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ON SOLUTIONS IN THE REGRESSIVE ISOLS

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Let $f(x)$ be a recursive function and let $D_f(X)$ denote the Nerode canonical extension of f to the isols. Let A and Y be particular isols such that $D_f(A) = Y$. The main results in the paper deal with the following problem: if one of the isols A and Y is regressive, what regressive property if any will the other isol have. It is shown that if A is a regressive isol then Y will be also. Also, it is possible for Y to be a regressive isol while A is not. In this event there exist regressive isols B with $D_f(B) = Y$ and $B \leqq_A A$. Extensions of these results for recursive functions of more than one variable are discussed in the last section of the paper.

1. Introduction. We will assume that the reader is familiar with the primary definitions and results of the papers listed as references. We will cite some particular definitions and results that have a special role in the paper. E will denote the set of nonnegative integers, Λ the collection of isols, Λ^* the collection of isolic integers, and Λ_R the collection of regressive isols. If f is a partial function from a subset of E into E then δf will denote its domain. If $f: E^n \rightarrow E$ is a recursive function then D_f will denote the canonical extension of f to the isols. Two sets α and β will be *separated*, written $\alpha | \beta$, if there exist disjoint r.e. supersets of α and β. $j(x, y)$ will denote the familiar recursive pairing function defined by,

$$j(x, y) = x + 1/2(x + y)(x + y + 1) ,$$

and k and l the associated functions with the property $j(k(x), l(x)) = x$. $[\rho_x]$ will be the canonical enumeration for the collection of all finite subsets of E, [6]. Associated with this enumeration is the recursive function $r(x)$ having the property $r(x) = \text{card } \rho_x$. We will use a $\sum$ to stand for union among sets (and also $a +$ for a union of two sets).

2. Recursive functions of one variable. Let $f: E \rightarrow E$ be a recursive function. If f is a combinatorial function then its extension D_f will map Λ into Λ, and if f is an increasing function then D_f will map Λ_R into Λ_R. Each combinatorial function of one variable will be increasing, but not conversely. The condition needed for D_f to map Λ_R into Λ_R is that f be an eventually increasing function, [1].

THEOREM 1. *Let $f: E \rightarrow E$ be a recursive function and A and Y be* *isols such that* $D_f(A) = Y$. *If* A *is a regressive isol then* Y *will be regressive also.*

Proof. Assume A is a regressive isol. Let

$$g(0) = 0\ ,$$
$$g(n+1) = f(n) + g(n)\ .$$

Then g will be an increasing and recursive function. Hence its canonical extension D_g will map Λ_R into Λ_R. Since

$$g(n+1) = f(n) + g(n)\ ,$$

it follows from the Nerode metatheorem for such identities (combining [12, Theorem 10.1] and the representation of the canonical extension of a recursive function [11, 4]), that

$$(1) \qquad D_g(A+1) = D_f(A) + D_g(A)\ .$$

Because A is a regressive isol and g is increasing and recursive, each of the isols $A + 1$, $D_g(A+1)$ and $D_g(A)$ will also be regressive. In addition, $Y = D_f(A)$ is an isol and from (1) it then follows

$$(2) \qquad Y \leqq D_g(A+1) \text{ and } D_g(A+1) \in \Lambda_R\ .$$

In view of a result due to Dekker [4, P8 (a)], (2) implies that Y will be a regressive isol.

REMARK. If f is a recursive function of one variable then although its canonical extension may not map every isol onto an isol, its value may be an isol for some. In addition, it may also occur that the value of $D_f(A)$ will be a regressive isol for an isol A which is non-regressive. An example of such a recursive function will be given in the following section. We want to show next that if this possibility does occur, then there will be a regressive isol B such that $D_f(B) = D_f(A)$. The following lemma essentially gives this result, once the connection is made between the canonical extensions of recursive functions and recursive combinatorial functions.

