S |\text{grad } \gamma| dt = 2S.$$

Combining equations (4) and (5) we find

$$\begin{aligned} 0 &\leq \liminf_{j \rightarrow \infty} \left(\int_0^S |\text{grad } \gamma_{n_j} - \text{grad } \gamma| dt \right)^2 \\ &\leq \limsup_{j \rightarrow \infty} \left(\int_0^S |\text{grad } \gamma_{n_j} - \text{grad } \gamma| dt \right)^2 \\ &= 4S \left(2S - \liminf_{j \rightarrow \infty} \int_0^S |\text{grad } \gamma_{n_j} + \text{grad } \gamma| dt \right) \leq 0, \end{aligned}$$

which yields (2).

EXAMPLE 2.5. Let $\gamma_n: [0, 2\pi] \rightarrow \mathcal{C}$ be given by $\gamma_n(s) = (e^{ins})/n$. Note these functions converge uniformly to the constant function $\gamma(t) = 0$. Although (1) holds, (3) clearly does not, and

$$\int_0^{2\pi} |\text{grad } \gamma_n(t) - \text{grad } \gamma(t)| dt = \int_0^{2\pi} |e^{int}| dt = 2\pi.$$

COROLLARY 2.6. Let $\{\gamma_n(s)\}$ be a sequence of rectifiable curves in E^m such that $\gamma_n(0) \rightarrow \gamma_0$ and $\gamma_n(s_n) \rightarrow \gamma_s$, $0 < s_n \leq S < \infty$. Then there

is a subsequence $\{\gamma_{n_j}\}$ and a curve γ containing γ_0 and γ_S such that

$$(6) \quad \liminf_{j \rightarrow \infty} \int_{\gamma_{n_j}[s_{n_j}]} f dH^1 \geq \int_{\gamma} f dH^1,$$

for every nonnegative lower semicontinuous function $f: E^m \rightarrow E^1$.

Proof. Let γ_n^* be the restriction of γ_n to $[0, s_n]$. Extend γ_n^* to $[0, S]$ by setting $\gamma_n^*(t) = \gamma_n^*(s_n)$, for $s_n \leq t \leq S$. Each γ_n^* is Lipschitzian with constant 1, so, as in Theorem (2.3), some subsequence converges uniformly on $[0, S]$ to a curve γ containing γ_0 and γ_S , and having Lipschitz constant 1. By passing to a subsequence, we can assume $s_{n_j} \rightarrow s^*$ in $[0, S]$. Then $\gamma(s^*) = \gamma_S$ since

$$|\gamma(s^*) - \gamma_{n_j}^*(s_{n_j})| \leq |\gamma(s^*) - \gamma_{n_j}^*(s^*)| + |s^* - s_{n_j}| \rightarrow 0.$$

For every $\varepsilon > 0$, $s_{n_j} > s^* - \varepsilon$ for large j , so by Fatou's Theorem

$$\begin{aligned} \liminf_{j \rightarrow \infty} \int_{\gamma_{n_j}[s_{n_j}]} f dH^1 &\geq \liminf_{j \rightarrow \infty} \int_0^{s^* - \varepsilon} f(\gamma_{n_j}^*(t)) dt \\ &\geq \int_0^{s^* - \varepsilon} f(\gamma(t)) dt \geq \int_{\gamma[s^* - \varepsilon]} f dH^1, \end{aligned}$$

from which the result follows.

THEOREM 2.7. *Let $\{\gamma_n(t)\}$ be a sequence of curves in E^m such that $H_*^1(\gamma_n \cap \text{cl } B(0, k)) \leq L_k < \infty$, for all $n, k = 1, 2, \dots$, and $\gamma_n(0) \rightarrow \gamma_0$. Then some subsequence $\{\gamma_{n_j}\}$ converges uniformly on compact subsets to a curve γ containing γ_0 such that*

$$(7) \quad \liminf_{j \rightarrow \infty} \int_{\gamma_{n_j}} f dH^1 \geq \int_{\gamma} f dH^1,$$

for every nonnegative lower semicontinuous function $f: E^m \rightarrow E^1$.