LEMMA. *Let* $f, g\colon E \to E$ *be recursive combinatorial functions and* A *and* Y *be isols which satisfy the identity,*

$$(1) \qquad D_f(A) = Y + D_g(A)\ .$$

If Y *is a regressive isol, then there will also exist a regressive isol* B *with,*

$$(2) \qquad D_f(B) = Y + D_g(B)\ .$$

Proof. Assume that Y is a regressive isol, and consider separately the following three cases.

Case 1. A is finite. Then A will be regressive and we may set $B = A$.

Case 2. A is infinite and Y is finite. Let $Y = p \in E$. Set

$$h(x) = p + g(x), \text{ for } x \in E .$$

Then h will be a recursive combinatorial function, since the function g is recursive and combinatorial. By a theorem of Myhill and Nerode [11, Theorem 7], we also obtain,

$$D_h(A) = Y + D_g(A) . \tag{3}$$

Combining (1) and (3) implies

$$D_f(A) = D_h(A) , \tag{4}$$

and since A is an infinite isol, it follows from (4) and a theorem due to Myhill [8], that there will be infinitely many numbers n that satisfy

$$f(n) = h(n) . \tag{5}$$

Let m be the smallest number that satisfies (5), and let $B = m$. Then B will be a regressive solution to (2), since

$$\begin{aligned} D_f(m) &= f(m) \\ &= h(m) \\ &= D_h(m) \\ &= p + D_g(m) \\ &= Y + D_g(m) . \end{aligned}$$

Case 3. Both A and Y are infinite isols. Let φ_f and φ_g be the normal combinatorial operators, and let $[c_i]$ and $[d_i]$ be the sequences of combinatorial coefficients that are associated with the functions f and g respectively. Let $\alpha \in A$ and $\eta \in Y$. Then α and η will each be infinite and isolated sets, and also η will be regressive. We will assume that

$$\eta | \alpha \text{ and } \eta | \varphi_g(\alpha) , \tag{6}$$

for otherwise an easy modification may be made in the proof. Based on their respective definitions, each of the functions c_i and d_i will be recursive, and also

$$\varphi_f(\alpha) = (j(x, y) \mid \rho_x \subseteqq \alpha \text{ and } y < c_{r(x)}) ,$$
$$\varphi_g(\alpha) = (j(x, y) \mid \rho_x \subseteqq \alpha \text{ and } y < d_{r(x)}) .$$

From (1) and (6) it follows also,

(7) $$\varphi_f(\alpha) \cong \eta + \varphi_g(\alpha) .$$

Let p be a partial recursive function that establishes (7), i.e., p will be defined on $\varphi_f(\alpha)$, will be one-to-one, and will map

(8) $$p\colon \varphi_f(\alpha) \to \eta + \varphi_g(\alpha) ,$$

one-to-one and onto.

Let y_x be a regressive function that ranges over the set η.

Our first aim is to define two particular sequences of subsets of α and of η respectively, whose corresponding terms will share the property appearing in (8). With each number n we will associate two sets α_n a subset of α, and η_n a subset of η. These sets are meant to be the collections of those members of α and η respectively, that we can effectively find if we start with the value of y_n and use only the regressive property of the function y_x, the separability property in (6), and the recursive and partial recursive properties that appear in (8). Note that the inverse function p^{-1} of p will be well-defined and partial recursive. The particular definition for these sets is as follows; for $n \in E$, the members of α_n and η_n are determined by repeated applications of the six rules below,

(i) $y_n \in \eta_n$,
(ii) if $y_k \in \eta_n$ then $(y_0, \cdots, y_k) \subseteqq \eta_n$,
(iii) if $y_k \in \eta_n$ and $p^{-1}(y_k) = j(x, u)$, then $\rho_x \subseteqq \alpha_n$,
(iv) if $a_1, \cdots, a_k \in \alpha_n$, $\rho_x = (a_1, \cdots, a_k)$, $y < c_k$, $pj(x, y) \in \eta$ and $pj(x, y) = y_m$, then $y_m \in \eta_n$,
(v) if $a_1, \cdots, a_k \in \alpha_n$, $\rho_x = (a_1, \cdots, a_k)$, $y < c_k$ and $pj(x, y) = j(u, v)$, then $\rho_u \subseteqq \alpha_n$,
(vi) if $a_1, \cdots, a_k \in \alpha_n$, $\rho_x = (a_1, \cdots, a_k)$, $y < d_k$ and $p^{-1}j(x, y) = j(u, v)$, then $\rho_u \subseteqq \alpha_n$.

Note that each of the sets η_n will be non-empty, in view of (i). It may occur that some of the sets α_n are empty, however this will be true for at most only finitely many of the α_n. It is easy to see upon a moments reflection that from the value of the number y_n one can effectively enumerate all of the members in each of the sets α_n and η_n. It follows that each of the sets α_n and η_n (for any number n) will be r.e. subsets of α and η respectively. Since α and η are each isolated sets, we see that each of the sets α_n and η_n will be finite. It will be useful to list some of these properties and also some that can be arrived at in an easy manner from the six rules above.

$$(\forall n)[\eta_n \neq \emptyset] \text{ and } (\forall n)(\exists k)[\alpha_{n+k} \neq \emptyset]\ . \tag{9}$$

$$(\forall n)(\exists t)[t \geqq n \text{ and } \eta_n = (y_0, \cdots, y_t)]\ . \tag{10}$$

$$\alpha_0 \subseteqq \alpha_1 \subseteqq \alpha_2 \subseteqq \cdots \text{ and } \sum_0^\infty \alpha_n \subseteqq \alpha\ . \tag{11}$$

$$\eta_0 \subseteqq \eta_1 \subseteqq \eta_2 \subseteqq \cdots \text{ and } \sum_0^\infty \eta_n = \eta\ . \tag{12}$$

In addition, note that the six rules (i) – (vi) have been so defined so have the following property; if one would simply know only the value of y_n, then the totality of those members of α and η that could be found by using only the recursive and regressive features present in (8) would be the two sets α_n and η_n respectively. It follows from this property that, for $n \in E$

$$p: \varphi_f(\alpha_n) \rightarrow \eta_n + \varphi_g(\alpha_n), \text{ one-to-one and onto.} \tag{13}$$

For each number $n \in E$, let the

torre number of η_n = the largest number t with $\eta_t = \eta_n$.

In view of (i) and the fact that each of the sets η_n is finite, it follows that there will be infinitely many torre numbers. In addition it is easy to see that if t is the torre number of η_n, then $t \geqq n$ and $\eta_t = \eta_n = (y_0, \cdots, y_t)$. Let t_x denote the strictly increasing function that ranges over the set of all torre numbers. Then

$$\eta_{t_x} = (y_0, \cdots, y_{t_x})\ , \tag{14}$$

$$\eta_{t_0} \subseteqq \eta_{t_1} \subseteqq \eta_{t_2} \subseteqq \cdots\ , \tag{15}$$

$$t_x < k \leqq t_{x+1} \Longrightarrow \eta_k = \eta_{t_{x+1}}, \text{ and} \tag{16}$$

$$\eta = \sum_0^\infty \eta_{t_n}\ . \tag{17}$$

In addition, by combining the remark prior to (13) with (16) and the fact that y_n is a regressive function, we can also see that y_{t_x} will be a regressive function (of x). This turns out to be a very useful property. Another fact that is important to note here is property A given below; it follows from (13), (16), the definitions of η_n and its torre number, and the regressive property of y_{t_x}.

Property A. If we are given the value of y_k then we can effectively determine whether $k \leqq t_0$ or there is a number x such that $t_x < k \leqq t_{x+1}$. In the former event we could also find the value of y_{t_0}, and in the latter event both of the numbers y_{t_x} and $y_{t_{x+1}}$ could be found.