Proof. There exists an integer K such that γ_0 and all $\gamma_n(0)$ lie in $B(0, K)$. In each closed ball $\text{cl } B(0, k)$, $k \geq K$, reparametrize a restriction of γ_n by arc length

$$\gamma_{kn}: [0, s_{kn}] \rightarrow \gamma_n,$$

where $0 < s_{kn} \leq L_k$ is either the first real number such that $\gamma_{kn}(s_{kn})$ lies in $\partial B(0, k)$ or $H_*^1(\gamma_n)$, if no such number exists. If denumerably many γ_n lie in some $B(0, k)$ the proof follows by Corollary 2.6. Otherwise, delete all γ_n which lie in $B(0, k + 1)$. Then a subsequence of $\{\gamma_{(K+1)n}(1)\}$ converges to a point p_1 in $\text{cl } B(0, K + 1)$, and Theorem 2.3 yields a subsequence $\{\gamma_{(K+1)n_i}\}$ converging uniformly on $[0, 1]$ to a curve γ^1 containing γ_0 and p_1 . Delete all γ_{n_i} lying in $B(0, K + 2)$. A sub-

sequence of $\{\gamma_{(K+2)n_i}(2)\}$ converges to a point p_2 in $\text{cl } B(0 + 2)$, implying a subsequence, which we also denote by $\{\gamma_{(K+2)n_j}\}$, converges uniformly on $[0, 2]$ to a curve γ^2 containing γ_0 and p_2 . Continuing in this manner we note that γ^k is an extension of γ^j , for $k > j$, hence there is a $\gamma: [0, \infty) \rightarrow E^m$ and a subsequence $\{\gamma_{n_j}\}$ obtained by Cantor's diagonalization process such that $\{\gamma_{n_j}\}$ converges uniformly to γ on compact subsets of $[0, \infty)$. By Theorem 2.3 we have that for every real number $S > 0$,

$$\liminf_{j \rightarrow \infty} \int_{\gamma_{n_j}[S]} f dH^1 \geq \int_{\gamma[S]} f dH^1,$$

for every nonnegative lower semicontinuous function, hence the proof is complete.

REMARK 2.8. Observe, from the construction above, that γ is bounded if denumerably many γ_n lie in some $B(0, k)$, and is unbounded otherwise, as a consequence of the hypothesis $H_*^1(\gamma_n \cap \text{cl } B(0, k)) \leq L_k < \infty$.

Theorem 2.7 is true if we replace this condition by the requirement that $H_*^1(\gamma_n \cap \text{cl } B(0, k)) < \infty$, for all positive integers n and k , since if denumerably many γ_n lie in some $B(0, k)$ and no uniform bound exists on their lengths, an argument similar to the rest of the proof above, using curves of length $\geq j$, sequences of points $\{\gamma_n(j)\}$, $j = 1, 2, \dots$, and diagonalization, yields a subsequence $\{\gamma_{n_j}\}$ converging uniformly on compact subsets of $[0, \infty)$ to a curve γ for which (7) holds. Of course, γ might then be a constant function as in Example 2.5. Moreover, it is no longer true that γ is unbounded if only finitely many γ_n lie in each $B(0, k)$, as is seen in the next example.

EXAMPLE 2.9. In E^2 , select the points

$$a_n = \left(\frac{n-1}{n}, \frac{n-1}{n} \right), b_n = \left(1, \frac{n}{n+1} \right),$$

$$c_n = \left(n+1, \frac{n}{n+1} \right), n = 1, 2, \dots$$

Let γ_n be the polygonal arc obtained by joining the points $a_1, b_1, a_2, b_2, \dots, a_n, b_n, c_n$ be straight line segments in their given order. Clearly $H_*^1(\gamma_n \cap \text{cl } B(0, k)) < \infty$, for all n , and γ_n lies in $B(0, k)$ iff $n \leq k - 2$. However, if we parametrize these arcs by arc length, then $\{\gamma_n\}$ converges uniformly on compact subsets of $[0, \infty)$ to the polygonal arc γ joining the points $a_1, b_1, a_2, b_2, \dots$.

LEMMA 2.10. Let K be a closed subsets of the bounded arcwise connected set A in E^m , y a point in $A - K$, Γ the family of curves

joining y to K in A , and $f: E^m \rightarrow E^1$ a positive lower semicontinuous function. Then there is a curve γ_f in Γ such that

$$(8) \quad \int_{\gamma_f} f dH^1 = \inf_r \int_r f dH^1.$$

Proof. Assume the right side of (8) equals $M < \infty$ as otherwise any γ will do. Let $\{\gamma_n\}$ be a minimizing sequence of curves in Γ . Since $f(x) \geq a > 0$ on $\text{cl } A$, for sufficiently large n we have $H_*^1(\gamma_n) \leq 2M/a$. Parametrizing these rectifiable curves by arc length so that $\gamma_n(0) = y$ and $\gamma_n(s_n)$ belongs to K , for $s_n \leq 2M/a$, Corollary (2.6) and the compactness of K imply the existence of a curve γ_f in Γ such that

$$M \leq \int_{\gamma_f} f dH^1 \leq \liminf_{j \rightarrow \infty} \int_{\gamma_{n_j}} f dH^1 = M.$$

REMARK 2.11. If A is unbounded, the same result may be obtained by requiring that the lower semicontinuous function f be bounded below, by a positive constant, on A .