Combining (11), (13) and (15) gives,

$$\alpha_{t_0} \subseteqq \alpha_{t_1} \subseteqq \alpha_{t_2} \subseteqq \cdots, \text{ and} \tag{18}$$

$$p\colon \varphi_f(\alpha_{t_x}) \longrightarrow \eta_{t_x} + \varphi_g(\alpha_{t_x}) , \tag{19}$$

one-to-one and onto, for each number x. Since φ_f and φ_g are combinatorial operaors, the inclusions appearing in (18) also imply that

$$\varphi_f(\alpha_{t_x}) \subseteqq \varphi_f(\alpha_{t_{x+1}}) ,$$

and

$$\varphi_g(\alpha_{t_x}) \subseteqq \varphi_g(\alpha_{t_{x+1}}) .$$

Therefore, in view of (15) and (19), we obtain for each number $x \in E$,

$$\begin{aligned} &p\colon (\varphi_f(\alpha_{t_{x+1}}) - \varphi_f(\alpha_{t_x})) \\ &\longrightarrow (\eta_{t_{x+1}} - \eta_{t_x}) + (\varphi_g(\alpha_{t_{x+1}}) - (\varphi_g(\alpha_{t_x})) , \end{aligned} \tag{20}$$

one-to-one and onto.

We now begin to design a regressive set β whose recursive equivalence type will have the desired properties of the lemma. First with each number y_{t_x} a particular finite set β_x will be associated. Let the functions w_x and e_x be defined by

$$\begin{aligned} w_x &= \text{cardinality of } \alpha_{t_x} , \\ e_0 &= w_0 , \\ e_{n+1} &= w_{n+1} - w_n . \end{aligned}$$

Since y_{t_x} is a regressive function and since from the value of y_{t_x} we can determine the complete set α_{t_x} (refer to the remarks appearing before (13)), we see that from the value of y_{t_x} alone, each of the numbers w_x and e_x can be computed. Hence each of the mappings $y_{t_n} \to w_n$ and $y_{t_n} \to e_n$ will have a partial recursive extension; in the notation of [4] these properties are denoted respectively by

$$y_{t_n} \leqq^* w_n \text{ and } y_{t_n} \leqq^* e_n . \tag{21}$$

We will assume here that $e_0 \geqq 1$ (otherwise the proof would need to be slightly changed). Then, by (18), it will also follow that $e_n \geqq 1$ for each number n. For $n \in E$, let

$$\delta_n = [j(y_{t_n}, r) \mid r = 0, 1, \cdots, e_n - 1] . \tag{22}$$

Then $[\delta_n]$ will be a sequence of mutually disjoint nonempty sets. From

(21) and (22), we see that by knowing the value of y_{t_n} we can effectively find all the members of the set δ_n; this property will be denoted by writing

(23) $$y_{t_n} \leqq^* \delta_n .$$

For $n \in E$ set

(24) $$\beta_n = \delta_0 + \delta_1 + \cdots + \delta_n .$$

Then, in view of (23) and the regressiveness of y_{t_n}, it is possible to effectively find all the elements of β_n from the value of y_{t_n}. We will denote this property by

(25) $$y_{t_n} \leqq^* \beta_n .$$

In addition, note that

(26) $$\beta_0 \subseteqq \beta_1 \subseteqq \beta_2 \subseteqq \cdots, \text{ and}$$

(27) $$\text{card } \beta_x = \text{card } \alpha_{t_x} \text{ for every } x \in E .$$

Let

$$\beta = \sum_0^\infty \beta_n = \sum_0^\infty \delta_n .$$

We will assume here that the sets η and $\varphi_g(\beta)$ are separated (otherwise an easy change in the proof would be made), i.e.,

(28) $$\eta \mid \varphi_g(\beta) .$$

Let $B = \text{Req } \beta$. The remainder of the discussion now is toward showing that B will satisfy the desired requirements of the lemma, i.e., that B is a regressive isol and that B satisfies (2). Observe that by (28),

$$\eta + \varphi_g(\beta) \in Y + D_g(B) .$$

Hence in order to complete the proof, it suffices to show that

(29) β is a regressive and isolated set, and

(30) $$\varphi_f(\beta) \simeq \eta + \varphi_g(\beta) .$$

For (29): Note that β will be an infinite set, since $e_n \geqq 1$ for each number n. Also, it is easy to see that if β contains an infinite r.e. subset, then the set $(y_{t_0}, y_{t_1}, \cdots)$ would also then include an infinite r.e. subset. But then the set η would contain an infinite r.e. subset, yet we know that this cannot be true since it is an isolated set. And therefore we may conclude that β will be an isolated set. We know that the function y_{t_x} is regressive. If we combine this fact with (23) and the definition of β, then it is easy to see that β will be a regressive set, and in particular that a regressive enumeration of its members will be

$$j(y_{t_0}, 0), \cdots, j(y_{t_0}, e_0 - 1), j(y_{t_1}, 0), \cdots, j(y_{t_1}, e_1 - 1), \cdots .$$

For (30): Recall that

$$\beta = \sum_0^\infty \beta_n \quad \text{where} \quad \beta_n = \delta_0 + \cdots + \delta_n . \tag{31}$$

Because φ_f and φ_g are combinatorial operators, it follows from (26) and (31) that,

$$\varphi_f(\beta_0) \subseteqq \varphi_f(\beta_1) \subseteqq \cdots \text{ and } \varphi_f(\beta) = \sum_0^\infty \varphi_f(\beta_n) , \tag{32}$$

$$\varphi_g(\beta_0) \subseteqq \varphi_g(\beta_1) \subseteqq \cdots \text{ and } \varphi_g(\beta) = \sum_0^\infty \varphi_g(\beta_n) , \tag{33}$$

and also, in view of (19) and (27), that for $n \in E$,

$$\text{card } \varphi_f(\beta_n) = \text{card } \eta_{t_n} + \text{card } \varphi_g(\beta_n) . \tag{34}$$

Combining (15), (32), (33) and (34) gives

$$\text{card } \varphi_f(\beta_0) = \text{card } \eta_{t_0} + \text{card } \varphi_g(\beta_0), \text{ and} \tag{35}$$

$$\begin{aligned} \text{card } (\varphi_f(\beta_{k+1}) - \varphi_f(\beta_k)) &= \text{card } (\eta_{t_{k+1}} - \eta_{t_k}) \\ &+ \text{card } (\varphi_g(\beta_{k+1}) - \varphi_g(\beta_k)) . \end{aligned} \tag{36}$$

Now we can define a partial function,

$$q: \varphi_f(\beta) \longrightarrow \eta + \varphi_g(\beta) ,$$

based on the previous two equations. Let

$$q: \varphi_f(\beta_0) \overset{*}{\longrightarrow} \eta_{t_0} + \varphi_g(\beta_0) ,$$

$$q: (\varphi_f(\beta_{k+1}) - \varphi_f(\beta_k)) \overset{*}{\longrightarrow} (\eta_{t_{k+1}} - \eta_{t_k}) + (\varphi_g(\beta_{k+1}) - \varphi_g(\beta_k)) ,$$

where we write $\overset{*}{\longrightarrow}$ to mean that the related mapping is to be order preserving. From (35) and (36) it follows that the mapping q is well-defined, and from (12), (32) and (33) that q will map $\varphi_f(\beta)$ onto $\eta + \varphi_g(\beta)$ in a one-to-one manner. To verify (30), it suffices to prove that q will have a one-to-one partial recursive extension. Because the sets $\varphi_f(\beta)$ and $\eta + \varphi_g(\beta)$ are isolated, it follows from a theorem due to Dekker [4, Proposition 9(*b*)], that q will have a one-to-one partial recursive extension, if both q and q^{-1} have partial recursive extensions. It suffices therefore to verify this latter property, and this will be our approach here. We will consider first the mapping q.