One may also weaken the requirement on the lower semi-continuous function f by asking that it be nonnegative and satisfy

$$(9) \quad H^1(\{x: f(x) < \varepsilon\}) = o(1).$$

Then $M > 0$ and a minimizing sequence $\{\gamma_n\}$ can be chosen, for $M < \infty$ and sufficiently small ε , such that

$$H_*^1(\gamma_n) < o(1) + 2M/\varepsilon.$$

The proof follows as before. Condition (9) can not be removed entirely as is seen by letting A be the closed unit disk in \mathcal{D} , $K = \partial A$, $y = 0$, and f be the characteristic function on the complement of the set

$$\{z(t): z(t) = (1 - t^{-1})e^{i-t}, 1 \leq t < \infty\}.$$

3. Some compactness theorems. Let \mathfrak{A} be the set of functions $f: E^m \rightarrow E^1$ for which Theorem 2.3 (2.6, or 2.7) holds, and \mathfrak{B} the set of functions which permit the verification of Lemma 2.10. Clearly \mathfrak{A} and \mathfrak{B} properly contain the nonnegative and positive lower semicontinuous function respectively, since the function values may be changed on sets of H^1 -measure zero without affecting (1) or (8).

THEOREM 3.1. Let $\{f_k\}$ be a nondecreasing sequence of functions in \mathfrak{A} and $f(x) = \lim_k f_k(x)$. Then f is also in \mathfrak{A} . The same result also holds for \mathfrak{B} provided $f_1(x) \geq a > 0$ on $\text{cl } A$.

Proof. Let $\{\gamma_n\}$ be a sequence of curves satisfying the hypothesis

of Theorem (2.3). Then by the Lebesgue monotone convergence theorem and (2.3), we have

$$(10) \quad \int_{\gamma} f dH^1 \leq \limsup_{k \rightarrow \infty} \left(\liminf_{n \rightarrow \infty} \int_{\gamma_n} f_k dH^1 \right) \leq \liminf_{n \rightarrow \infty} \int_{\gamma_n} f dH^1,$$

implying that f lies in \mathfrak{A} . Let M equal the right side in equation (8). There is nothing to prove if $M = \infty$, so let $M < \infty$. For each f_k there is a curve γ_k such that

$$\int_{\gamma_k} f_k dH^1 = \inf_{\Gamma} \int_{\Gamma} f_k dH^1 = M_k \leq M.$$

Since

$$(11) \quad M_j = \int_{\gamma_j} f_j dH^1 \leq \int_{\gamma_k} f_j dH^1 \leq M_k, \quad j \leq k,$$

the sequence $\{M_k\}$ has a limit $M^* \leq M$. Moreover

$$aH_*^1(\gamma_k) \leq \int_{\gamma_k} f_k dH^1 \leq M,$$

so the curves $\{\gamma_k\}$ satisfy the hypothesis in Corollary (2.6). Hence there is a curve γ such that (6) holds for each f_k . Thus by (10) and (11)

$$\begin{aligned} M &\leq \int_{\gamma} f dH^1 \leq \limsup_{j \rightarrow \infty} \left(\liminf_{k \rightarrow \infty} \int_{\gamma_k} f_j dH^1 \right) \\ &\leq \limsup_{j \rightarrow \infty} \left(\liminf_{k \rightarrow \infty} M_k \right) = M^* \leq M. \end{aligned}$$

Now let A be a subset of E^m , $0 < S < \infty$, $f: E^m \rightarrow E^1$ a nonnegative lower semicontinuous function, and

$$\mathfrak{D}_A = \mathfrak{D}_A(f, S) = \left\{ \gamma: \gamma(0) \in A, |\gamma(s) - \gamma(0)| \leq \int_{\gamma[0,s]} f dH^1, 0 \leq s \leq S \right\}.$$

Then \mathfrak{D}_A is a subset of the Banach space of all continuous functions on $[0, S]$ with the sup norm.

THEOREM 3.3. *If A is compact, then \mathfrak{D}_A is compact.*

Proof. Let $\{\gamma_n\}$ be a sequence of curves in \mathfrak{D}_A . By Theorem (2.3), some subsequence, which will also be denoted by $\{\gamma_n\}$, converges uniformly on $[0, S]$ to a curve γ , with $\gamma(0)$ in A , and satisfies

$$\int_{\gamma[0,s]} f dH^1 \leq \liminf_{n \rightarrow \infty} \int_{\gamma_n[0,s]} f dH^1 \leq |\gamma(s) - \gamma(0)|, \quad 0 \leq s \leq S.$$

Reparametrizing γ by arc length ($H_*^1(\gamma) \leq S$) and extending it to $[0, S]$, as in the proof of (2.6) shows that γ belongs to \mathfrak{D}_A .

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