Let $w \in \varphi_f(\beta)$. We now describe a procedure whereby, with the possible exception of finitely many such w, one can effectively compute the value of $q(w)$. From w first find the particular numbers x and u with

$$(37) \qquad w = j(x, u),\ \rho_x \subseteqq \beta \text{ and } u < c_{r(x)}\ .$$

Note that if ρ_x is nonempty then each of its members can also be found. Moreover, since φ_f is a normal combinatorial operator, it follows that for all but possibly finitely many $w \in \varphi_f(\beta)$ the corresponding finite set ρ_x appearing in (37) will be nonempty. From now on let us assume that ρ_x is nonempty. Members of ρ_x will be of the form $j(y_{t_k}, v)$, and for each such member we can find the corresponding values of y_{t_k} and v. In addition, the values of t_k and k can also be determined, by using the regressive properties of y_n and y_{t_n}. Let k^* denote the largest value of k such that $j(y_{t_k}, v) \in \rho_x$, for some number v. Then, it is easy to show that

$$\begin{aligned} &w \in \varphi_f(\beta_0) &&, \text{ if } k^* = 0, \text{ and} \\ &w \in \varphi_f(\beta_{k^*}) - \varphi_f(\beta_{k^*-1}), &&\text{ if } k^* \geqq 1\ . \end{aligned}$$

We know, by (25), that from the value of $y_{t_{k^*}}$ we can effectively find all the members of the set β_{k^*}. In addition, note that if $k^* \geqq 1$ then also the members of the set β_{k^*-1} can be found, for we may regress down from $y_{t_{k^*}}$ to $y_{t_{k^*-1}}$ and apply (25). In a similar manner, in view of (14), it follows that from the value of $y_{t_{k^*}}$ we can find all the members in the set

$$\begin{aligned} &\eta_{t_0} &&, \text{ if } k^* = 0, \text{ and} \\ &\eta_{t_{k^*}} - \eta_{t_{k^*-1}}, &&\text{ if } k^* \geqq 1\ . \end{aligned}$$

Finally, by combining these properties with the fact that the normal operators φ_f and φ_g are each recursive, it can be seen that the members in each of the sets below can be effectively determined,

$$\begin{gathered} \varphi_f(\beta_0) \text{ and } \eta_{t_0} + \varphi_g(\beta_0), \qquad \text{if } k^* = 0 \text{ and ,} \\ \varphi_f(\beta_{k^*}) - \varphi_f(\beta_{k^*-1}) \quad \text{and} \\ (\eta_{t_{k^*}} - \eta_{t_{k^*-1}}) + (\varphi_g(\beta_{k^*}) - \varphi_g(\beta_{k^*-1})), \quad \text{if } k^* \geqq 1\ . \end{gathered}$$

It follows directly from this property and the definition of q, that the value of $q(w)$ can now be computed. Therefore, there will be a procedure that is effective and which will enable one to compute $q(w)$ for all but a possible finite number of $w \in \varphi_f(\beta)$. It is readily seen that this feature implies that the mapping q will have a partial recursive extension.

An approach very similar to the previous one can be employed to show that the mapping q^{-1} will also have a partial recursive. For this reason we will omit the main details for doing this, and will only mention the two essentially new observatians that we would have been required to make. The first is that given any number $w \in \eta + \varphi_g(\beta)$ one can effectively determine whether $w \in \eta$ or $w \in \varphi_g(\beta)$. This property follows from the separability of the sets η and $\varphi_g(\beta)$ given in (28). The other observation is that if $w \in \eta$, then one can effectively find the particular numbers s, k^*, t_{k^*} and $y_{t_{k^*}}$ that are related to w in the following way, $w = y_s$ and

$$\begin{aligned} &w \in \eta_{t_{k^*}} &&, \text{ if } k^* = 0 , \\ &w \in (\eta_{t_{k^*}} - \eta_{t_{k^*-1}}), &&\text{ if } k^* \geqq 1 . \end{aligned}$$

This particular property follows from (14), (16), Property A and the regressive properties of the functions y_n and y_{t_n}. The importance of the second property lies in the fact that it means that from the value of any $w \in \eta$, one can effectively find $y_{t_{k^*}}$, and therefore also determine the appropriate sets,

$$\begin{aligned} &\beta_{t_0} \text{ and } \eta_{t_0} &&, \text{ if } k^* = 0 , \\ &\beta_{t_{k^*}}, \beta_{t_{k^*-1}}, \eta_{t_{k^*}} \text{ and } \eta_{t_{k^*-1}}, &&\text{ if } k^* \geqq 1 . \end{aligned}$$

It is then with these two observations that a similar approach, as with q, will lead to showing that q^{-1} will have a partial recursive extension.

In view of the remarks made up to this point, we see that the mapping

$$q\colon \varphi_f(\beta) \longrightarrow \eta + \varphi_g(\beta)$$

will have a one-to-one partial recursive extension. This verifies (30) and complets the proof of the lemma.

THEOREM 2. *Let $f\colon E \rightarrow E$ be a recursive function and A and Y be isols such that*

(1) $$D_f(A) = Y .$$

If Y is a regressive isol, then there will also exist regressive isols B such that,

$$D_f(B) = Y .$$

Proof. Let us assume that Y is a regressive isol. Let f^+ and f^- be the *positive* and *negative* recursive and combinatorial functions that are associated with f (refer to [11]). Then for every number $x \in E$, $f(x) = f^+(x) - f^-(x)$, and also

$$D_f(A) = D_f + (A) - D_f - (A) .$$

Therefore, by (1), it also follows that

$$D_f + (A) = Y + D_f - (A) .$$

If we now apply the previous lemma to this equation, we see that there will be a regressive isol B such that

$$D_f + (B) = Y + D_f - (B) ,$$

and from this identity it also follows that $D_f(B) = Y$.

REMARK. Theorem 2 is our principal result and it is easy to observe that it follows almost directly from the lemma. It turns out that, as a consequence of the manner in which the lemma was proved, a slightly stronger form of both the lemma and the theorem can be established. We would like to state without a proof the particular form that is related to the theorem. It involves the Nerode canonical extension of the familiar binary relation $\leqq$ (among numbers) to the isols. The extension procedure is introduced in [12], and for the relation $\leqq$ its extension will be denoted by $\leqq_\Lambda$. It can be shown that the regressive isol B constructed in the proof of the lemma (in each of the cases considered there) is related to the isol A by $B \leqq_\Lambda A$. Based on this fact one can obtain the following result.

THEOREM A. *Let $f: E \rightarrow E$ be a recursive function and A and Y be isols such that $D_f(A) = Y$. If Y is a regressive isol, then there will exist regressive isols B such that $B \leqq_\Lambda A$ and $D_f(B) = Y$.*

3. An example. It is possible that the canonical extension of a recursive function may map an isol that is nonregressive onto an isol that is infinite and regressive. We would like to give an example of such a function. First some definitions are needed.

If α and β are two sets of numbers, then $\alpha \leqq^* \beta$ will mean that either α is a finite set and card $\alpha \leqq$ card β, or else both α and β are infinite sets and there is a partial recursive function p such that, $\alpha \subseteqq \delta p$, $p(\alpha) = \beta$ and p is one-to-one on α. If A and B are two isols then $A \leqq^* B$ will mean that there are sets $\alpha \in A$ and $\beta \in \beta$ such that $\alpha \leqq^* \beta$. Let *min* (a, b) denote the familiar recursive function minimum (a, b), and let $D_{\min}$ denote its canonical extension to Λ^2. min (a, b) is not an almost combinatorial function, and therefore its canonical extension will not map Λ^2 into Λ. On the otherhand, it is proved in [3] that $D_{\min}$ will map Λ_R^2 into Λ_R. In addition, by combining results in [3] and [4], one obtains for $A, B \in \Lambda_R$,

$$D_{\min}(A, B) = A \Longleftrightarrow A \leqq^* B .$$

Concerning isols and regressive isols the following property due to Dekker [4] is also needed; if S and T are any isols, then

$$(*) \qquad S \leqq T \text{ and } T \in \Lambda_R \Longrightarrow S \in \Lambda_R .$$

In the result below we will construct the kind of example that was described earlier. We note that the functions $j(x, y)$, $k(x)$ and $l(x)$ that appear in its proof refer to those particular recursive functions introduced in §1.

THEOREM 3. *There is a recursive function $h(x)$ and an isol C such that $D_h(C) \in \Lambda_R$ and yet $C \notin \Lambda_R$.*

Proof. Define

$$h(x) = \min(k(x), l(x)) .$$

Then h will be a recursive function, and for $a, b \in E$

$$hj(a, b) = \min(a, b) .$$

Therefore also,

$$D_h D_j(U, V) = D_{\min}(U, V), \text{ for } U, V \in \Lambda .$$

Select $A, B \in \Lambda_R$ such that

$$(1) \qquad A \leqq^* B \quad \text{and} \quad A + B \notin \Lambda_R ;$$

the existence of such a pair of regressive isols is proved in [2]. Then it follows

$$D_h D_j(A, B) = D_{\min}(A, B) = A ,$$

and in addition, if we let $C = D_j(A, B)$, then also

$$(2) \qquad D_h(C) = A \in \Lambda_R .$$

The function $j(x, y)$ is recursive and combinatorial, and therefore its canonical extension will map Λ^2 into Λ. In particular, we see that

$$(3) \qquad C = D_j(A, B) \in \Lambda .$$

Let us now verify

$$(4) \qquad C = D_j(A, B) \in \Lambda_R \Longrightarrow A + B \in \Lambda_R .$$

First consider the implications,

$$\begin{aligned} D_j(A, B) \in \Lambda_R &\Longrightarrow 2D_j(A, B) \in \Lambda_R \\ &\Longrightarrow 2A + (A + B)(A + B + 1) \in \Lambda_R \\ &\Longrightarrow A + B \in \Lambda_R . \end{aligned}$$

The first two implications are clear. The last one follows from (*) and the property,

$$A + B \leqq 2A + (A + B)(A + B + 1) .$$

Together they imply (4). In view of (1), (3) and (4) we obtain $C \in \Lambda - \Lambda_R$, and if we combine this property with (2) the desired result follows directly.

N. B. The fact that the familiar j function is combinatorial we first learned from some unpublished notes of Erik Ellentuck. Once this property is pointed out it is easy to show, and we will leave it for the reader.

4. Recursive functions of several variables. We would like to describe some of the results that can be obtained for recursive functions of more than one variable that are similar to those given in §2. First let us note some features that distinguish the one and more than one variable cases. We know that for a recursive combinatorial function of one variable, its canonical extension will map regressive isols onto regressive isols. On the otherhand, even for recursive combinatorial functions of two variables, it need not be true that their canonical extension will map pairs of regressive isols onto regressive isols. For example, Dekker showed in [4] that it is possible for both the sum and the product of two regressive isols to be an isol that is non-regressive. The characterization of those recursive functions of two variables whose canonical extensions will map regressive isols to regressive isols was given by Mathew Hassett in [9]. The following is a special case of a theorem also due to Hassett [8].

THEOREM B. (Hassett) *Let $f: E^n \rightarrow E$ be a recursive and combinatorial function. Let $A_1, \cdots, A_n$ be n regressive isols whose sum $A_1 + \cdots + A_n$ is also regressive. Then the value of $D_f(A_1, \cdots, A_n)$ will be a regressive isol.*

Note that when $n = 1$ in Theorem B one obtains the earlier result mentioned about recursive combinatorial functions of one variable. Based upon the procedure for representing the canonical extension of a recursive function (in terms of the canonical extensions of recursive combinatorial functions) and applying Theorem *B*, analogues of Theorems

1 and 2 can be obtained for functions of more than one variable. We conclude the paper with statements of these two theorems.

THEOREM C. *Let $f: E^n \to E$ be a recursive function and $A_1, \cdots, A_n$ and Y be isols with $D_f(A_1, \cdots, A_n) = Y$. If the sum $A_1 + \cdots + A_n$ is regressive, then the isol Y will also be regressive.*

THEOREM D. *Let $f: E^n \to E$ be a recursive function and $A_1, \cdots, A_n$ and Y be isols with $D_f(A_1, \cdots A_n) = Y$. If Y is regressive, then there will be regressive isols $B_1, \cdots, B_n$ such that the sum $B_1 + \cdots + B_n$ will be regressive and also $D_f(B_1, \cdots, B_n) = Y$.*

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